# Appendix A

# Geometry tools

From *Algebra: For the Enthusiastic Beginner* (Draft version, July 2024)


The goal of this appendix is to introduce a number of standard results in geometry, so that we can invoke them when needed throughout the book. For example, we used the Pythagorean theorem, along with similar triangles, in Sections 6.7.2 and 6.7.3. And we used the base-times-height expression for the area of a rectangle on numerous occasions. We'll derive most of the results from scratch, although in a few cases we'll have to simply state things without proof. We'll use algebra freely here (including solving for variables), so it's assumed that you've already read up through Section 5.1. There won't be as many exercises in this appendix, since the main goal is just to get the basics down and then apply them elsewhere in the book.

We'll begin in Section A.1 with a discussion of *angles* and the properties they satisfy. We'll talk about the various types of *triangles* in Section A.2, and then go into more detail on *similar triangles* in Section A.3. Sections A.4 and A.5 cover the *areas* and *volumes* of various shapes. Section A.6 presents the *Pythagorean theorem*, along with a number of proofs. We'll finish up in Section A.7 with a discussion of a few triangles with particularly nice shapes. In the end, this appendix covers quite a bit more geometry than we'll actually use in the book. But if we're going to derive a few of the basic results, why not derive lots of them!

## A.1 Angles

### A.1.1 Degrees

The standard way of measuring angles is with *degrees*. (Another common way is with *radians*, but we'll stick with degrees here.) A degree is defined so that there are 360 degrees in a full revolution, that is, in a full circle.

> FACT 1: There are 360 degrees in a full circle, as shown in the last angle in Fig. A.1.

Examples of various angles are shown in Fig. A.1. The little circle on the upper-right of each number is the degree symbol. So $20^\circ$ is read as "20 degrees." You can think of angles as pie pieces. For example, since $18 \cdot 20 = 360$, it takes 18 of the 20° pie pieces to make a whole pie. Similarly, since $8 \cdot 45 = 360$, it takes eight of the 45° pieces to make a whole pie.

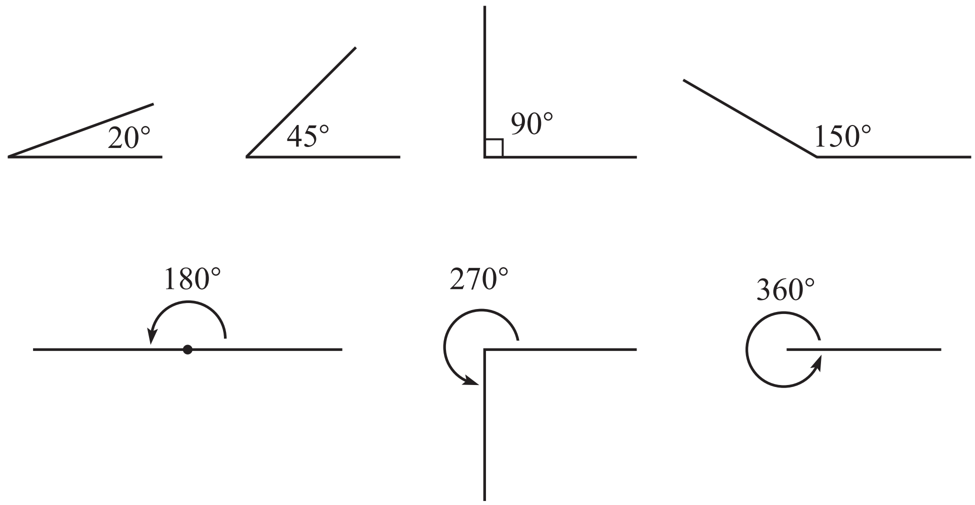


**Figure A.1:** Examples of some angles.

A 90° angle, which is called a *right angle*, is a quarter of a full revolution, because 90 is 1/4 of 360. It takes four of the 90° pieces to make a whole pie. A right angle is denoted by a little square in the corner, as shown in Fig. A.1. A right angle is what you have at each corner of a square or rectangle.

The tip of an angle (where the sides of the pie piece meet) is called the *vertex*. In Fig. A.1 the location of the vertex is clear in all of the angles except 180°. For that angle, someone needs to tell you where the vertex is. We've indicated it with a dot, which we could have put anywhere on the line.

Why did someone long ago define a degree so that there are 360 of them in a full circle? Why choose 360 and not some other number, like 100, or 42, or 137? It's unclear if anyone knows the answer for sure, but some possible reasons are: (1) There are 365 days in a year, which is close to 360. (2) The Sumerians and Babylonians used a sexagesimal number system, which is based on the number 60. And 360 is a nice multiple of 60. (3) 360 is divisible by many numbers (24 of them, as you can verify by using the technique in Exercise 2.20). In particular, 360 is divisible by all of the integers up to 10 except 7. So if you divide 360 into 5 equal angles, or 8, or 9, etc., you'll end up with a nice integer number of degrees in each angle.

Angles are often labeled with Greek letters, such as $\theta$ (theta), $\alpha$ (alpha), or $\beta$ (beta). But any kind of letter is fine. Sometimes we'll use $a$, $b$, $c$, etc. The "$\angle$" symbol stands for "angle." So if we have a triangle with vertices (the "corners") labeled as $A$, $B$, $C$, then the angle at vertex $A$ can be denoted by $\angle A$. Or you can just call it $\alpha$ or $a$ or whatever you want, as long as you've defined it clearly in a diagram or with words.

Another notation for the angle at $A$ in a triangle with vertices $A$, $B$, $C$ is $\angle BAC$ (or $\angle CAB$). The order of the three letters $BAC$ tells you that if you march from $B$ to $A$ to $C$, you'll create two segments ($BA$ and $AC$) that form an angle with its vertex at $A$. Likewise for the three letters $CAB$. The letter corresponding to the location of the angle is always the middle one. The various notations that are commonly used for the angle at $A$ in triangle $ABC$ are shown in Fig. A.2.

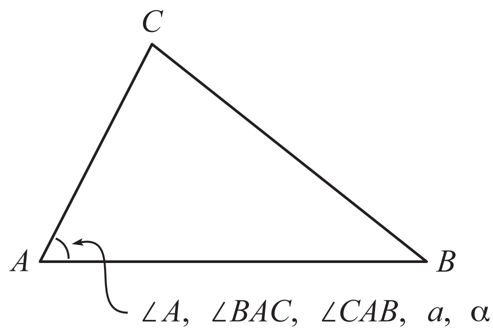


**Figure A.2:** Various ways to label an angle.

### A.1.2 Vertical angles

Consider the angles shown in Fig. A.3, formed by two straight lines crossing each other. Four angles are produced at the crossing, and we claim that they come in two equal pairs, as indicated by the two $\alpha$'s and the two $\beta$'s. The name for the two equal angles in a pair is *vertical angles*. The term "opposite angles" would be more descriptive, but "vertical angles" was first used a couple millennia ago, and it stuck.

FACT 2: Vertical angles are equal, as indicated in Fig. A.3.

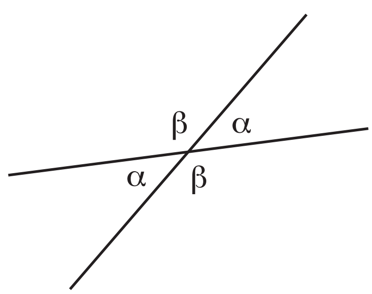


**Figure A.3:** The two $\alpha$'s are a pair of vertical angles, as are the two $\beta$'s.

We'll show below that vertical angles are indeed equal, but first we need to make an observation: In Fig. A.4(a), the angles $\alpha$ and $\beta$ add up to 180°. This is true because together they form a straight line, and we saw in Fig. A.1 that a straight line corresponds to an angle of 180°. Two angles like these that together form a straight line (and hence add up to 180°) are called *supplementary* angles:

$$\alpha + \beta = 180^\circ \quad \text{(Supplementary angles, Fig. A.4(a))} \tag{A.1}$$

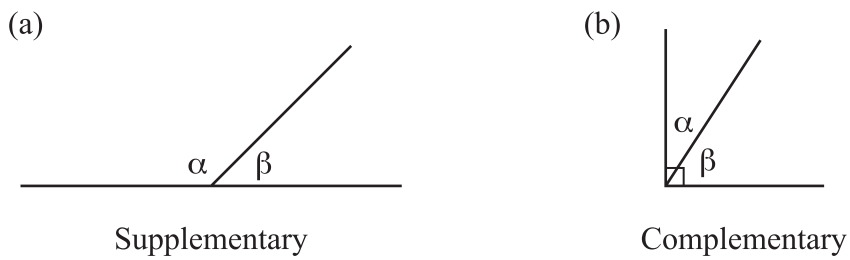


**Figure A.4:** (a) Supplementary angles add up to 180°. (b) Complementary angles add up to 90°.

We can draw the tilted line in Fig. A.4(a) at any angle we want and make $\beta$ be whatever we want, and $\alpha$ will always be $180° - \beta$. The figure we drew happens to have $\beta = 45°$ and $\alpha = 135°$. But the angles can take on any values that add up to 180°.

A related definition is the following: Two angles that together form a right angle (and hence add up to 90°) are called *complementary* angles, as shown in Fig. A.4(b).

$$\boxed{\alpha + \beta = 90° \quad \big(\text{Complementary angles, Fig. A.4(b)}\big)} \tag{A.2}$$

Supplementary and complementary angles don't actually need to be right next to each other, where they would form a straight line or a right angle. You can draw a 53° angle on one piece of paper, and a 127° angle on another piece of paper across the room, and they're still supplementary angles since they add up to 180°.

Proof that vertical angles are equal: To show that the angles in each vertical pair in Fig. A.3 are equal, we'll start by labeling the four angles in Fig. A.5 with four different letters, because at the moment we don't yet know anything about how they relate to each other.

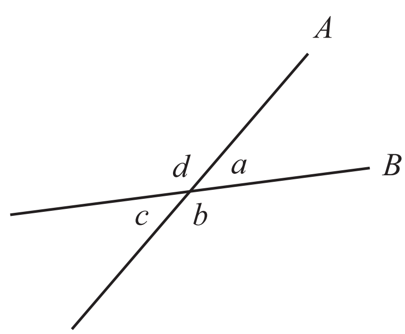


**Figure A.5:** Proving that vertical angles are equal: $a$ and $b$ are supplementary, as are $b$ and $c$. So $a$ must equal $c$.

The angles $a$ and $b$ are supplementary because together they form line $A$. So $a+b = 180°$, and hence $a = 180° - b$. Likewise, angles $b$ and $c$ are supplementary because together they form line $B$. So $c + b = 180°$, and hence $c = 180° - b$. Since the expressions we have just obtained for $a$ and $c$ are equal (they are both $180° - b$), we conclude that $a = c$. That is, the angles in that vertical pair are equal. Their common value is labeled as $\alpha$ above in Fig. A.3.

A similar procedure using the facts that $a + b = 180°$ (from looking at line $A$) and $a + d = 180°$ (from looking at line $B$) tells us that $b = d$. So the angles in that vertical pair are also equal. Their common value (which is labeled as $\beta$ above in Fig. A.3) is $180° - a$, or equivalently $180° - \alpha$. ■

## A.1.3 Parallel lines, alternate interior angles

### Parallel lines

Parallel lines are defined as follows:

> DEFINITION: Two straight lines are *parallel* if they never intersect when extended infinitely in both directions.

Examples of three different groups of parallel lines are shown in Fig. A.6. Given one line, we can draw an arbitrary number of other lines parallel to it, as suggested by the group of eight lines on the right; we could have drawn 1000 parallel lines if we wanted to. We've given the lines a finite length (of course) in the figure, but the segments represent lines that extend infinitely in both directions.

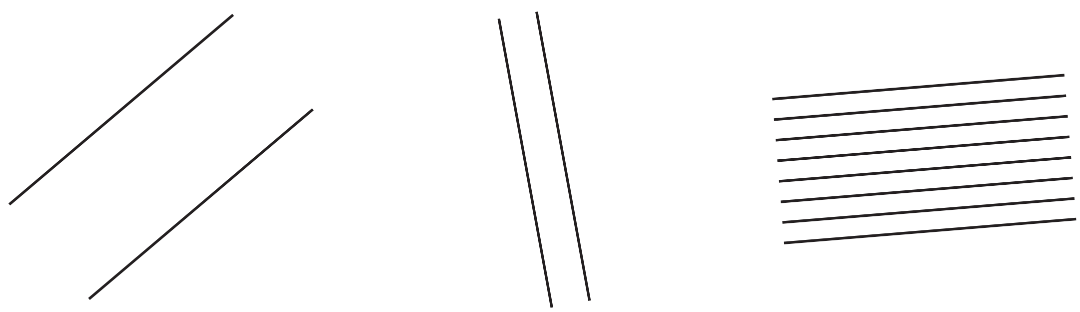

**Figure A.6:** Examples of parallel lines.

A comment on terminology: The word "line" technically means an *infinite* line. And most people also take it to mean an infinite *straight* line. So the phrase "infinite straight line" is redundant; it means the same thing as "line." However, many people (including me) often get a little sloppy and use the term "line" when talking about paths with *finite* length, which should more properly be called *segments*. And they also often use "line" when talking about curved paths. So someone might call a squiggle a "curved line," even though lines are technically straight. These different uses of "line" usually don't cause any confusion, because it's generally clear from the context what the shape looks like.

Note that a line (and more generally any mathematical curve, finite or infinite) has length but not width. Of course, when you draw a line/curve on a piece of paper, it necessarily has a non-zero width (the width of your pencil lead). If you drew it with truly zero width, you wouldn't be able to see it. But for any line you see drawn on a page, you should pretend that the width is essentially zero. Similarly, a *point* is something with no length or width in any direction. But again, if you drew a point that way, you wouldn't be able to see it.

### Alternate interior angles

Fig. A.7 shows two parallel lines $A$ and $B$ that are cut by a third line $C$. The two angles $\alpha$ shown are called *alternate interior angles*. Likewise, the $\beta$'s are a second pair of alternate interior angles. The word "alternate" indicates that the angles are on opposite sides of line $C$, and the word "interior" indicates that they lie between lines $A$ and $B$.

> FACT 3: If lines $A$ and $B$ in Fig. A.7 are parallel, then alternate interior angles are equal. That is, the two $\alpha$ angles are equal, and the two $\beta$ angles are equal.

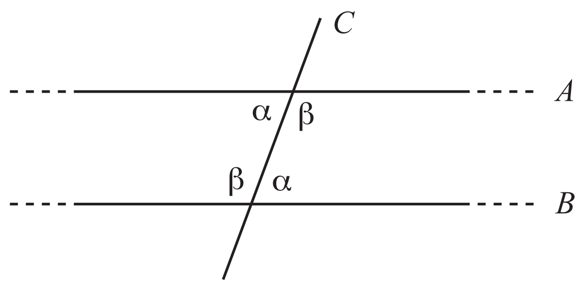


**Figure A.7:** If lines $A$ and $B$ are parallel, then alternate interior angles are equal.

The "reverse" statement (with the "if" and "then" parts switched) is also true: If alternate interior angles are equal in a setup like Fig. A.7, then lines $A$ and $B$ are parallel. Putting the two statements together, we see that "parallel lines" and "equal alternate interior angles" go hand-in-hand. You can't have one without the other:

> FACT 3′: Whenever you have parallel lines, you also have equal alternate interior angles. And whenever you have equal alternate interior angles, you also have parallel lines.

If we combine Fact 3 with our earlier Fact 2 about equal vertical angles, we see that there are actually *four* angles that equal $\alpha$ (and likewise four angles that equal $\beta$) in Fig. A.7, as shown in Fig. A.8. This means that the words "alternate interior angles" in Fact 3 (and Fact 3′) could just as well be replaced with "alternate *exterior* angles," because the $\alpha$ at the bottom left in Fig. A.8 equals the $\alpha$ at the top right. These angles are "alternate" with respect to line $C$, but they're now "exterior" to lines $A$ and $B$.

"PROOF" THAT PARALLEL LINES IMPLY EQUAL ALTERNATE INTERIOR ANGLES: How do we prove Fact 3? Well, technically we can't. Euclid (the founder of geometry) considered Fact 3 (or rather, an equivalent statement) to be a *postulate* (or *axiom*), which is something you accept as true without proof. The reason he had to just accept it is that not all "spaces" are

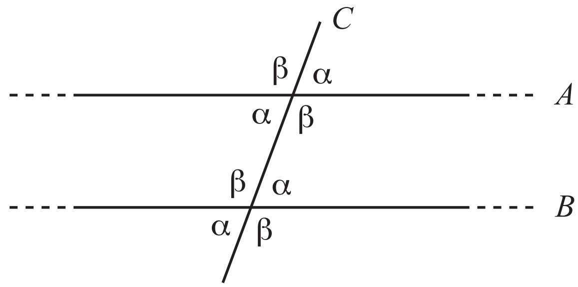


**Figure A.8:** If lines $A$ and $B$ are parallel (which means that alternate interior angles are equal), then since vertical angles are also equal, it follows that alternate exterior angles are likewise equal.

flat like a piece of paper. Some are curved, like the surface of a sphere (and there are many other types of curved surfaces). In curved spaces, there are complications with Fact 3 (or its "reverse" statement). However, for a flat space like a piece of paper (which is the kind of space we'll deal with in this book), there are intuitive "proofs" of Fact 3 that are plenty convincing:

- One "proof" is to imagine sliding the picture in Fig. A.8 diagonally upward along line $C$, until line $B$ ends up on top of line $A$ (since $A$ and $B$ are parallel), with $C$ remaining on top of itself. (Pretend that we don't yet know that the angles are all equal to $\alpha$ or $\beta$. We just have eight unknown angles at this point.) The lower-right angle $\alpha$ will then end up exactly on top of the upper-right angle $\alpha$ (because $B$ ends up on top of $A$, and $C$ ends up on top of itself). So these two angles must be equal (meaning that we can indeed label them both as $\alpha$). Fact 2 on vertical angles then tells us that the other two angles labeled as $\alpha$ in Fig. A.8 are in fact also equal to $\alpha$. The alternate interior angles $\alpha$ in Fig. A.7 are therefore equal, as we wanted to show. The same reasoning holds for the four $\beta$ angles.
- Another "proof" is to put a dot on line $C$ in Fig. A.7, halfway between lines $A$ and $B$, and to then imagine rotating the picture by half a turn around the dot. Lines $A$ and $B$ will exchange places, and line $C$ will end up on top of itself. So the lower-right angle $\alpha$ will end up exactly on top of the upper-left angle $\alpha$. These two angles must therefore in fact be equal, as we wanted to show. ■

> Is one of the angles superior?
> Or perhaps just a little inferior?
> "No!" we proclaim,
> "They're exactly the same
> If they're alternate *and* they're interior."

The above rotation type of proof also works for Fact 2 on vertical angles. If you rotate the picture in Fig. A.3 by half a turn around the point where the two lines intersect, the two angles $\alpha$ will end up exactly on top of each other. So they must be equal. Likewise for the $\beta$'s.

There isn't anything surprising about the equal $\alpha$'s or $\beta$'s in Figs. A.3 and A.8. You can pretty much assume that if two angles in figures like these look like they're equal, they are.

## A.2 Triangles

### A.2.1 180° in a triangle

Fact 3 allows us to prove a very useful result for triangles:

FACT 4: The sum of the angles in any triangle is 180°.

PROOF THAT TRIANGLE ANGLES ALWAYS ADD UP TO 180°: In Fig. A.9 we have labeled the angles of triangle $ABC$ as $\alpha$, $\beta$, and $\gamma$. And we have drawn a line through $C$ parallel to $AB$. Due to the two parallel lines in the figure, Fact 3 tells us that the alternate interior angles $\alpha$ produced by the line $AC$ are equal, as shown. Likewise, the alternate interior angles $\beta$ produced by the line $BC$ are equal.

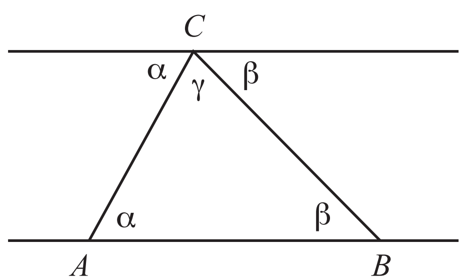


**Figure A.9:** Proof that the sum of the angles in a triangle is 180°.

We now note that since the three angles at $C$ together form a straight line, they must add up to 180°. That is,

$$\boxed{\alpha + \beta + \gamma = 180^\circ} \tag{A.3}$$

And this is just the statement that the sum of the three angles in triangle $ABC$ equals 180°, as we wanted to show. For the specific triangle we've drawn in Fig. A.9, the three angles happen to be $\alpha = 61^\circ$, $\beta = 45^\circ$, and $\gamma = 74^\circ$. These do indeed add up to 180°. ■

Some special cases of the fact that the sum of the angles is always 180° are shown in Fig. A.10: (a) If all three angles are equal, then they are all equal to $180^\circ/3 = 60^\circ$. (b) If two of the angles are very close to 90°, then the third angle must be very small. (c) If one angle is very close to 180°, then the two other angles must be very small.

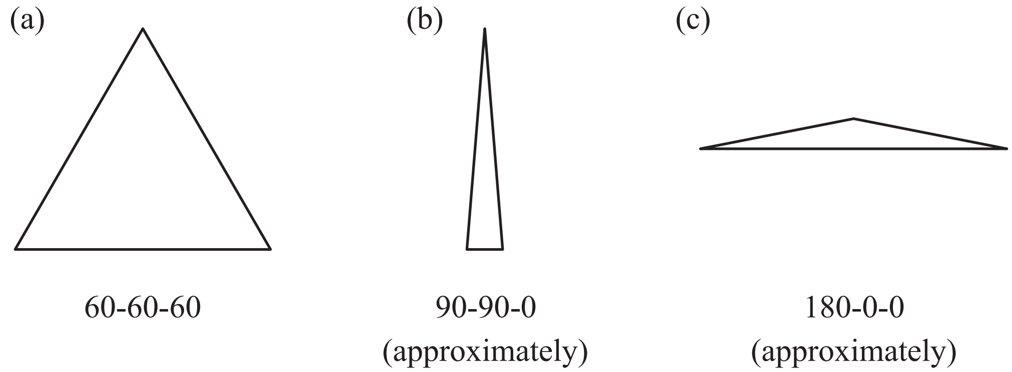


**Figure A.10:** Special cases of triangle shapes.

### A.2.2 Types of triangles

#### Acute, obtuse, right

Any triangle falls into one of the following three categories, which are shown in Fig. A.11:

(a) ACUTE TRIANGLE: All of the angles are smaller than 90°.

(b) OBTUSE TRIANGLE: One of the angles is larger than 90°. In Fig. A.10, the first two triangles are acute, and the third one is obtuse. Note that it is impossible for two angles to be larger than 90°, because that would make the sum of the angles be larger than the required 180°.

(c) RIGHT TRIANGLE: One of the angles is equal to 90°. A right triangle is the borderline case between acute and obtuse; one angle is exactly 90°. The diagonal side of a right triangle is called the *hypotenuse*, and the other two sides (adjacent to the right angle) are called the *legs*. See Section A.6 for a detailed discussion of right triangles and the Pythagorean theorem.

Note that if two of the angles equal 90°, then the third angle is zero, since the sum must be 180°. So we simply have a straight line instead of an actual triangle. Fig. A.10(b) would be a straight vertical line if both of the bottom angles were exactly equal to 90°.

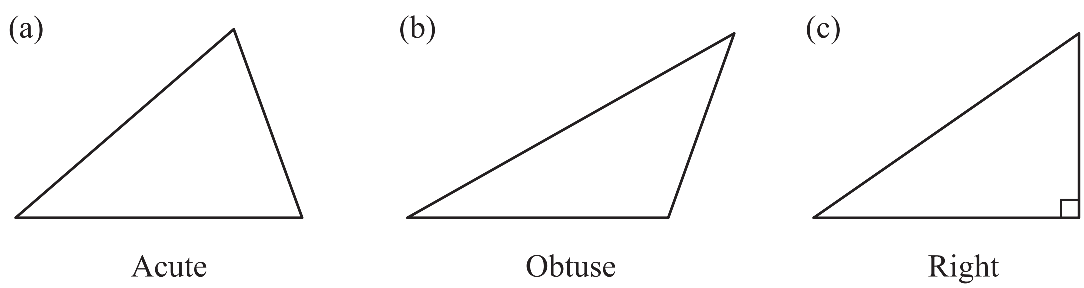


**Figure A.11:** Three types of triangles. (a) Acute: All angles are smaller than 90°. (b) Obtuse: One angle is larger than 90°. (c) Right: One angle is equal to 90°.

#### Equilateral, isosceles

An *equilateral* triangle is one in which all three sides have the same length. The first triangle in each of Fig. A.10 and Fig. A.12 is equilateral.

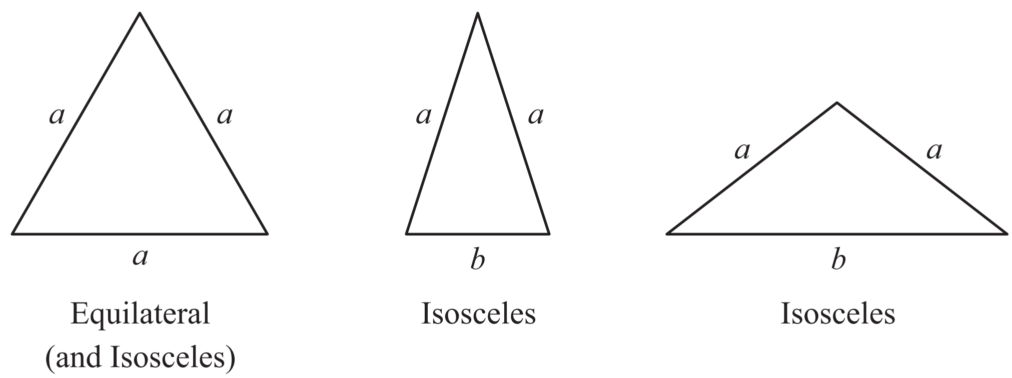


**Figure A.12:** An equilateral triangle has all three sides equal. An isosceles triangle has two sides equal.

An *isosceles* triangle is one that has two sides of the same length. This common length may be longer or shorter than the third side (or equal), as shown in the second and third triangles in Fig. A.12. An equilateral triangle is necessarily also an isosceles triangle (there's nothing wrong with the third side of an isosceles triangle being equal to the other two). But an isosceles triangle is not necessarily equilateral, as is evident from the second and third triangles in Fig. A.12.

In an equilateral triangle, all three angles are equal (to 60°). In an isosceles triangle, the two angles opposite the two equal sides are equal; these are called the "base angles."

### A.2.3 Congruent, similar triangles

#### Congruent

Two triangles are *congruent* if they have the same shape and size. Equivalently, two triangles are congruent if the three side lengths of one are equal to the three side lengths of the other. Fig. A.13 shows three triangles that are congruent to each other. They all have the same side lengths $a$, $b$, and $c$. Relative to the first triangle, the second one is shifted to the right, and also rotated. But that's fine; neither of these motions affects the side lengths, so the triangles are still congruent. The third triangle is obtained by flipping the first one over; it's the "mirror image." This is again fine – the flipping motion doesn't affect the side lengths.

In short, you can imagine a triangle made out of cardboard, and you can move it around wherever you want. You can even move it so that it doesn't lie in the plane of the page, although then it's hard to draw. But no matter where the cardboard triangle ends up in 3-dimensional space, and no matter how it is oriented, it will be congruent to the original triangle.

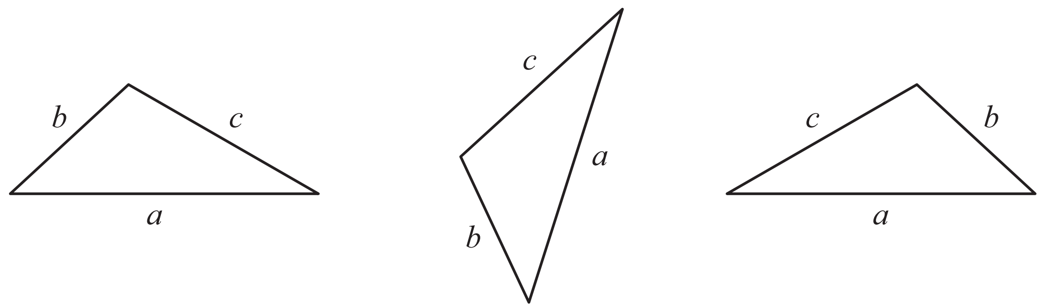


**Figure A.13:** Three triangles that are congruent to each other. They all have the same three side lengths.

### Similar

Two triangles are *similar* if they have the same shape but not (necessarily) the same size. Equivalently, two triangles are similar if the three angles of one are equal to the three angles of the other. Equivalently again, two triangles are similar if the side lengths of one are all the same multiple of the side lengths of the other; one triangle is simply a scaled-up version of the other. Imagine zooming in on your computer screen. This zooming changes the size, but not the shape. Fig. A.14 shows two similar triangles, where the common ratio of the corresponding side lengths is 2. But the 2 here could be anything. Also, as in Fig. A.13, we can shift, rotate, or flip either of the triangles in Fig. A.14, and they will still be similar.

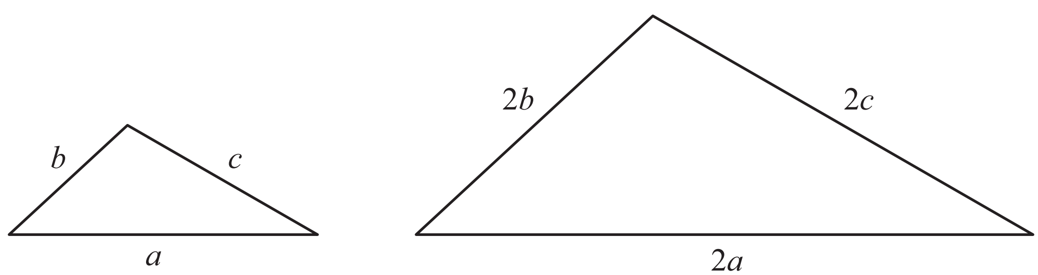


**Figure A.14:** Two similar triangles. The side lengths of one are all the same multiple (2 here) of the side lengths of the other. The two triangles have the same shape, and they have the same three angles.

In the special case where the zooming factor is 1 (that is, we haven't actually zoomed at all), the similar triangles are also congruent. We see that two congruent triangles are necessarily also similar (there's nothing wrong with the zooming factor being 1). But two similar triangles are not necessarily congruent, as is evident from Fig. A.14. We'll discuss similar triangles in more detail in Section A.3.

## A.2.4 Triangle inequality, degenerate triangles

A *degenerate* triangle is one that isn't actually a triangle, due to the fact that it has no height (or no width, depending on how you look at it). Equivalently, a degenerate triangle is one where two of the sides lie in a straight line along the third side. Equivalently again, one of the vertices lies on the "opposite" side. Imagine taking the squat triangle in Fig. A.15(a) and lowering the top vertex until it lies along the bottom side. This yields the degenerate "triangle" in Fig. A.15(b), which is actually just a straight line. We have put dots at the locations of the vertices, because otherwise you wouldn't be able to tell where the middle vertex is. All you would see is a featureless straight line.

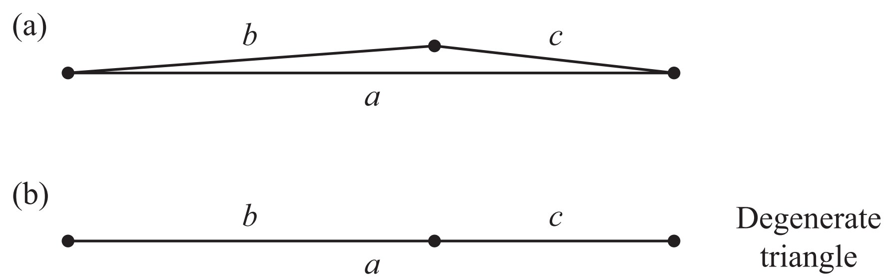


**Figure A.15:** Producing a degenerate triangle by lowering the top vertex until it lies on the bottom side.

The concept of a degenerate triangle (which, again, isn't much of a triangle) allows us to deduce the *triangle inequality*:

FACT 5: (Triangle inequality) The sum of any two sides of a triangle is always greater than or equal to the third side.

In particular, for the triangles in Fig. A.15, the triangle inequality says that

$$\boxed{b + c \geq a} \qquad \text{(Triangle inequality)} \tag{A.4}$$

Note: When talking about triangles (or any shape), the word "side" can be used in two different ways: It can refer to the side itself, or to the *length* of the side. In the first of these usages, you would say something like, "The length of the side is 3 inches," while in the second you would say, "The side is 3 inches." Both usages are fine. In the above statement of Fact 5, we used the second one.

PROOF OF THE TRIANGLE INEQUALITY: To show that the inequality in Eq. (A.4) holds, we need to consider two cases. First, for the degenerate triangle in Fig. A.15(b), we have $b + c = a$, because the $b$ and $c$ segments together exactly add up to the $a$ segment. So equality holds in Eq. (A.4) in this case.

Second, for the "normal" triangle in Fig. A.15(a), the sum $b + c$ is strictly greater than $a$, due to the fact that the shortest distance between any two arbitrary points is along the straight line between them. This means that since $a$ is the straight-line shortest distance between the left and right vertices, and since the $b + c$ distance between these two vertices isn't straight, the sum $b + c$ must be larger than $a$. That is, $b + c > a$. We therefore see that the $b + c \geq a$ triangle inequality in Eq. (A.4) is indeed true, with equality holding in the case of a degenerate triangle, and strict inequality holding in all other cases. ■

The triangle inequality holds for any grouping of the sides. So in addition to Eq. (A.4), it is also true that $a + b \geq c$ and $a + c \geq b$. These inequalities are clear by looking at Fig. A.15. The first one comes from the fact that $c$ represents the shortest distance between the right two vertices. And the second one comes from the fact that $b$ represents the shortest distance between the left two vertices.

If someone gives you three sticks with lengths 4, 5, and 6, then you can make a triangle with them, because the triangle inequality holds for all three possible groupings, as you can quickly verify. To construct the actual triangle, you can connect the sticks of lengths 4 and 5 with a hinge. If you start with the hinge wide open at 180° and then gradually decrease the angle, the third side will eventually have length 6, at which point you can insert the third stick. For the full range of the hinge (from 180° down to 0), the length of the third side will sweep through all values from $5 + 4 = 9$ down to $5 - 4 = 1$. So if the three sticks instead had lengths 4, 5, and 10, you wouldn't be able to make a triangle with them, because 10 is larger than $4 + 5$.

### A.2.5 Quadrilaterals, $n$-gons

Fact 4 tells us that the sum of the angles in a triangle is 180°. What if we have a shape with more than three sides, like a square or a rectangle? Well, the four corners of a square or a rectangle are all right angles (90°), so their sum is $4 \cdot 90° = 360°$. But what about the irregular 4-sided shape in Fig. A.16(a)? (A general 4-sided shape is called a *quadrilateral*.) What is the sum of its four angles?

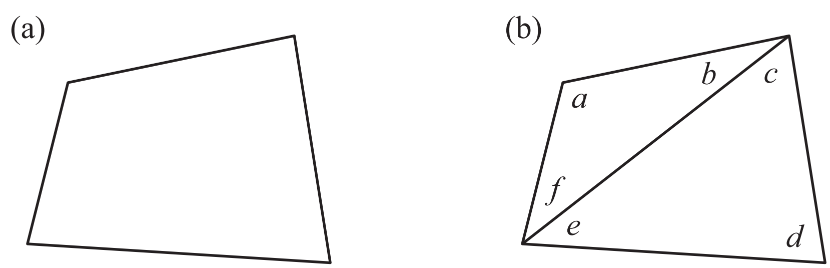


**Figure A.16:** Proving that the sum of the angles in a quadrilateral is 360°.

It turns out that the sum of the four angles in *any* arbitrarily shaped quadrilateral (assuming it lies in a plane) is always 360°. We can demonstrate this by dividing the quadrilateral into two triangles, as shown in Fig. A.16(b). We can then write the sum of the four angles in the quadrilateral as

$$S_{\text{quad}} = a + (b + c) + d + (e + f). \tag{A.5}$$

Grouping these angles in a helpful manner yields

$$S_{\text{quad}} = (a + b + f) + (c + d + e) = 180° + 180° \quad \Longrightarrow \quad \boxed{S_{\text{quad}} = 360°} \tag{A.6}$$

where we have used the fact that the sum of the angles in a triangle is 180°. In short, a quadrilateral is made of two triangles, so the sum of the angles is $2 \cdot 180° = 360°$.

The quadrilateral in Fig. A.16 is a convex one, but a similar proof also works in the concave case (you can think about this). A *convex* polygon is one with the property that if you wrap a rubber band around it, the band touches all of the vertices. A *concave* polygon has at least one vertex that the band doesn't touch, as shown on the right in Fig. A.17. Equivalently, a more mathematical-sounding definition is that a convex polygon is one where all of the angles are less than or equal to 180°. A concave polygon has at least one interior angle that is greater than 180°, as shown. You can remember which name goes with which shape by noting that if the two figures in Fig. A.17 each represent a huge boulder, then the con*cave* one has a *cave* that you can climb into.

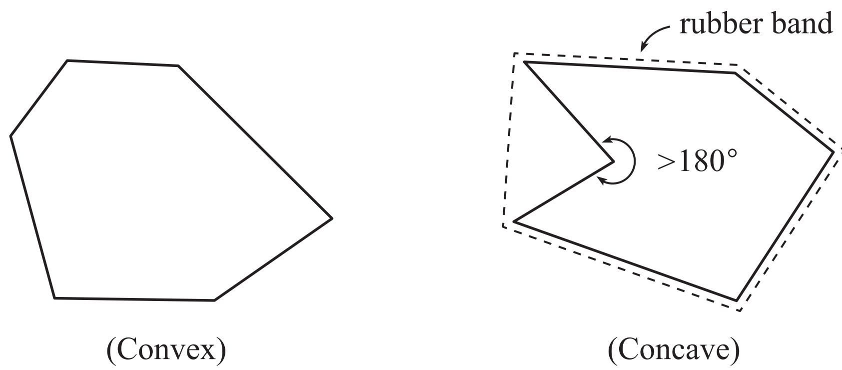


**Figure A.17:** Convex and concave polygons. A concave polygon has the property that a rubber band wrapped around it misses at least one vertex. Equivalently, at least one interior angle is greater than 180°.

**Exercise A.1** What is the sum of the angles in a general $n$-sided polygon (or "$n$-gon" for short), such as the one shown in Fig. A.18? (We have chosen $n = 9$ for concreteness. You can assume the polygon is convex.) Derive a formula for the sum of the angles in terms of $n$, in two different ways by making use of the two figures shown.

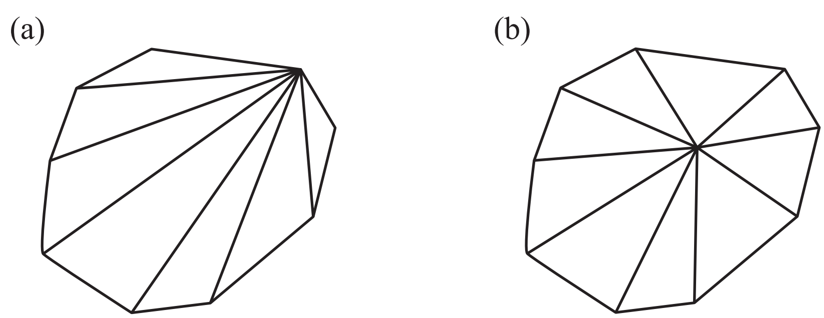


**Figure A.18:** Two ways to find the sum of the angles in an $n$-sided polygon.

## A.3 Similar triangles

We gave a few different definitions of similarity above in Section A.2.3. Let's repeat them here. Two triangles are similar if:

1. They have the same *shape*, and differ only in *size*.

2. They have the same three angles.

3. Each triangle is a scaled-up or scaled-down version of the other. That is, the side lengths of one triangle are all the same multiple of the side lengths of the other.

These three definitions all say the same thing; any one of them implies the other two. As far as the third definition goes, you can imagine zooming in or out on your computer. Zooming in is the same as scaling up (making larger), and zooming out is the same as scaling down (making smaller). The important point is that any zooming you do changes only the size, not the shape. If you zoom in on a triangle, you increase all the side lengths by the same factor, and you don't change the shape (that is, you don't change the angles).

Other shapes can be similar too; they don't need to be triangles. You can have a wacky 7-sided figure, and if you zoom in and make it larger, the new 7-sided shape will be similar to the old one. But we'll focus on triangles here.

Fig. A.19 shows two similar triangles. With regard to the above three definitions: (1) the triangles have the same shape, (2) they have the same three angles, and (3) the right one is a scaled-up version of the left one (by a factor of 2). These three statements are all equivalent.

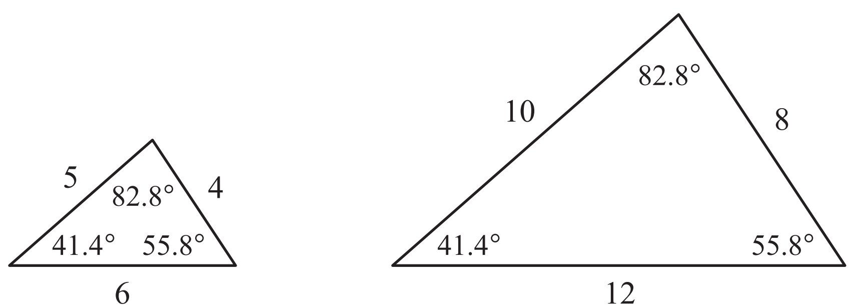


**Figure A.19:** Two similar triangles. They have the same shape, the same angles, and one is a scaled-up version of the other (by a factor of 2 here).

Although we have chosen the larger triangle in Fig. A.19 to be 2 times as large as the smaller one, any other factor would work just as well. Even a factor of 1 is fine. In that case, the triangles not only have the same shape, they also have the same size. So they are congruent. If the left triangle is very tiny, say $1/100$ the size of the right one, then if we zoom in on the left triangle by a factor of 100, it will look exactly like the right one.

The above three equivalent definitions of similarity are summarized in the following limerick:

Objects possess similarity
If their angles lack any disparity.
Their shapes must agree,
And if one is quite wee,
You can zoom in until there is parity.

The third definition of similarity (the "scaled-up/down" one) allows us to make some useful statements. Since the larger triangle in Fig. A.19 is twice as large as the smaller one, every side of the larger triangle is *twice as long as the corresponding side* of the smaller triangle. We can check this fact by writing down the three ratios of corresponding sides, and noting that they are all equal to 2:

$$\frac{10}{5} = \frac{8}{4} = \frac{12}{6} = 2. \tag{A.7}$$

See the "ratio" entry in the Glossary for a discussion of that term.

There is also another set of true statements we can make. In the smaller triangle in Fig. A.19, the ratio of the upper-left side to upper-right side is $5/4$. And in the larger triangle, the ratio of the same two sides is $10/8$, which also equals $5/4$. So the two ratios are the same. Looking at the ratios of the three different pairs of sides (upper-left/upper-right, upper-left/bottom, upper-right/bottom), we can write down the following three true statements:

$$\frac{5}{4} = \frac{10}{8}, \qquad \frac{5}{6} = \frac{10}{12}, \qquad \frac{4}{6} = \frac{8}{12}. \tag{A.8}$$

The lefthand sides of these three equations are the ratios of two given sides in the smaller triangle, and the righthand sides are the ratios of the corresponding two sides in the larger triangle. We see that when we scale up the smaller triangle (by a factor of 2 in the present case), the ratios don't change.

Fig. A.20 shows the general case where a triangle is scaled up by a factor of $N$, instead of the specific factor of 2 in Fig. A.19. The $N$ in Fig. A.20 happens to be 1.47, but that isn't important here.

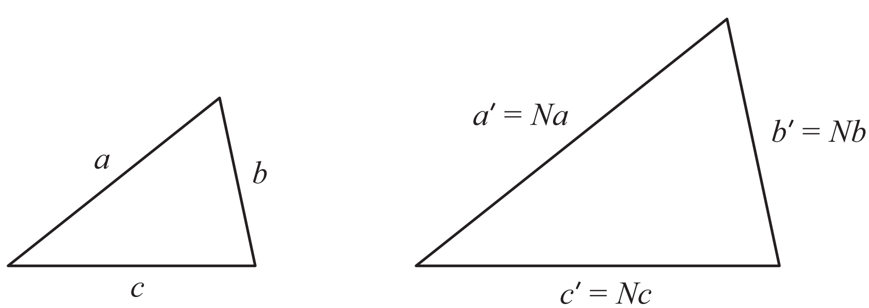


**Figure A.20:** Two similar triangles, where the scaling factor is $N$.

Eq. (A.7) is now replaced with this more general version:

$$\boxed{\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} = N} \qquad \text{(Ratios } \textit{between} \text{ triangles)} \tag{A.9}$$

And Eq. (A.8) is replaced with this more general version:

$$\boxed{\frac{a}{b} = \frac{a'}{b'}, \qquad \frac{a}{c} = \frac{a'}{c'}, \qquad \frac{b}{c} = \frac{b'}{c'}} \qquad \text{(Ratios } \textit{within} \text{ triangles)} \tag{A.10}$$

The first equality here is true because $a'/b' = Na/Nb$, and the $N$'s cancel, so we're left with $a/b$. Likewise for the other two equalities. In short, scaling up/down a triangle doesn't change the ratios, because all sides are scaled up or down by the same factor.

Each of the ratios in Eq. (A.9) involves sides in both triangles. That is, the ratios compare sides *between* the two triangles. (Each ratio is associated with a given pair of corresponding sides.) In contrast, each of the ratios in Eq. (A.10) involves sides *within* a particular triangle. (Each ratio is associated with a specific triangle.) Depending on what the setup is, one set of equations might be more natural to use than the other. But it doesn't matter which set you use, because they say the same thing in the end. This is true because, for example, if we take the $a/b = a'/b'$ relation in Eq. (A.10) and multiply both sides by $b'/a$, we obtain

$$\frac{a}{b} = \frac{a'}{b'} \quad \Longrightarrow \quad \frac{\cancel{a}}{b} \cdot \left(\frac{b'}{\cancel{a}}\right) = \frac{a'}{\cancel{b'}} \cdot \left(\frac{\cancel{b'}}{a}\right) \quad \Longrightarrow \quad \frac{b'}{b} = \frac{a'}{a}, \tag{A.11}$$

which is just the (reverse of the) first equality in Eq. (A.9). Likewise for the other sets of equations. Eqs. (A.9) and (A.10) therefore contain the same information, simply expressed in a different manner.

Personally, I usually think in terms of the "within" ratios in Eq. (A.10). That is, I'll say something like, "Left side over bottom side in this triangle equals left side over bottom side in that triangle." See, for example, Eq. (6.39) in Chapter 6. But you could just as well use the "between" ratios in Eq. (A.9) and say something like, "Left side of this triangle over left side of that triangle equals bottom side of this triangle over bottom side of that triangle."

The ratios in Eqs. (A.9) and (A.10) can be written very cleanly with the " : " ratio notation:

$$\boxed{a : b : c = a' : b' : c'} \qquad \text{(Similar triangles)} \tag{A.12}$$

This is shorthand for the statement that $a$, $b$, and $c$ are in the same ratio as $a'$, $b'$, and $c'$. A numerical example is $5 : 7 : 11 = 15 : 21 : 33$. Every number in the second set is 3 times the corresponding number in the first set. This means that pretty much any equality of ratios you think is true is in fact true. For example,

$$\begin{aligned} \frac{5}{15} &= \frac{7}{21}, \quad \text{or more generally} \quad \frac{a}{a'} = \frac{b}{b'}, \\ \frac{5}{7} &= \frac{15}{21}, \quad \text{or more generally} \quad \frac{a}{b} = \frac{a'}{b'}. \end{aligned} \tag{A.13}$$

Likewise for the other pairs of letters.

# A.4 Areas

## A.4.1 Rectangles

Fig. A.21 shows a rectangle whose sides have lengths 7 and 4. We haven't said what units of length we're using, so the 7 here could be 7 inches, or 7 feet, or 7 miles, or 7 whatevers. To be concrete, let's say it's 7 inches, although it isn't important for the present purposes. We won't keep repeating the word "inches" (or whatevers) when we state a length, but it's understood to be there.

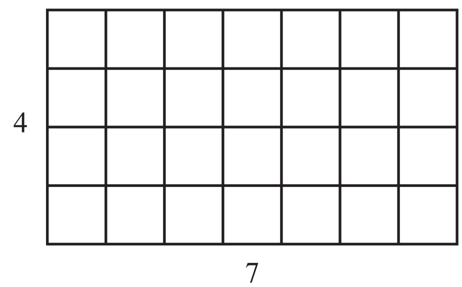


**Figure A.21:** The area of a 7-by-4 rectangle is $7 \cdot 4 = 28$.

What is the area of our 7-by-4 rectangle? (You can just as well call it a 4-by-7 rectangle.) To answer this, we first need to define what we mean by "area." In the case of Fig. A.21, we simply mean the number of little 1-by-1 squares that fit in the rectangle. And since we have 7 columns of 4 squares each (or 4 rows of 7 squares each), the number of little squares is $7 \cdot 4 = 28$. The area is therefore 28. To be more precise with the units included, the area is

$$(7\,\text{inch})(4\,\text{inch}) = 28\ \text{inch}^2. \tag{A.14}$$

An $\text{inch}^2$ is an inch multiplied by an inch, which we call (quite reasonably) a "square inch." A square inch is the area of each of the little 1-by-1 squares.

What if a shape can't be divided evenly into little 1-by-1 squares? In the 7.5-by-4.5 rectangle shown in Fig. A.22, we have the same $7 \cdot 4 = 28$ nice 1-by-1 squares that we had above, but we also have 7 half-squares along the top edge and 4 half-squares along the right edge. Additionally, there is one quarter-square in the upper-right corner.

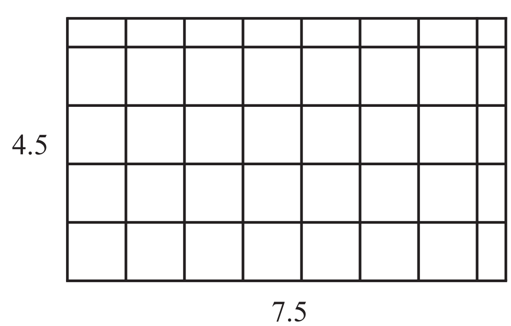


**Figure A.22:** The area of a 7.5-by-4.5 rectangle is $(7.5)(4.5) = 33.75$.

Since it takes two half squares to make a whole 1-by-1 square, each half-square has 0.5 of the area of a whole square. And likewise the quarter square has 0.25 of the area of a whole square. So the total area of the 7.5-by-4.5 rectangle is

$$7 \cdot 4 + 7(0.5) + 4(0.5) + 0.25 = 28 + 3.5 + 2 + 0.25 = 33.75. \qquad \text{(A.15)}$$

Note that the lefthand side here is exactly what you obtain when you apply FOIL to the product $(7.5)(4.5)$, with 7.5 written as $7 + 0.5$, and 4.5 written as $4 + 0.5$:

$$(7 + 0.5)(4 + 0.5) = 7 \cdot 4 + 7(0.5) + 4(0.5) + (0.5)^2. \qquad \text{(A.16)}$$

The area is therefore simply the product of the side lengths, $(7.5)(4.5)$, which is what you would expect. This is consistent with our earlier FOIL discussion of Fig. 3.2. It's just that now we can have partial 1-by-1 squares. In any case, the (possibly non-integer) area equals the number 1-by-1 squares that are needed if you want to cover the given area. Imagine pieces of cardboard that you can cut up into half-squares or quarter-squares, etc.

There's nothing special about the "leftover" bits of each length being 0.5, as they are in Fig. A.22. You can have messy lengths like 7.83 and 4.19, and the area is still the product of the side lengths, $(7.83)(4.19)$. If you want to cover the area, then for the leftover bits, you'll need to use a bunch of tiny cardboard squares whose side length is $0.01 = 1/100$. You'll need a lot of them, but that's fine. They'll get the job done.

Therefore, the area of any arbitrary rectangle with side lengths $a$ and $b$ is

$$\boxed{\text{Rectangle area} = ab = (\text{base})(\text{height})} \qquad \text{(A.17)}$$

A square is a special case of a rectangle where the side lengths $a$ and $b$ are equal. (All squares are rectangles, but only special rectangles are squares.) With $a = b$ in Eq. (A.17), the area of a square with side $a$ is $a^2$.

The *perimeter* of a rectangle (or a polygon in general) is the sum of the lengths of all the edges. So the perimeter of an $a$-by-$b$ rectangle is $a + b + a + b = 2a + 2b$.

### A.4.2 Triangles

What is the area of the triangle in Fig. A.23, with base $a$ and height $b$? (The "base" is the bottom side, and the height is how far the top vertex is above the base.) This triangle is a *right triangle*, which means that it has a 90° angle.

We can determine the area of this right triangle by noting that it is half of a rectangle. If we draw a diagonal line between two opposite corners of a rectangle with sides $a$ and $b$, we obtain two identical right triangles, as shown in Fig. A.24. So twice the area of each triangle equals the area of the rectangle, which itself is $ab$. The area of each triangle is therefore half of this:

$$\boxed{\text{Area of right triangle} = \frac{ab}{2}} \qquad \text{(A.18)}$$

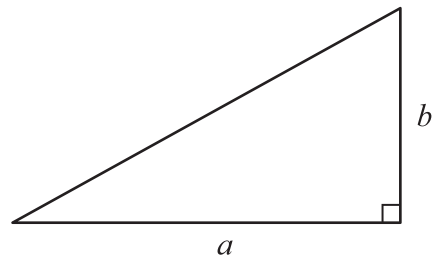


**Figure A.23:** A right triangle with base $a$ and height $b$.

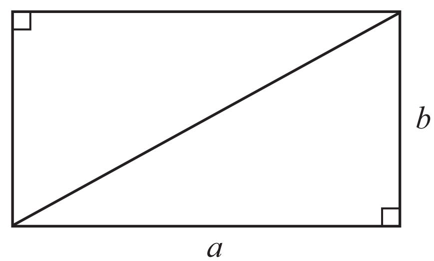


**Figure A.24:** The area of each triangle is half the area of the rectangle, hence $ab/2$.

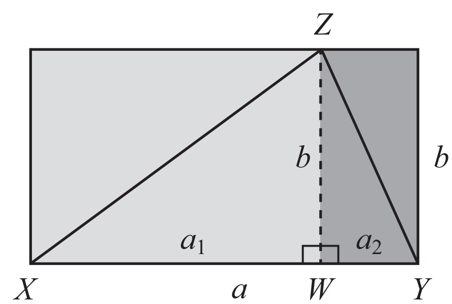


**Figure A.25:** The area of triangle $XYZ$ is half the area of the "bounding" rectangle.

What if we don't have a right triangle, but instead a general triangle like the one shown in Fig. A.25, with vertices at $X$, $Y$, and $Z$? The key point to realize is that we can consider this triangle to be two right triangles side by side, as we have indicated with the shading.

From Eq. (A.18), the area of the light-shaded right triangle $XWZ$ is $a_1 b/2$ (half of the light rectangle), and the area of the dark-shaded right triangle $YWZ$ is $a_2 b/2$ (half of the dark rectangle). The desired area of triangle $XYZ$ is the sum of these two areas, which gives

$$\text{Area} = \frac{a_1 b}{2} + \frac{a_2 b}{2} = \frac{(a_1 + a_2)b}{2} = \frac{ab}{2}, \tag{A.19}$$

where we have used the fact that $a_1 + a_2 = a$. Just like with each triangle in Fig. A.24, the area of triangle $XYZ$ in Fig. A.25 is half the area of the overall "bounding" rectangle.

The base of triangle $XYZ$ in Fig. A.25 is $a$ (the length of side $XY$), and the height to the opposite vertex $Z$ is $b$. So the general way of stating the $ab/2$ result in Eq. (A.19) is to say that the area of a triangle is half the base times the height:

$$\boxed{\text{Area} = \frac{(\text{base})(\text{height})}{2}} \tag{A.20}$$

Acute, or obtuse, or just right,
It's an easy result to recite:
You just need to say
That a triangle's *A*
Is a half of the base times the height.

The dashed segment $WZ$ representing the height in Fig. A.25 is called the *altitude*. In the case of the right triangle in Fig. A.23, the altitude is simply the vertical side with length $b$. (And the base is $a$.) Alternatively, if you tilt your head sideways, you can consider the $b$ side to be the base, and the $a$ side to be the altitude. Eq. (A.20) then gives the area as $ba/2$. This is the same as $ab/2$, of course, because multiplication is commutative. You'd better get the same area, independent of how you calculate it!

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**Exercise A.2** In finding the area of a triangle, there is one more case we need to consider, because the reasoning associated with Fig. A.25 works only if the angles at $X$ and $Y$ are both less than or equal to 90° (equivalently, only if $Z$ lies somewhere above the $XY$ side). What is the area of the triangle $XYZ$ show in Fig. A.26? The angle at $Y$ is larger than 90°, and $Z$ does *not* lie above the $XY$ side. The shading in the figure gives a hint for how to find the area.

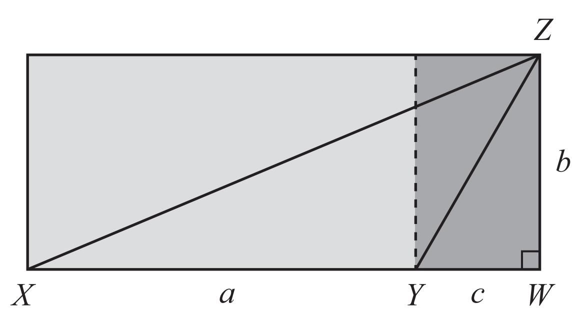


**Figure A.26:** Showing that the area of triangle $XYZ$ is still half the base times the height.

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### A.4.3 Circles

#### Definition of $\pi$

A circle is the set of all points that are the same distance $r$ (called the *radius*) from a given point (the center). The *diameter* $d$ is twice the radius $r$; see Fig. A.27. Equivalently, the diameter is the length of any segment connecting two opposite points on the circle. So any diameter passes through the center. The *circumference* $C$ is the perimeter of the circle. For other shapes like triangles and rectangles, we just say "perimeter," but for circles we use the special name "circumference."

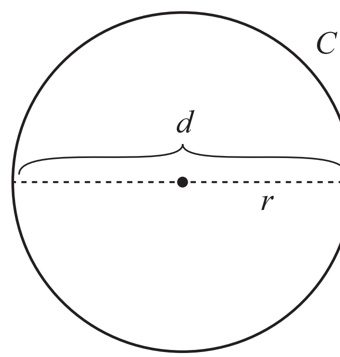


**Figure A.27:** The radius $r$, diameter $d$, and circumference $C$ of a circle.

As with the two usages of the word "side" we discussed in the paragraph following Eq. (A.4), the words "radius," "diameter," and "circumference" can refer to the segments (or circle) themselves, or to their lengths. So you can say either, "The radius has a length of 3 inches," or "The radius is 3 inches." Both usages are fine.

A very important number associated with circles is $\pi$ (the Greek letter pi), whose value is approximately 3.14, although the digits go on forever. A more accurate value is 3.14159, but rarely do you need that many digits. $\pi$ is defined to be the ratio of the circumference $C$ to the diameter $d$. That is,

$$\pi = \frac{C}{d}, \quad \text{or equivalently} \quad C = \pi d. \tag{A.21}$$

Both of these relations say that the circumference $C$ is a little more than 3 times the diameter $d$. If you want to work in terms of the radius $r$, then since the diameter is twice the radius (that is, $d = 2r$), you can write $C = \pi d$ as $C = \pi(2r)$, or

$$\boxed{C = 2\pi r} \qquad \text{(Circumference of a circle)} \tag{A.22}$$

Not only do the digits of $\pi = 3.14159\ldots$ go on forever, they do so without any pattern. In contrast, although the digits of, say, $1/7 = 0.142857\ldots$ also go on forever, they have a definite pattern, because the 6-digit sequence "142857" keeps repeating itself. At present, $\pi$ has been calculated to 100 trillion digits. And the record for the most digits memorized is 70,000! (And possibly more.) Personally, I have accomplished the amazing feat of memorizing the first six quadrillion digits of $1/7$. Impressive, right?

REMARK: A *rational* number is one that can be written as a fraction $a/b$, where $a$ and $b$ are integers. For example, $3/7$ and $11/8$ are rational. An *irrational* number is one that cannot be written this way, as is the case with $\sqrt{2}$; see the "Proof that $\sqrt{2}$ is irrational" subsection on page 95. The digits of irrational numbers go on forever without repeating. A *transcendental* number (which includes $\pi$) is even less orderly than an irrational number, in that not only do its digits go on forever without repeating, the number can't be written as the solution to an algebraic equation (effectively an equation with integer coefficients, such as $2x^7 - 3x^5 - 6x + 1 = 0$). In contrast, the irrational (but not transcendental) number $\sqrt{2}$ is the solution to the equation $x^2 - 2 = 0$. No such equation has $\pi$ as a solution. All transcendental numbers are irrational, but not all irrational numbers (for example, $\sqrt{2}$) are transcendental. ♣

She grimaced and let out a sigh,
When he said, "I'll transcribe all of pi."
Her reply wasn't gentle:
"But pi's transcendental!
So don't even give it a try!"

### Area

Eq. (A.22) gives the *circumference* of a circle as $C = 2\pi r$. Is there a way we can use this $C = 2\pi r$ relation to calculate a circle's *area* in terms of $r$ (and $\pi$)? We could try to cover the area of the circle with little squares (or partial squares), as we did in Fig. A.22. But there's no simple way to get a handle on the number of squares, so it's difficult to produce a general formula this way. In short, squares don't have much to do with circles; they don't fit naturally inside. So trying to deduce a general formula about circles by using squares isn't the best idea. Are there any other shapes we can divide a circle into, that fit more nicely than squares do? Indeed there are. We'll present two methods for finding the area of a circle.

FIRST METHOD: If you look at the spokes of a bicycle wheel (the front wheel, not the back one; you can ponder why there's a difference), you will see that a circle can be broken up into many thin triangles. This is shown in Fig. A.28, where we have drawn 36 triangles. The thin slivers technically aren't true triangles, because the short side is slightly curved since it's a little piece of the circumference. But if we have a thousand (or a million, etc.) triangles instead of 36, they will be extremely thin, which means that the (very) short side of each one will be essentially straight. So we pretty much have true triangles.

Let's now take all of the thin triangles and stack them side by side in an alternating opposite manner (pointing up, pointing down). This produces the nearly rectangular shape shown at the bottom of Fig. A.28. As we make the number of triangles larger and larger, the left and right sides of the "rectangle" become more and more vertical. And the slight bumpiness of the top and bottom sides (due to the slight curvature of the triangles' short sides) gets smaller and smaller. So we approach a perfect rectangle. We have therefore turned the hard-to-deal-with circle into an easy-to-deal-with rectangle.

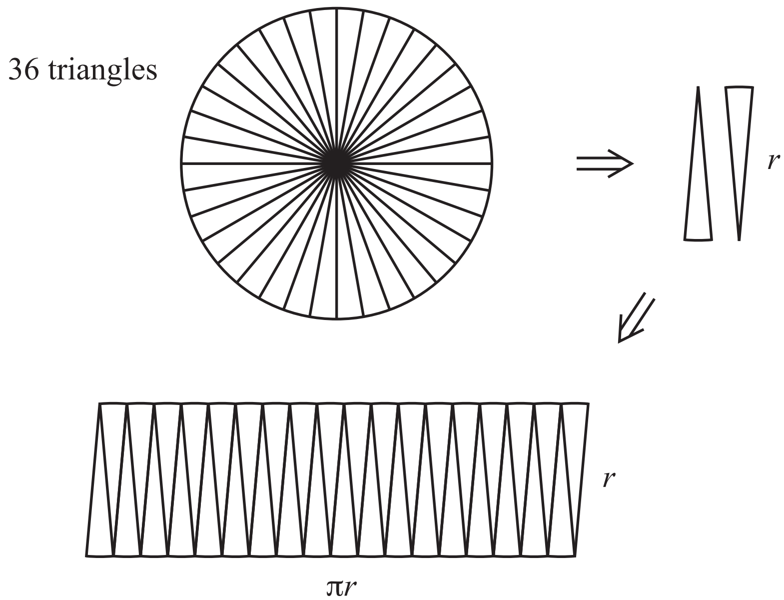


**Figure A.28:** Finding the area of a circle by slicing it into thin pie pieces, and then stacking them side by side to make the rectangle shown, with area $(\pi r)r = \pi r^2$.

For a very large number of triangles, the height of the rectangle approaches the radius $r$. And the width approaches half the circumference, because half of the circumference ends up on the top of the rectangle, and half ends up on the bottom (there are 18 short triangle sides on each of the top and bottom of the rectangle in Fig. A.28). So the width of the rectangle is $(2\pi r)/2 = \pi r$. The rectangle's area, which equals the desired area of the circle, is therefore

$$\text{Area} = (\text{width})(\text{height}) = \pi r \cdot r \quad \Longrightarrow \quad \boxed{A = \pi r^2} \qquad \text{(Area of circle)} \tag{A.23}$$

There is actually no need to stack the triangles alternately to form the rectangle in Fig. A.28 (although it does make for a nice picture!). Instead, we can simply let there be $N$ triangles (where $N$ is large), each with a short base $b$ along the circumference; so $Nb$ equals the circumference $C$. Since each triangle has a base $b$ and a height equal to the radius $r$ (assuming the triangles are very thin), each area is $br/2$. The total area of all $N$ triangles, which equals the area $A$ of the circle, is therefore

$$A = N \cdot \frac{br}{2} = \frac{(Nb)r}{2} = \boxed{\frac{Cr}{2}} = \frac{(2\pi r)r}{2} = \boxed{\pi r^2} \tag{A.24}$$

The $Cr/2$ expression in the middle here is a nice result in its own right, so we have boxed that too. It says that a circle's area is related to its circumference by the multiplicative factor of $r/2$.

In the above reasoning, the bases along the circumference don't even need to have the same length $b$, as long as they are all small (so that the pie pieces can be thought of as triangles with straight bases). The only fact that matters is that the sum of all the bases equals the circumference $C$. Using this fact, if we have $N$ tiny bases $b_1, b_2, \ldots, b_N$, then Eq. (A.24) gets modified to

$$A = \frac{b_1 r}{2} + \frac{b_2 r}{2} + \cdots + \frac{b_N r}{2} = \frac{(b_1 + b_2 + \cdots + b_N)r}{2} = \frac{Cr}{2} = \frac{(2\pi r)r}{2} = \pi r^2. \qquad (A.25)$$

Second method: In the above derivation of the $\pi r^2$ area of a circle, we found it natural to divide the circle into many thin triangles. There is another natural way of dividing a circle into smaller regions: We can use thin rings, as shown in Fig. A.29. We've drawn 15 rings, but as with the thin triangles above, in the end we'll imagine a thousand or a million very thin rings.

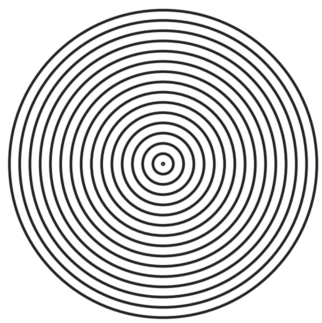

**Figure A.29:** Slicing a circle into thin rings.

Imagine straightening out all of the thin rings and laying them down horizontally like boards, and then stacking them on top of each other to create the triangular object shown in Fig. A.30. If we have a million very thin boards, the "steps" along the hypotenuse of the triangle won't be noticeable; we'll simply have a right triangle with an essentially smooth straight-line hypotenuse. So in this second derivation, we've turned the hard-to-deal-with circle into an easy-to-deal-with triangle.

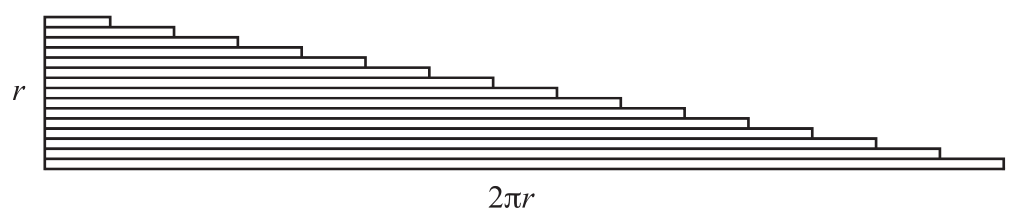


**Figure A.30:** Unrolling the rings, and stacking the "boards."

The base of the triangle (the length of the longest board) is the circumference of the circle, which equals $2\pi r$. The height of the triangle is the sum of the widths of all the boards, which is simply the radius $r$ of the circle. The desired area of the circle equals the area of the triangle, which has a base $2\pi r$ and a height $r$. So from Eq. (A.20) the area of the circle is

$$\text{Area} = \frac{(2\pi r)(r)}{2} = \pi r^2, \qquad (A.26)$$

in agreement with Eq. (A.23). Note that since the rings are very thin, we can indeed straighten them out without having to stretch (much) the inner circumference of each ring (or compress the outer circumference). You can easily roll and unroll a piece of paper since it's so thin, but you can't do that with a thick board!

> Observe the success that it brings
> If you slice up a circle in rings.
> You'll reap the rewards
> When you stack them like boards
> If you've made them as narrow as strings.

You'll use the above thin-ring technique again later on in Exercise 11.19. But in that exercise, you'll add up the areas of all the rings algebraically. In the present treatment, we added them up geometrically by noting that they form a triangle when stacked on top of each other.

REMARK: The above two derivations (involving triangles and boards) of the $\pi r^2$ area of a circle are very similar in the end, as can be seen by looking at Fig. A.31. The top picture is obtained by standing all 36 of the triangles in Fig. A.28 upright, and the bottom picture is obtained by shifting the boards in Fig. A.30 sideways so that they form a symmetric (isosceles) triangle. Both pictures make essentially the same use of the $A = bh/2$ area of a triangle to obtain $A = (2\pi r)r/2 = \pi r^2$; see Eqs. (A.24) and (A.26). The objects have the same total base $2\pi r$ (the circumference) and the same height $r$ (the radius), and they consist of only triangles. So they have the same total area. Interestingly, the top picture is what you get when you unroll an equatorial slice of a grapefruit (after cutting it along a radial line), as you can verify. And the bottom picture is what you get when you unroll an equatorial slice of an onion (after cutting it along a radial line). ♣

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**Exercise A.3** We showed above (in two different ways) how to derive the $A = \pi r^2$ area of a circle from the $C = 2\pi r$ circumference. We can also go the other way: We can derive the $C = 2\pi r$ circumference from the $A = \pi r^2$ area, as follows. Consider the two concentric circles in Fig. A.32 with radii $r$ and $r + \epsilon$, where $\epsilon$ is very small. The area of the thin shaded ring between the circles can be found in two ways: (1) It equals the area of the larger circle minus the area of the smaller circle. (2) It also equals the area of the thin rectangle you obtain if you unroll the ring. Equating these two expressions for the shaded area will tell you what the circumference must be. (*Hint*: If $\epsilon$ is very small, you can ignore one of the terms in your calculation.)

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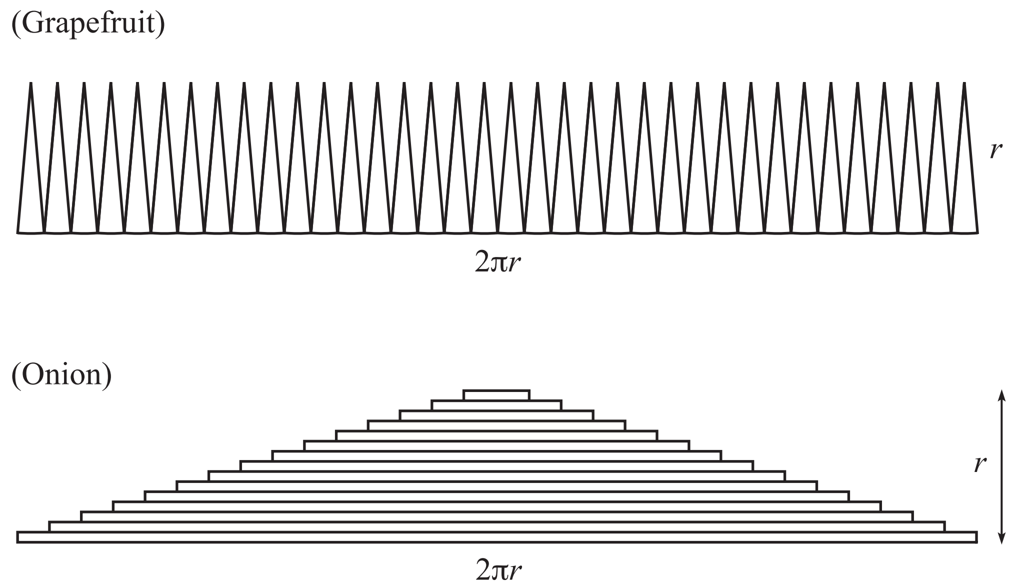


**Figure A.31:** The grapefruit and onion views of a circle.

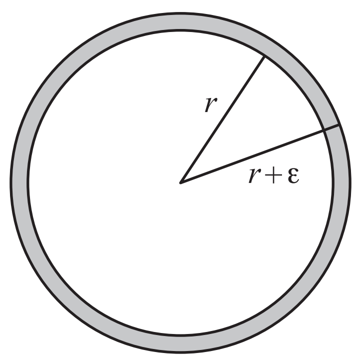


**Figure A.32:** The setup for deriving the $C = 2\pi r$ circumference from the $A = \pi r^2$ area.

### A.4.4 Cones

What is the lateral area of the cone in Fig. A.33? By "lateral" area we mean just the titled side area, and *not* the bottom circular face (whose area is simply $\pi r^2$). We're assuming the cone is symmetric, with the tip lying directly over the center of the circular base. If this isn't the case, then there isn't a nice simple formula for the lateral area.

Our strategy for finding the lateral area will be to divide the area into many thin triangles, one of which is the shaded triangle in the figure. The height of each thin triangle is the slant height $s$ of the cone. This is true because every point on the circumference of the base is the same distance $s$ from the tip of the cone, since we're assuming the cone is symmetric.

Let the small base of each thin triangle be $b$, as shown. Then the area of each triangle is $bs/2$. If there are $N$ triangles, then $Nb$ equals the circumference $C = 2\pi r$ of the base. The

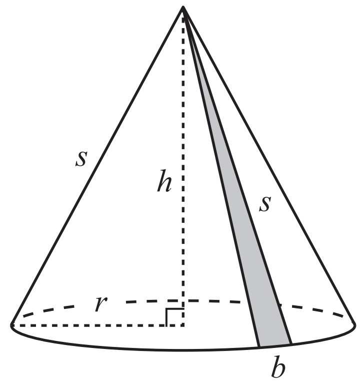


**Figure A.33:** The lateral area of a cone can be built up from many thin triangles.

total area of all the thin triangles (which is the desired lateral area of the cone) is therefore

$$A_{\text{lateral}} = N \cdot \frac{bs}{2} = \frac{(Nb)s}{2} = \boxed{\frac{Cs}{2}} = \frac{(2\pi r)s}{2} = \boxed{\pi r s} \tag{A.27}$$

The intermediate $Cs/2$ result here says that the lateral area equals half the circumference $C$ times the slant height $s$.

Remember that the $\pi rs$ result is just the *lateral* area of the cone. The *total* area, including the circular base, is obtained by adding the $\pi r^2$ base area to the $\pi rs$ lateral area, which yields $\pi rs + \pi r^2 = \pi r(s + r)$.

If the above strategy of adding up thin triangles looks familiar, it's because it's essentially the same as the strategy we used to find the area of a circle. Eq. (A.27) is simply a copy of Eq. (A.24), with one of the $r$'s replaced by $s$ (the first $r$ that appears, not the $r$ that comes from $C = 2\pi r$). The only modification needed for the cone is that the common height of all the thin triangles is now the slant height $s$, instead of the radius $r$ of the base. If you look at the cone from above, and if you close one eye so that you don't have any depth perception, then the division of the cone into thin triangles looks exactly like the circle in Fig. A.28. It's just that now the radial spokes represent the slant height $s$, instead of the radius $r$ of the base. Basically, a circle is just a flat cone with zero height. Alternatively, you can think of a cone as a pointy "circle" with nonzero height.

If we apply the Pythagorean theorem (see Section A.6) to the right triangle in Fig. A.33, we obtain $s = \sqrt{r^2 + h^2}$. So the lateral area in Eq. (A.27) can also be written as $\pi r\sqrt{r^2 + h^2}$. When $h = 0$, this correctly reduces to the $\pi r^2$ area of a circle.

Note that analogous to Eq. (A.25), the bases $b$ in the above derivation don't all need to have the same length, as long as they are all small.

Although the result in Eq. (A.27) for the lateral *area* of a cone holds only for symmetric cones (and not for asymmetric cones, or for pyramids, etc.), the formula for the *volume* of a cone that we'll derive below in Eq. (A.35) holds for *any* shape of cone or pyramid (as long as the base is flat).

**Exercise A.4** Imagine cutting the cone in Fig. A.33 along a straight line from the tip to an arbitrary point on the circumference of the base. You can then unroll the cone into a flat shape (as you might do with a conical party hat; we're assuming the circular base doesn't exist). The resulting shape is a *sector* of a circle, which is a fancy name for a pie piece (perhaps a very wide piece). Use the fact that the sector is a certain fraction of a full circle to find the (lateral) area of the cone.

### A.4.5 Spheres

What is the surface area of a sphere with radius $r$? Before answering this, we should explain what we mean by the area of a curved surface. Since a sphere is curved, you can't cover it nicely with little squares, as we did for the rectangle in Section A.4.1, because the flat squares will necessarily wrinkle a little bit when placed on the curved surface. (The cone in the preceding section was effectively flat, in that we could easily cover it with triangles or unroll it flat onto a table.)

However, if we make the squares very small when covering a sphere, they won't need to wrinkle much, since a sphere looks essentially flat on a small enough length scale. (You can't tell that the earth is curved if all you can see is your back yard.) So instead of using squares that have a side length of, say, 1 inch, we can use tiny squares with a side length of, say, $1/1000$ of an inch. It takes $1000^2 = 1{,}000{,}000$ of these tiny squares to make a square inch. So if it takes, for example, 5,380,000 tiny squares to cover the surface of a sphere, then the area is (essentially) equal to 5.38 square inches.

This concept of tiny squares isn't restricted to curved surfaces. If you draw on a piece of paper a flat shape with an irregular curved boundary, you're not going to be able to fit 1-inch squares into it nicely. So you'll need to use tiny squares instead. The smaller you make them, the closer they'll come to filling up the entire area, without leaving (much) missing area near the curved boundary.

Having defined what we mean by the area of a curved surface, we can now state that the area of a sphere with radius $r$ is given by

$$\boxed{A_{\text{sphere}} = 4\pi r^2} \tag{A.28}$$

There are various ways to derive this formula. The following exercise gives one method, although it assumes knowledge of the volume of a sphere.

**Exercise A.5** In Exercise A.3 you derived the circumference $C$ of a circle from its area $A$. Use a similar argument to derive the $4\pi r^2$ area $A$ of a sphere from its volume $V$. (You can accept here that the volume of a sphere is $V = 4\pi r^3/3$, which you'll derive from scratch later on in Exercise 11.21.) For this exercise, the two circles in Fig. A.32 now represent two spheres, and the shaded ring represents a thin spherical shell. So we're now concerned with the *volume* of this thin shaded shell, instead of the *area* of a thin ring.

We'll talk about general volumes below in Section A.5, but for now just note that the volume of the thin shaded shell can be found in two ways: (1) It equals the volume of the larger sphere minus the volume of the smaller sphere. (2) It also equals the area of the sphere times the thickness $\epsilon$ of the thin shell (see the remark about this in the solution). Equating these two expressions for the shaded volume will tell you what the area of the sphere must be. (*Hint*: If $\epsilon$ is very small, you can ignore two of the terms in your calculation.)

## A.5 Volumes

### A.5.1 Blocks, etc.

Fig. A.34 shows a rectangular block whose sides have lengths 6, 3, and 4. As with the rectangle in Fig. A.21, the units of these lengths could be inches, or feet, or whatever. Let's call them inches again.

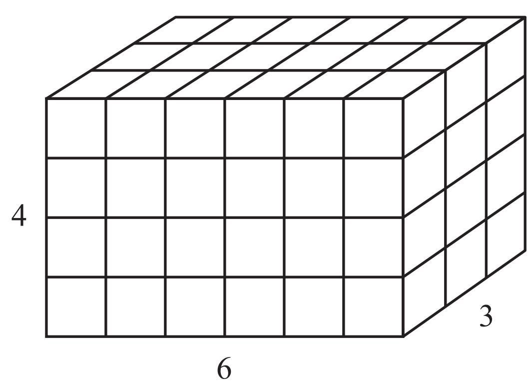


**Figure A.34:** The volume of the 6-by-3-by-4 block is $6 \cdot 3 \cdot 4 = 72$.

What is the volume of the 6-by-3-by-4 rectangular block? Just like with the rectangle in Fig. A.21, where the area was the number of little 1-by-1 squares that fit in the rectangle, the volume of our rectangular block is the number of little 1-by-1-by-1 *cubes* that fit in the block. There are $6 \cdot 3 = 18$ cubes in each layer, and the block is 4 layers tall, so the total number of little cubes is $18 \cdot 4 = 72$. The volume of the block is therefore 72. To be more precise with the units included, the volume is

$$(6\,\text{inch})(3\,\text{inch})(4\,\text{inch}) = 72\ \text{inch}^3. \qquad \text{(A.29)}$$

An $\text{inch}^3$ is an inch multiplied by an inch multiplied by an inch, which we call (quite reasonably) a "cubic inch." A cubic inch is the volume of each of the little 1-by-1-by-1 cubes.

As with the rectangles in Section A.4.1, the above reasoning also works if the sides of the block don't have integer lengths. So in general the volume of a rectangular block with side lengths $a$, $b$, and $c$ is (analogous to Eq. (A.17))

$$\boxed{\text{Volume of rectangular block} = abc} \tag{A.30}$$

This is just the base area $ab$ times the height $c$. Alternatively, you can tilt your head sideways and say that the block has a base area of $bc$ and a height of $a$. And a different sideways look gives a base area of $ac$ and a height of $b$. No matter how you look at it, the base area times the height yields the volume $abc$.

A cube is a special case of a block where the side lengths are all equal. With $a = b = c$ in Eq. (A.30), the volume of a cube with side $a$ is $a^3$.

The *surface area* of a rectangular block is $2ab + 2ac + 2bc$, because the six rectangular faces of the block come in three pairs: two are $a$-by-$b$ rectangles, two are $a$-by-$c$, and two are $b$-by-$c$.

We can also find the volume of the more general object shown in Fig. A.35. The top and bottom faces (with common area $A$) are flat, and the "ribbon" around the side is vertical. (Imagine that you're on a flat plateau surrounded by a vertical cliff.) The curve that defines the base can take on any planar (flat) shape. But the "roof" of the object must have exactly the same shape.

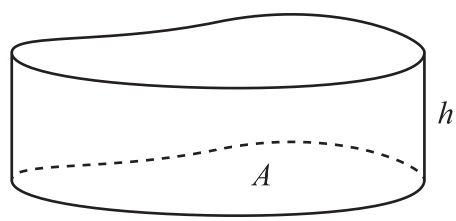


**Figure A.35:** The volume of this object is $Ah$.

If the height of the object is $h$, then the volume is

$$\boxed{\text{Volume} = (\text{base area})(\text{height}) = Ah} \tag{A.31}$$

This follows from the fact that the base area $A$ signifies how many 1-by-1 *squares* fit in the area, and then the height $h$ indicates how many layers of 1-by-1-by-1 *cubes* fit in the whole volume. So we have $h$ layers of $A$ cubes (ignoring the units of these two quantities). The total number of cubes is therefore $Ah$.

In the special case of a rectangular block, where the base area is $A = ab$ and the height is $h = c$, the $Ah$ volume in Eq. (A.31) correctly reduces to the $abc$ volume in Eq. (A.30). So Eq. (A.30) is a special case of the more general result in Eq. (A.31).

What if we have a volume with a complicated warped shape? (The shape in Fig. A.35 is still quite orderly, with its flat top and bottom and its vertical side surface.) In Section A.4.5 we noted that for irregular curvy *areas* in a plane, the area can be found (in principle) by filling it with a set of many tiny squares (with side length, say, $1/000$ of an inch), and then calculating how many 1-by-1 squares the set is equivalent to. The same process works for volumes. If we have an irregular warped volume, we can fill it with a set of many tiny *cubes*, and then calculate how many 1-by-1-by-1 cubes the set is equivalent to.

### A.5.2 Cones, pyramids

Later on in Example 11.6 we'll show that the volume of a cone whose circular base has radius $R$, and whose height is $H$, is $V = \pi R^2 H/3$. Since $\pi R^2$ is the area of the circular base, this result can be written as

$$\boxed{V_{\text{cone}} = \frac{(\text{base area})(\text{height})}{3} = \frac{Ah}{3}} \tag{A.32}$$

It turns out that this formula holds for *any* type of cone/pyramid, no matter what planar (flat) shape the base has. It can be circular, square, triangular, or even have an irregular curvy boundary. The only requirement is that it is flat.

Terminology: The terms "cone" and "pyramid" are used in many different ways. A *cone* can be defined as an object formed by connecting a given point in space (the tip, or vertex) with all points on a planar base of any shape (not necessarily circular). A *pyramid* is a more specific object: it is a cone with a polygonal base. So all of its side faces are triangles. All pyramids are cones, but not all cones are pyramids. For example, a cone with a circular base isn't a pyramid (although technically you can consider any curvy base to be a polygon with an infinite number of sides).

In practice, however, most people take the word "cone" to mean an object with a circular base, and furthermore one whose tip is directly over the center of the base. But whatever name you want to use, the volume of any cone/pyramid is always $1/3$ of the base area times the height, as given in Eq. (A.32).

---

**Example A.1** A special case of Eq. (A.32) is the shaded pyramid in Fig. A.36, which shows a cube with its four diagonals drawn. These diagonals divide the cube into six identical pyramids, one of which is indicated by the shading. Each of the six pyramids has one of the six faces of the cube as its base, and the center of the cube as its tip. Let's show that the volume of the shaded pyramid is consistent with Eq. (A.32).

**Solution** If the cube has side length $a$, its volume is $a^3$. Therefore, since the six pyramids are all identical, they must each have a volume of $V = a^3/6$. The base area of each pyramid is $a^2$, and the height to the tip at the center of the cube is $a/2$. So if we write the $a^3/6$ volume suggestively as $(1/3)(a^2)(a/2)$, we have

$$V = \frac{a^3}{6} = \frac{1}{3} \cdot a^2 \cdot \frac{a}{2} = \frac{1}{3}(\text{base area})(\text{height}) = \frac{Ah}{3}, \tag{A.33}$$

which is consistent with Eq. (A.32).

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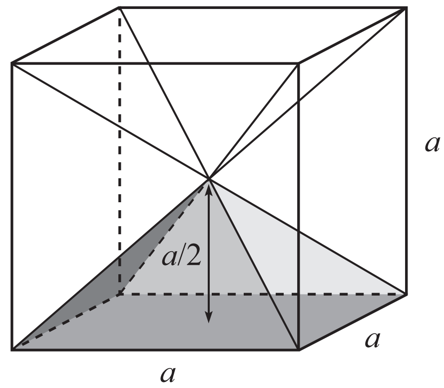


**Figure A.36:** The volume of the shaded pyramid is $1/6$ of the volume of the cube, which yields $a^3/6$.

Fig. A.36 is the starting point for a general derivation of Eq. (A.32) for an arbitrarily shaped cone/pyramid. The reasoning can be broken down into the following four steps, which involve morphing the specific pyramid in Fig. A.36 into an arbitrary cone/pyramid. Since this derivation works for arbitrarily shaped (planar) bases, it is more general than the derivation in Example 11.6 for circular cones.

1. Initial pyramid: The first step is to show, as we did in Eq. (A.33), that the volume of the specific pyramid in Fig. A.36 is $Ah/3$. The key was noting that six of the pyramids fit into the cube.

2. Stretching/squashing: Now imagine slicing the shaded pyramid in Fig. A.36 into thin horizontal square pancakes, as shown in Fig. A.37(a); this is a side view. (Assume that there are a billion pancakes, so that the leftover regions at the edges are tiny and can be ignored.) Then imagine stretching or squashing the pyramid vertically by raising or lowering the tip, as shown in Fig. A.37(b). (The specific stretching factor in this figure happens to be about 1.5.) If we double the height, for example, then each of the pancakes doubles in height, and hence also in volume due to Eq. (A.31). The volume of the pyramid therefore doubles too. And since the height has also doubled, the $V = Ah/3$ formula is still true. Both $V$ and $h$ have doubled, so the formula still contains the factor of $1/3$.

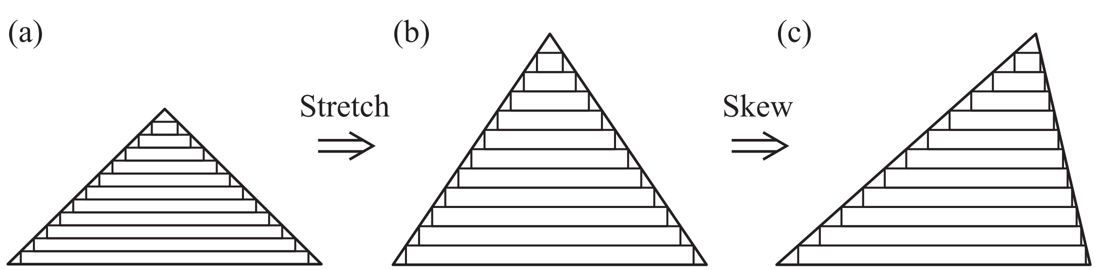


**Figure A.37:** The stretching and skewing actions don't affect the validity of the $V = Ah/3$ formula.

3. Skewing: Next, imagine "skewing" the pyramid by moving the tip sideways (keeping the same height), so that we now have an off-center pyramid, as shown in Fig. A.37(c).

The pancakes haven't changed, they've simply shifted sideways. So we still have the same volume, along with the same height. The $V = Ah/3$ formula is therefore still true for the off-center pyramid.

4. BUILDING UP: Finally, if we have a cone/pyramid with an irregular base, like the one shown in Fig. A.38, we can build it up from a large number of thin pyramids with tiny square bases (one of which is shown) that fill up the overall base. Therefore, since the $V = Ah/3$ formula holds for each of the thin square pyramids (from the above stretching/skewing reasoning), it also holds for the overall cone, because the base area of the overall cone is the sum of all the little square areas.

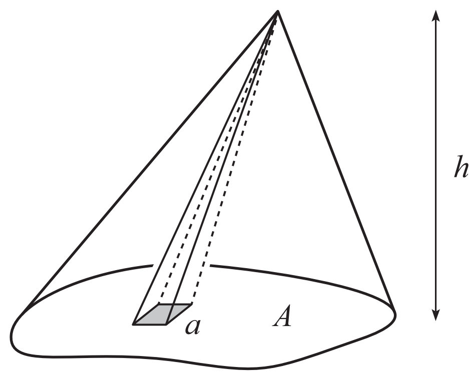


**Figure A.38:** Any cone/pyramid with an irregular base can be built up from a large number of thin pyramids with tiny square bases.

To be more explicit about this, let there be $N$ tiny square bases, each with a small area $a$ (so $Na$ equals the base area $A$). Since each thin square pyramid has a volume of $ah/3$ (from the above reasoning), the total volume of the overall cone is

$$V = N \cdot \frac{ah}{3} = \frac{(Na)h}{3} = \frac{Ah}{3}, \tag{A.34}$$

as we wanted to show. (This equation is analogous to Eq. (A.24). And similarly, there is no need for the tiny areas $a$ to be equal; an equation analogous to Eq. (A.25) also holds here.) We therefore see that building up a general cone from many thin square pyramids doesn't change the factor of 3 that appears in the denominator. So our general formula is

$$\boxed{\text{Volume of arbitrary cone/pyramid} = \frac{(\text{base area})(\text{height})}{3} = \frac{Ah}{3}} \tag{A.35}$$

The four steps in the above derivation can be summarized as follows. We can create an arbitrary cone/pyramid by: (1) starting with the specific pyramid in Fig. A.36 whose volume we showed takes the form of $V = Ah/3$, and then (2) stretching/squashing it vertically, and (3) skewing it by moving the tip sideways, and (4) building up an arbitrary cone from a large number of square (and skewed) thin pyramids, as shown in Fig. A.38. None of these changes affects the 3 that appears in the denominator of the $V = Ah/3$ result in Eq. (A.33) for the specific pyramid in Fig. A.36. So the $V = Ah/3$ formula also holds for any general cone like the one in Fig. A.38, as Eq. (A.35) states.

> We've seen how it can be inferred
> That no matter which change has occurred,
> The volume relation
> Has no variation;
> The factor is always one third.

### A.5.3 Spheres

The volume of a sphere with radius $r$ is

$$\boxed{V_{\text{sphere}} = \frac{4}{3}\pi r^3} \tag{A.36}$$

You'll derive this formula from scratch later on in Exercise 11.21. Another derivation is presented in Exercise 11.20, although that one requires knowledge of the $4\pi r^2$ area of a sphere.

The goal of this section is *not* to derive the formula in Eq. (A.36) from scratch, but rather just to produce a formula that relates the surface area $A$ and volume $V$ of a sphere. The result will be similar to the $A = Cr/2$ result in Eq. (A.24), which relates the circumference $C$ and area $A$ of a circle. Said in another way, in this section we'll give a derivation of Eq. (A.36), but it is one that requires knowledge of the surface area $A = 4\pi r^2$ of a sphere. (We'll also assume that we know the formula for the volume of a cone/pyramid in Eq. (A.35), since we *did* derive that from scratch.)

The strategy for relating the $A$ and $V$ of a sphere is basically the same as the strategy for relating the $C$ and $A$ of a circle (which led to Eq. (A.24)), except that we'll now view the top picture in Fig. A.28 as a 3-D object depicting a sphere that is divided into many thin *cones/pyramids*. The bases of the cones are tiny patches (their exact shape doesn't matter) on the surface of the sphere, and the tip of every cone is the center of the sphere. So all of the cones' heights are equal to the radius $r$.

Let there be $N$ cones, each with a small base area $a$ (so $Na$ equals the area $A$ of the sphere). Since each cone has base area $a$ and height $r$, Eq. (A.35) gives the volume as $ar/3$. The total volume of all $N$ cones (which equals the volume $V$ of the sphere) is therefore

$$V = N \cdot \frac{ar}{3} = \frac{(Na)r}{3} = \boxed{\frac{Ar}{3}} = \frac{(4\pi r^2)r}{3} = \boxed{\frac{4}{3}\pi r^3} \tag{A.37}$$

where we have invoked the fact that the area of a sphere is $4\pi r^2$. The $Ar/3$ expression in the middle of Eq. (A.37) is the main result we're after. The factor of $1/3$ that it contains is the same factor of $1/3$ that appears in the volume of a cone in Eq. (A.35).

This $V = Ar/3$ relation that we have obtained for a sphere is analogous to the $A = Cr/2$ relation in Eq. (A.24) for a circle. The factor of $1/2$ there came from the $1/2$ in the formula for the area of a triangle in Eq. (A.20). To summarize, the middle expressions in Eqs. (A.24) and (A.37) are

$$A_{\text{circle}} = \frac{C_{\text{circle}}\, r}{2} \qquad \text{and} \qquad V_{\text{sphere}} = \frac{A_{\text{sphere}}\, r}{3}. \tag{A.38}$$

The 2 here comes from the fact that a circle can be built up from triangles, and the 3 comes from the fact that a sphere can be built up from cones.

As with a circle, the bases of the cones in the above reasoning don't need to have the same area $a$, as long as they are all small. The only fact that matters is that the sum of all the base areas equals the area $A$ of the sphere. All of the $b_i r/2$ terms in Eq. (A.25) now become $a_i r/3$ terms, with the sum of all the $a_i$'s being equal to the area $A$ of the sphere.

If you work backwards through the reasoning that led to Eq. (A.37), and if you assume that you know both the $V$ and $A$ of a sphere, then you can deduce that the volume of a cone must be $1/3$ the base times the height. In short, Eq. (A.37) contains information about $V_{\text{sphere}}$, $A_{\text{sphere}}$, and $V_{\text{cone}}$. So if you know two out of three of these, you can deduce the third. Likewise for $A_{\text{circle}}$, $C_{\text{circle}}$, and $A_{\text{triangle}}$ in Eq. (A.24).

The classic derivation of the volume of a sphere uses *Cavalieri's principle*, which states that if two objects have equal cross-sectional areas at every corresponding height, then they have the same volume. You are encouraged to investigate this method on your own; there are many good articles on the web.

## A.6 The Pythagorean theorem

### A.6.1 The theorem

The Pythagorean theorem is one of the most beautiful theorems in mathematics. It is simple to state, easy to use, and highly accessible – it doesn't require a huge amount of mathematical machinery to prove. The theorem deals with *right triangles*, which are ones that have a 90° angle (called a "right angle"). The two perpendicular sides adjacent to the 90° angle are called the *legs*. The third side (opposite the 90° angle) is called the *hypotenuse*. So in Fig. A.39 the hypotenuse has length $c$, and the legs have lengths $a$ and $b$. As with words like "radius" and "diameter," the words "hypotenuse" and "leg" can refer either to the segments themselves, or to their lengths. So you can also say, "The hypotenuse is $c$."

The Pythagorean theorem states that the sides of a right triangle are related by

$$\boxed{a^2 + b^2 = c^2} \qquad \text{(Pythagorean theorem)} \tag{A.39}$$

This statement of the Pythagorean theorem was certainly known before Pythagoras' time, although it is unclear who first proved it, and when. In any case, we can only wonder what Pythagoras' first encounter with the theorem looked like. . .

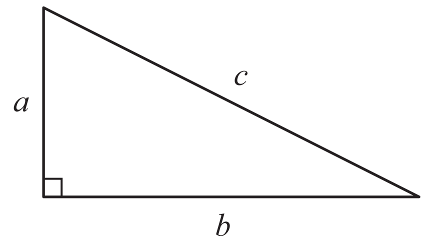


**Figure A.39:** A right triangle has a 90° (right) angle. The two perpendicular sides adjacent to the right angle are called the legs, and the diagonal side is called the hypotenuse.

Pythagoras wept and despaired
As he added the legs and compared.
Then he jumped up with glee,
"Though they don't add to $c$,
It's a match if the lengths are all squared!"

We'll present five proofs of the theorem in Section A.6.3, but there are a few things we should discuss first. If you draw a triangle with a random shape, the odds are that it won't be a right triangle. That is, most random sets of three numbers $a, b, c$ don't satisfy Eq. (A.39). Only special sets do, and thereby yield a right triangle. The simplest set of *integers* that satisfy the theorem is 3, 4, 5. These lengths produce a right triangle because

$$3^2 + 4^2 = 5^2 \quad \Longrightarrow \quad 9 + 16 = 25. \tag{A.40}$$

Most right triangles don't have integer lengths for all three sides. Or said in another way, if you pick integers for two sides of a right triangle, the third side probably won't be an integer. For example, if we pick the two legs to be 1 and 1, then the hypotenuse is given by

$$1^2 + 1^2 = c^2 \quad \Longrightarrow \quad c^2 = 2 \quad \Longrightarrow \quad c = \sqrt{2} \approx 1.414, \tag{A.41}$$

which isn't an integer. Or if we pick the hypotenuse to be 8 and one leg to be 5, then the other leg is given by

$$a^2 + 5^2 = 8^2 \quad \Longrightarrow \quad a^2 + 25 = 64 \quad \Longrightarrow \quad a^2 = 39 \quad \Longrightarrow \quad a = \sqrt{39} \approx 6.245, \tag{A.42}$$

which isn't an integer.

If all three sides of a right triangle are integers, then we call the set of these integers a *Pythagorean triple* (or just a *triple*, for short). People often list the integers of a triple inside parentheses, like "$(a, b, c)$." For example, in addition to the Pythagorean triple $(3, 4, 5)$ mentioned above, a few other triples are $(6, 8, 10)$, $(5, 12, 13)$, and $(8, 15, 17)$ because, as you can verify,

$$6^2 + 8^2 = 10^2, \qquad 5^2 + 12^2 = 13^2, \qquad 8^2 + 15^2 = 17^2. \tag{A.43}$$

A quick way of producing new triples from other known triples is to use the fact that any integer multiple of the three numbers in a triple yields three new numbers that are again a triple. This is true because if $(a, b, c)$ is a triple, then we can multiply both sides of the Pythagorean theorem in Eq. (A.39) by $s^2$ (where $s$ is an integer) to obtain another true statement. This multiplication by $s^2$ yields

$$a^2 + b^2 = c^2 \implies s^2a^2 + s^2b^2 = s^2c^2 \implies (sa)^2 + (sb)^2 = (sc)^2. \tag{A.44}$$

And this is just the statement that $(sa, sb, sc)$ is a Pythagorean triple, as we wanted to show. For example, the $(6, 8, 10)$ triple mentioned above is the $(3, 4, 5)$ triple multiplied by $s = 2$.

It makes intuitive sense that multiplying each side of a right triangle by $s$ produces another right triangle. If you're given a right triangle, and if you scale it up uniformly by multiplying all of the sides by the same factor, then the new triangle will be similar to the old one (it will have the same shape); recall the discussion of similarity in Section A.3. So it will still be a right triangle. Even if $s$ isn't an integer, you'll still end up with a right triangle. But if the sides aren't integers, we don't call it a Pythagorean triple.

Note that the Pythagorean theorem in Eq. (A.39) is *symmetric* in $a$ and $b$. That is, both $a$ and $b$ are raised to the same power (namely 2), and the two terms have the same coefficient (namely 1). This symmetry follows from the fact that it can't matter which leg you arbitrarily choose to label as $a$, and which one you label as $b$. If someone claimed that the theorem took the form of, say $a^2 + 2b^2 = c^2$, then you would get a different result for $c$ if you switched your $a$ and $b$ labels (which you're free to do). But the length $c$ of the hypotenuse has a definite value and can't depend on which letter you feel like writing next to each leg. So this "theorem" can't be correct.

### A.6.2 General form of triples

It turns out that there is a very simple and general way to produce Pythagorean triples, beyond the easy ones that are simply integer multiples of other triples. We claim that if we start with any two integers $m$ and $n$, then the following three integers $a, b, c$ are a Pythagorean triple, that is, they satisfy the Pythagorean theorem:

$$\boxed{a = m^2 - n^2, \qquad b = 2mn, \qquad c = m^2 + n^2} \tag{A.45}$$

You can verify this claim in the following exercise.

---

**Exercise A.6** Show that the $a$, $b$, $c$ expressions in Eq. (A.45) satisfy Eq. (A.39), by calculating the sum $a^2 + b^2$ and showing that the result equals $c^2$.

---

> For integers $m$, $n$, who knew
> That $b$ is their product times 2?
> And $a$? It's a fact:
> Form the squares and subtract.
> And for $c$, instead add up the two.

It turns out that not only does Eq. (A.45) generate Pythagorean triples, it generates *all* of them. That is, there are no triples that aren't of the form in Eq. (A.45). Every triple has an associated $(m, n)$ pair. The proof of this statement ("If three numbers are a triple, then they take the form of Eq. (A.45)") is more involved than the proof of the original statement ("If three numbers take the form of Eq. (A.45), then they are a triple"), which you completed in the above exercise. So we'll just accept it here.

Table A.1 lists the triples that Eq. (A.45) generates for various $(m, n)$ pairs. Some of the triples are multiples of others. For example, $(24, 10, 26)$ is 2 times $(5, 12, 13)$, although in a different order. You should pick a few of the triples and verify that they do indeed satisfy the Pythagorean theorem.

| | | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $m$ | $n$ | $m^2 - n^2$ | $2mn$ | $m^2 + n^2$ |
| 2 | 1 | 3 | 4 | 5 |
| 3 | 1 | 8 | 6 | 10 |
| 3 | 2 | 5 | 12 | 13 |
| 4 | 1 | 15 | 8 | 17 |
| 4 | 2 | 12 | 16 | 20 |
| 4 | 3 | 7 | 24 | 25 |
| 5 | 1 | 24 | 10 | 26 |
| 5 | 2 | 21 | 20 | 29 |
| 5 | 3 | 16 | 30 | 34 |
| 5 | 4 | 9 | 40 | 41 |

**Table A.1:** Pythagorean triples for various $(m, n)$ pairs.

If you stare at Table A.1 long enough, you'll see some patterns, one of which is the following. Look at the cases where $n = m - 1$. So $(m, n)$ takes the form of $(2, 1)$, $(3, 2)$, $(4, 3)$, $(5, 4)$, etc. The $a$ values associated with these pairs are the odd numbers 3, 5, 7, and 9, which equal $m + n$. And in each case, you will observe that $b$ and $c$ differ by 1 and add up to $a^2$. For example, in the $(5, 4)$ case we have $40 + 41 = 9^2$. And in the $(4, 3)$ case we have $24 + 25 = 7^2$. You can prove that this holds in general in the following exercise.

---

**Exercise A.7** If $n = m - 1$, calculate the values of $a$, $b$, and $c$ in Eq. (A.45) in terms of $m$. Then show that: (1) $a$ is odd, (2) $a$ equals $m + n$, (3) $b$ and $c$ differ by 1, and (4) $b$ and $c$ add up to $a^2$.

---

### A.6.3 Five proofs

Below are five proofs of the Pythagorean theorem (and there are many others too!). They all take the form of exercises, so that you have the chance to do them yourself. It would be a shame to miss out on the fun! Here's the statement of the theorem:

**Pythagorean theorem:** *If the sides of a right triangle are a, b, and c, with c being the hypotenuse, then* $a^2 + b^2 = c^2$.

Here are the proofs:

---

**Exercise A.8 (Proof 1)** Prove the Pythagorean theorem by using the fact that the area of the overall square in Fig. A.40 equals the sum of the areas of the four triangles plus the area of the smaller square. (There's a bit of an optical illusion in this figure. The sides of the overall square are indeed horizontal and vertical, even if they don't look it!)

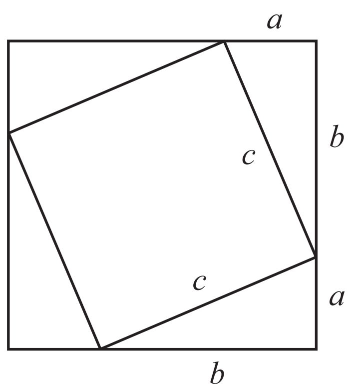


**Figure A.40:** The Pythagorean theorem follows from the fact that the area of the overall square equals the sum of the areas of the four triangles plus the area of the smaller square.

**Exercise A.9 (Proof 2)** Fig. A.41 shows a square with side length $c$ subdivided into four right triangles with legs $a$ and $b$ (and hypotenuse $c$), along with a square in the middle with side length $b - a$. Prove the Pythagorean theorem by using the fact that (as in the preceding proof) the area of the overall square equals the sum of the areas of the four triangles plus the area of the smaller square.

**Exercise A.10 (Proof 3)** This proof requires no algebra. It's basically a geometry-only interpretation of the first proof above. Perhaps it shouldn't count as a separate proof, but it's so slick, I think it should. It's the quickest and simplest proof of them all.

Fig. A.42 (which is the same as Fig. A.40, but with shading) shows four shaded triangles inside a square. Show how to rearrange the triangles in a way that makes it clear that the area of the white region (which is $c^2$) equals $a^2 + b^2$.

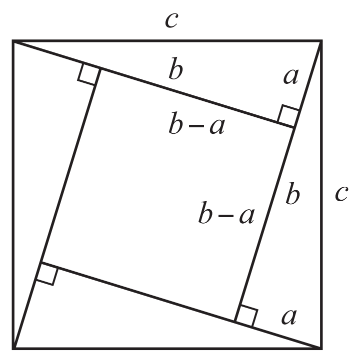


**Figure A.41:** The same strategy as in Fig. A.40 yields another proof of the Pythagorean theorem.

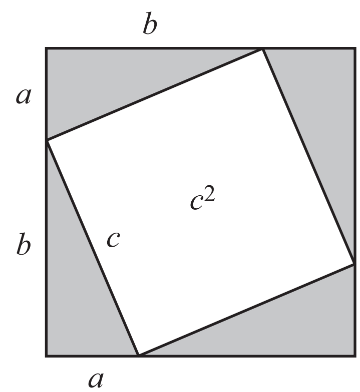


**Figure A.42:** The Pythagorean theorem can be proved by rearranging the triangles.

**Exercise A.11 (Proof 4)** Here's another geometry-only proof. Fig. A.43 shows a combo version of Figs. A.40 and A.41. Rearrange some of the triangles to show that $a^2 + b^2 = c^2$. We've given a hint by drawing two shaded squares with areas $a^2$ and $b^2$. (These shapes are indeed squares, since all sides are either $a$ or $b$.)

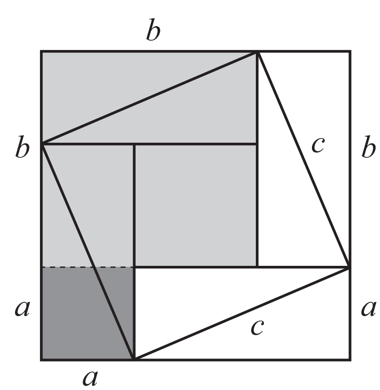


**Figure A.43:** Another proof by rearranging some triangles.

**Exercise A.12 (Proof 5)** The overall right triangle in Fig. A.44 has sides $a$, $b$, and $c$. The altitude to the hypotenuse is drawn. Explain why the two smaller right triangles are similar to (that is, they have the same angles as) the overall right triangle. Then use this similarity to find the two lengths into which $c$ is divided. The Pythagorean theorem will follow from the fact that these two lengths add up to $c$.

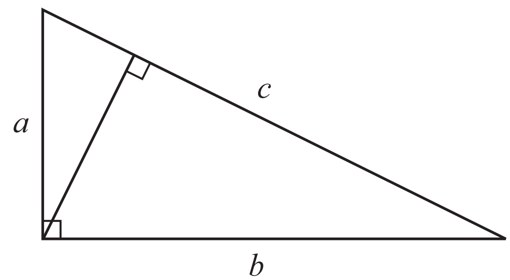


**Figure A.44:** A proof that uses similar triangles.

Having worked through all of these proofs, you can now say without a doubt that you understand the Pythagorean theorem! And that has its perks. . .

> It's quick to spot which kids are cool
> As they saunter the halls of the school.
> "*Who's* got the swagger? Us!
> *We* know Pythagoras!
> Sure, we're all square, but we rule!"

### A.6.4 The converse

The Pythagorean theorem says, "If a triangle is right, then its side lengths satisfy $a^2+b^2 = c^2$." The reverse/backward statement is also true. The mathematical term for the reverse/backward version of a claim is the *converse.* You just reverse the "if" and "then" parts. So the converse of the Pythagorean theorem is:

**Converse:** *If a triangle's side lengths satisfy $a^2 + b^2 = c^2$, then the triangle is right.*

As with all of the above proofs of the "forward" direction of the theorem, there are many different ways to prove the converse. We'll present just one proof here.

**Proof:** We're starting with the assumption that $a^2 + b^2 = c^2$, and our goal is to show that the triangle is right. That is, we want to show that it *cannot* look like either of the triangles (obtuse or acute) in Fig. A.45, where the $b$ side is tilted. So for both of these possibilities, our goal is to show that $x$ must be zero, meaning that $b$ is actually vertical. That is, the $x$ segment (or lack thereof) must in fact not exist. Equivalently, the top vertex is directly over the left end of the $a$ side.

We'll address the obtuse case here. (The acute case proceeds in the same manner, with only one small sign modification, as you can check.) This proof makes use of the "forward" direction of the Pythagorean theorem, so it assumes (quite correctly!) that we've already proved that.

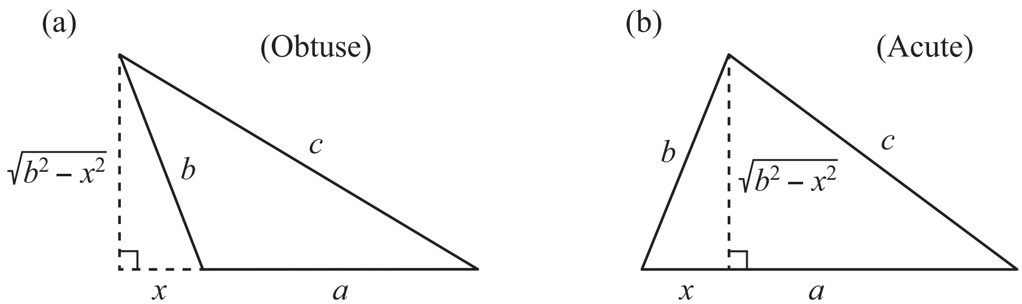


**Figure A.45:** Proving the converse of the Pythagorean theorem. The goal is to show that if $a^2+b^2 = c^2$, then $x = 0$ (that is, the triangle is right).

From the Pythagorean theorem, the vertical leg of the small right triangle in Fig. A.45(a) has length $\sqrt{b^2 - x^2}$, as shown. The Pythagorean theorem applied to the overall big right triangle then gives

$$(a + x)^2 + \left(\sqrt{b^2 - x^2}\right)^2 = c^2$$
$$\Longrightarrow \quad (a^2 + 2ax + \cancel{x^2}) + (b^2 - \cancel{x^2}) = c^2. \tag{A.46}$$

If we now use the given information that $a^2 + b^2 = c^2$, we can replace the $a^2 + b^2$ on the lefthand side of Eq. (A.46) with $c^2$, which yields

$$c^2 + 2ax = c^2 \quad \Longrightarrow \quad 2ax = 0. \tag{A.47}$$

We see that the product $2ax$ equals zero. And since neither 2 nor $a$ is zero, it must be the case that $x = 0$. In other words, the $b$ side is vertical, and the triangle is a right triangle, as we wanted to show. You are encouraged to work through the acute case, which is nearly the same. ∎

## A.7 Special triangles

In this section we'll discuss three specific triangles that have particularly nice shapes. They are shown in Fig. A.46. In each triangle, we have arbitrarily picked one of the sides to have length 1. Ignore any comparison of lengths among the different triangles. We'll explain below where the relative lengths *within* each triangle come from. That is, we'll explain where, for example, the $\sqrt{2}$ and $\sqrt{3}$ lengths come from. Let's look at the three triangles in turn. The first and third are right triangles.

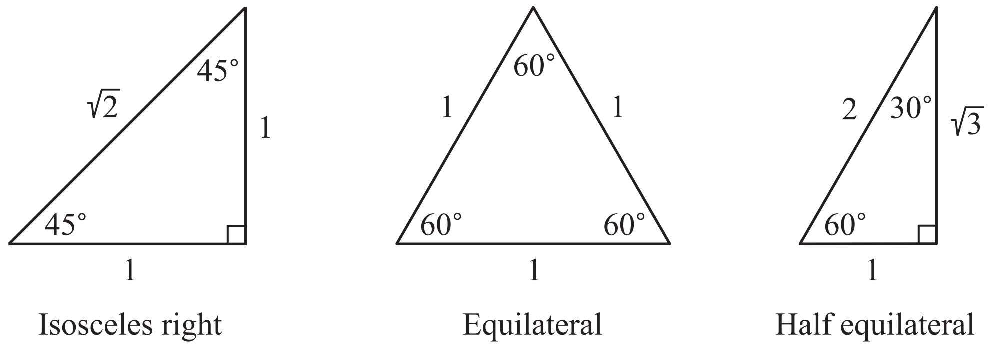


**Figure A.46:** Three nicely shaped triangles.

1. 45-45-90 (Isosceles right): The left triangle in Fig. A.46 is an isosceles right triangle. We have arbitrarily chosen the two equal sides (the legs) to have length 1. Since the triangle is isosceles, the angles opposite the equal sides are equal. We then quickly see that these angles must each be 45°, because the sum of the two 45° angles plus the 90° right angle correctly yields the 180° sum that any triangle must have. So an isosceles right triangle has angles of 45-45-90.

   If the two legs have length 1, what is the length of the hypotenuse? The Pythagorean theorem quickly gives the answer. With $a = 1$ and $b = 1$ in Eq. (A.39), the hypotenuse $c$ is

   $$1^2 + 1^2 = c^2 \implies c^2 = 2 \implies c = \sqrt{2}. \tag{A.48}$$

   The numerical value of $\sqrt{2}$ is about 1.414, and this is consistent with a visual inspection of Fig. A.46. The hypotenuse looks like it's about 1.5 times as long as each leg.

   We have found that the hypotenuse of the 45-45-90 triangle in Fig. A.46 is $\sqrt{2}$ times either leg. This means that in any other 45-45-90 triangle we might draw (which will be similar to the one in Fig. A.46), the hypotenuse will always be $\sqrt{2}$ times either leg. So if each leg is 5, then the hypotenuse is $5\sqrt{2}$, which is a hair more than 7. And if each leg is 100, then the hypotenuse is $100\sqrt{2}$.

2. 60-60-60 (Equilateral): The middle triangle in Fig. A.46 is an equilateral triangle, meaning that all three sides have the same length. Equivalently, it is an isosceles triangle for any two pairs of sides. All three angles are equal, and hence equal to 60° since $3 \cdot 60° = 180°$.

3. 30-60-90 (Half equilateral): If we draw the altitude of an equilateral triangle, it divides the triangle into two equal halves, as shown in Fig. A.47. The 60° angle at the top is divided into two 30° angles, so each of the two halves of the original triangle looks like the rightmost triangle in Fig. A.46. The three angles are the 90° at the foot of the altitude, the 30° at the top, and one of the original 60° angles. So a half-equilateral triangle has angles of 30-60-90.

   How do the side lengths of a half-equilateral triangle compare? If we arbitrarily choose the bottom (short) leg to be 1, then since this leg is half of a side of the original equilateral

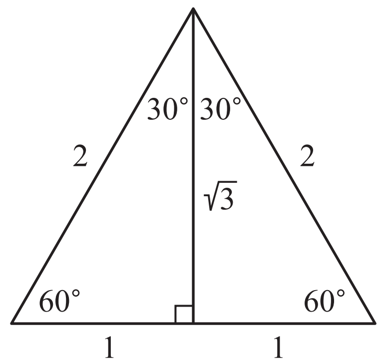


**Figure A.47:** Finding the side lengths of a 30-60-90 triangle.

triangle, we see that the hypotenuse (which is one of the sides of the original equilateral triangle) must have length 2. The third side (the longer, vertical leg) can then be found via the Pythagorean theorem. With $a = 1$ (short leg) and $c = 2$ (hypotenuse), the longer leg $b$ is given by

$$1^2 + b^2 = 2^2 \quad \Longrightarrow \quad 1 + b^2 = 4 \quad \Longrightarrow \quad b = \sqrt{3}. \tag{A.49}$$

The numerical value of $\sqrt{3}$ is about 1.732, and this is consistent with a visual inspection of Fig. A.46. The longer leg looks like it's somewhere between 1.5 and 2 times the shorter leg. It certainly has to be less than twice the shorter leg, because the hypotenuse is twice the shorter leg, and the hypotenuse is always longer than either leg.

The splitting-in-two method in Fig. A.47 (where one side was quick to find, and the other followed from the Pythagorean theorem) can be summarized in limerick form:

For 30 and 60 degrees,
Deducing the sides is a breeze.
Take a shape that you know,
And then split it like so,
And you'll find any length that you please.

As with the 45-45-90 triangle above, similarity tells us that in any 30-60-90 triangle we might draw, the hypotenuse will always be 2 times the shorter leg, and the longer leg will always be $\sqrt{3}$ times the shorter leg. (These facts in turn imply that the longer leg is always $\sqrt{3}/2 \approx 0.866$ times the hypotenuse.) So if the shorter leg is, say, 5, then the hypotenuse is 10, and the longer leg is $5\sqrt{3} \approx 8.66$. Given any one of the sides of a 30-60-90 triangle, we can determine the other two sides from these ratios.

If a triangle has two unequal angles, the ratio of these angles is *not* equal to the ratio of the sides opposite the angles. For example, in a 45-45-90 triangle, 90° is twice 45°, but $\sqrt{2}$ is not twice 1. And in a 30-60-90 triangle, 60° is twice 30°, but $\sqrt{3}$ is not twice 1. And 90° is 3 times 30°, but 2 is not 3 times 1. The ratio of the sides does have *something* to do with the angles. But the relation involves a trigonometric function, as you'll see when you learn about trigonometry.

## A.8 Exercise solutions

1. First derivation: In Fig. A.18(a) there are 6 lines drawn from a given vertex to all the others, except the adjacent two. These 6 lines divide the polygon into 7 triangles. For a general $n$-gon, the 6 lines become $n-3$ lines (to the $n$ vertices minus the given one and the two adjacent ones). And the 7 triangles become $n-2$ triangles.

   From the same reasoning as in the quadrilateral case in Fig A.16, the sum of all the angles in the polygon equals the sum of the angles in all the triangles. Therefore, since we have $n-2$ triangles, and since the sum of the angles in any triangle is 180°, the desired sum is

   $$\boxed{S = (n-2)180^\circ} \qquad \text{(Sum of the angles in an } n\text{-gon)} \tag{A.50}$$

   This formula correctly yields 180° when $n = 3$ (a triangle), and 360° when $n = 4$ (a quadrilateral). As with quadrilaterals, there is a similar proof for the concave case.

   Second derivation: In Fig. A.18(b) there are 9 (or more generally $n$) triangles. The sum of the angles in all of these triangles is $n \cdot 180^\circ$. However, each triangle includes an angle at the meeting point in the interior; call this point $P$. These angles are included in the sum of the triangle angles, but they should *not* be included in the sum of the polygon angles, because those include only the angles around perimeter of the polygon. So we need to subtract off all the angles at $P$. These add up to 360° since they make a full revolution. The sum $S$ of the polygon angles therefore equals the sum of the angles in the $n$ triangles, minus 360°. This gives

   $$S = n \cdot 180^\circ - 360^\circ = n \cdot 180^\circ - 2 \cdot 180^\circ = (n-2)180^\circ, \tag{A.51}$$

   in agreement with Eq. (A.50). This $(n-2)180^\circ$ expression is about as simple a result as you could imagine for quadrilaterals and general $n$-gons, given that it must yield 180° when $n = 3$.

   > Each triangle always agrees
   > On a hundred and eighty degrees.
   > But how to respond
   > To those quads and beyond?
   > With that $n$-minus-2, it's a breeze!

2. Since $Z$ now lies *outside* the rectangle that has segment $XY$ as its base (the light-shaded rectangle in Fig. A.26), we can't just add the areas of two triangles as we did in Eq. (A.19). What can we now do instead, to find the area of triangle $XYZ$?

   In Fig. A.25, triangle $XYZ$ was the *sum* of right triangles $XWZ$ and $YWZ$. But in Fig. A.26, triangle $XYZ$ is the *difference* of right triangles $XWZ$ and $YWZ$. That is, if we

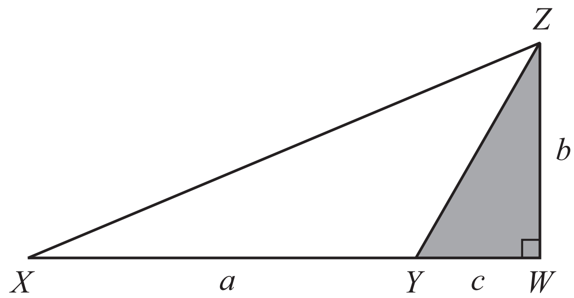


**Figure A.48:** Finding the area of a triangle in the case where $Z$ doesn't lie above the $XY$ side.

subtract the shaded right triangle $YWZ$ in Fig. A.48 from the overall right triangle $XWZ$, we end up with the desired triangle $XYZ$.

Using the result in Eq. (A.18) for the area of a right triangle, we have

$$A_{XYZ} = A_{XWZ} - A_{YWZ} = \frac{(a+c)b}{2} - \frac{cb}{2} = \frac{(ab + \cancel{cb}) - \cancel{cb}}{2} = \frac{ab}{2}. \tag{A.52}$$

The terms involving $c$ cancel, so the result is independent of $c$. That is, $c$ can be anything, and the area is still $ab/2$. Our $ab/2$ result here is the same as in Eq. (A.19), so we see that it doesn't matter if the $Z$ vertex lies above the $XY$ side. $Z$ can lie anywhere along the horizontal line that is a height $b$ above $XY$, and the area is still $ab/2$. Eq. (A.20) therefore holds for all possible triangles.

3. Using the $\pi r^2$ formula for the area of a circle and applying it to the two different values of the radii ($r$ and $r+\epsilon$), we see that the area of the thin ring is

$$\begin{aligned} A_{\text{ring}} &= A_{r+\epsilon} - A_r = \pi(r+\epsilon)^2 - \pi r^2 \\ &= \pi(\cancel{r^2} + 2r\epsilon + \epsilon^2) - \cancel{\pi r^2} = 2\pi r\epsilon + \pi\epsilon^2. \end{aligned} \tag{A.53}$$

This is the exact expression for the area of the ring. But when $\epsilon$ is very small, the $\pi\epsilon^2$ term is much smaller than the $2\pi r\epsilon$ term because it has an extra power of the small quantity $\epsilon$. (If $\epsilon = 10^{-100}$, then $\epsilon^2$ is a factor of $10^{-100}$ smaller than $\epsilon$.) So to a good approximation, we can ignore the $\pi\epsilon^2$. An *approximate* expression for the area of the ring is therefore $A_{\text{ring}} = 2\pi r\epsilon$, if $\epsilon$ is very small.

For our second expression for the area of the ring, we can unroll it into a long thin rectangle. The base of this rectangle is the circumference $C$, and the height is $\epsilon$. So the area is $A_{\text{ring}} = C\epsilon$. ($C$ could be the inner or outer circumference of the ring. It doesn't matter, since they're essentially equal when $\epsilon$ is very small.) This $C\epsilon$ result is an approximate expression for the area, because technically the inner part of the ring will need to stretch (or the outer part will need to compress) a tiny bit when it's unrolled.

Equating our two approximate expressions for $A_{\text{ring}}$ gives

$$C\cancel{\epsilon} = 2\pi r\cancel{\epsilon} \implies C = 2\pi r, \tag{A.54}$$

as desired.

4. The sector is shown in Fig. A.49. The shape is in fact a pie piece of a circle, because all points on the circumference of the base of the cone (which becomes the curved arc in Fig. A.49) are the same distance $s$ from the tip of the cone. So the tip is the center of the would-be complete circle in Fig. A.49.

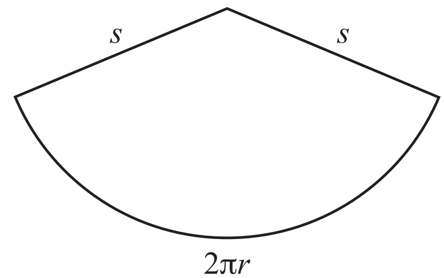


**Figure A.49:** Unrolling a cone produces a sector of a circle.

The circumference of the would-be complete circle is $2\pi s$. And the length of the curved arc is $2\pi r$, because that is the circumference of the base of the cone. The sector is therefore a fraction $2\pi r/2\pi s$ of the complete circle. And since the area of the complete circle is $\pi s^2$, the area of the sector (which equals the desired lateral area of the cone) is

$$A_{\text{sector}} = \pi s^2 \cdot \frac{2\pi r}{2\pi s} = \pi s^2 \cdot \frac{r}{s} = \pi r s, \tag{A.55}$$

in agreement with Eq. (A.27).

Note that after cutting the cone, it is indeed possible to unroll it flat on a table. This is a nice property of cones. It does *not* hold for spheres. If you cut a sphere and then try to flatten it out on a table, you won't be able to, without ripping/stretching/crumpling the paper. This is consistent with the fact that flat maps of a globe look stretched near the poles.

5. Using the $4\pi r^3/3$ formula for the volume of a sphere and applying it to the two different values of the radii ($r$ and $r+\epsilon$), we see that the volume of the thin shell is

$$V_{\text{shell}} = V_{r+\epsilon} - V_r = \frac{4}{3}\pi(r+\epsilon)^3 - \frac{4}{3}\pi r^3. \tag{A.56}$$

Expanding the cube gives

$$\begin{aligned}(r+\epsilon)^3 &= r^3 + 3r^2\epsilon + 3r\epsilon^2 + \epsilon^3 \\ &\approx r^3 + 3r^2\epsilon,\end{aligned} \tag{A.57}$$

where we have ignored the $\epsilon^2$ and $\epsilon^3$ terms because they are much smaller than the $\epsilon$ term, since we are assuming that $\epsilon$ is very small. (This is the same reasoning that allowed us to drop the $\epsilon^2$ term that appeared in Eq. (A.53).) The volume of the shell in Eq. (A.56) therefore becomes (approximately)

$$V_{\text{shell}} = \frac{4}{3}\pi(\cancel{r^3} + 3r^2\epsilon) - \cancel{\frac{4}{3}\pi r^3} = 4\pi r^2\epsilon. \tag{A.58}$$

The second way of viewing the volume of the thin shell is that it (approximately) equals the area $A$ of the sphere times the thickness $\epsilon$. So the volume is $V_{\text{shell}} = A\epsilon$. ($A$ could be the inner or outer area of the shell. It doesn't matter, since they're essentially equal when $\epsilon$ is very small.) This $A\epsilon$ result is an approximate expression for the volume, because if you cut the shell into many little patches (which are slightly curved) and then lay them flat, the inner part of each patch will need to stretch (or the outer part will need to compress) a tiny bit when it's flattened.

Equating our two approximate expressions for $V_{\text{shell}}$ gives

$$A\not\epsilon = 4\pi r^2 \not\epsilon \quad \Longrightarrow \quad A = 4\pi r^2, \tag{A.59}$$

as desired.

Remark: In the above solution, we used the fact that the volume of a thin shell equals the area times the thickness. This might not be quite as evident as the analogous statement that the area of the thin ring in Fig. A.32 equals the circumference times the width. In the case of a ring, it's easy to imagine unrolling it into a thin rectangle. However, in the case of a sphere, you can't flatten out a thin shell without ripping it. But what you can do is the following.

Consider a perfect sphere the size of the earth, and imagine covering it with water with a small depth $h$ (say, a foot). Then divide the earth into $N$ football-field sized patches, each with area $A$ (so $NA$ equals the area of the earth). As we'll see in Section A.5.1, the volume of water above each patch is the area $A$ of the patch times $h$. (Or at least it's *really* close to this. The top area of each patch is *slightly* larger than the bottom area $A$, since the volume isn't quite a perfect rectangular slab.) The total volume of water above all of the patches is then $N \cdot Ah = (NA)h =$ (area of earth)$h$. This is the area of the sphere times the thickness of the shell, as desired. The patch areas don't even need to all be the same. The only fact that matters is that the sum of their areas equals the area of the sphere. ♣

6. Plugging the $a$ and $b$ expressions from Eq. (A.45) into Eq. (A.39) gives

$$\begin{aligned} a^2 + b^2 &= (m^2 - n^2) + (2mn)^2 \\ &= (m^4 - 2m^2n^2 + n^4) + 4m^2n^2 \\ &= m^4 + 2m^2n^2 + n^4 \\ &= (m^2 + n^2)^2 \\ &= c^2, \end{aligned} \tag{A.60}$$

as desired.

7. If we plug $n = m - 1$ into the expressions for $a$, $b$, and $c$ in Eq. (A.45), we obtain

$$\begin{aligned} a &= m^2 - (m-1)^2 = \cancel{m^2} - (\cancel{m^2} - 2m + 1) = 2m - 1, \\ b &= 2m(m-1) = 2m^2 - 2m, \\ c &= m^2 + (m-1)^2 = m^2 + (m^2 - 2m + 1) = 2m^2 - 2m + 1. \end{aligned} \tag{A.61}$$

We want to show four things:

(1) $a$ is odd: Since $a$ takes the form of $2m - 1$ where $m$ is an integer, we see that $a$ is indeed odd. (Even numbers take the form of $2m$, and odd numbers take the form of $2m - 1$. Or equivalently $2m + 1$.)

(2) $a = m + n$: Since $n = m - 1$, the sum $m + n$ equals $m + (m - 1) = 2m - 1$, which equals the $a$ we found in Eq. (A.61), as desired.

Another method: The difference-of-squares result in Eq. (3.24) gives $a = m^2 - n^2 = (m + n)(m - n) = (m + n)(1) = m + n$, where we have used the given fact that $n$ is 1 less than $m$.

(3) $b$ and $c$ differ by 1: The forms we found for $b$ and $c$ in Eq. (A.61) tell us that $c = b+1$, so they do in fact differ by 1.

(4) $b + c = a^2$: The sum of $b$ and $c$ is

$$b + c = (2m^2 - 2m) + (2m^2 - 2m + 1) = 4m^2 - 4m + 1. \tag{A.62}$$

This is indeed equal to $a^2$ since

$$a^2 = (2m - 1)^2 = 4m^2 - 4m + 1. \tag{A.63}$$

Note that for *any* arbitrary values of $m$ and $n$, we have $b+c = 2mn+(m^2+n^2) = (m+n)^2$. But it's only in the special case where $n = m - 1$ that $a = m + n$ (and thus $b + c = a^2$).

8. The overall square has side length $a + b$, so its area is $(a + b)^2$. The area of the smaller square is $c^2$. And the area of each of the four triangles is $ab/2$ (since each one has a base of $a$ and a height of $b$, or vice versa). So the statement that the overall area equals the sum of the areas of the sub-regions is

$$\begin{aligned} (a + b)^2 &= 4 \cdot \frac{ab}{2} + c^2 \\ \implies \quad a^2 + \cancel{2ab} + b^2 &= \cancel{2ab} + c^2 \\ \implies \quad a^2 + b^2 &= c^2, \end{aligned} \tag{A.64}$$

as desired.

9. The area of the overall square is $c^2$, and the area of the smaller square is $(b - a)^2$. The area of each of the four triangles is $ab/2$ (since each one has a base of $a$ and a height of $b$, or vice versa). So the statement that the overall area equals the sum of the areas of the sub-regions is

$$\begin{aligned} c^2 &= 4 \cdot \frac{ab}{2} + (b - a)^2 \\ &= \cancel{2ab} + (b^2 - \cancel{2ab} + a^2) \\ &= b^2 + a^2, \end{aligned} \tag{A.65}$$

as desired.

10. Fig. A.50 shows the rearrangement. The two white squares that are formed have side lengths $a$ and $b$, because those are the legs of the four shaded rectangles in the original figure. So the area of the white region is $a^2 + b^2$. And since the rearrangement doesn't change the white area (because it doesn't change the shaded area), we conclude that $c^2 = a^2 + b^2$.

    Note that for this proof, we don't even need to know that the area of a triangle is $bh/2$, as we did in the first two proofs above. All we need to know is that the area of a square is the side length squared.

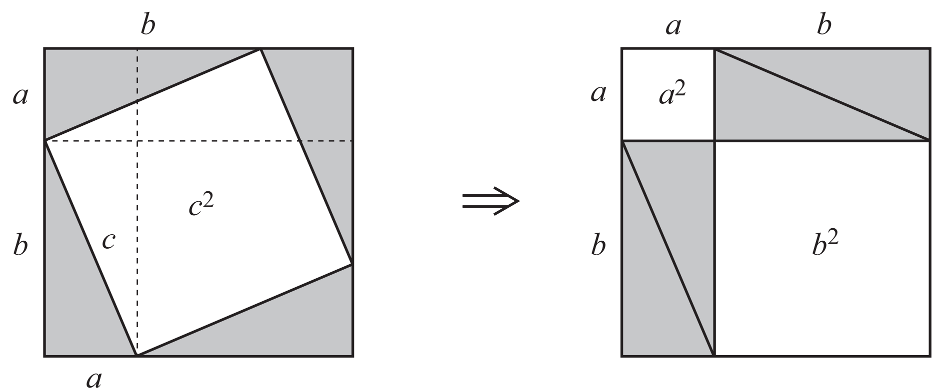


**Figure A.50:** Showing how to rearrange the triangles.

11. We just need to move the two dark-shaded triangles in Fig. A.51, as indicated. (It doesn't matter which one goes where, since they're identical.) The total shaded area (light and dark) is the same in the two figures. And since this area is $a^2 + b^2$ in the left figure (from looking at Fig. A.43), and $c^2$ in the right figure, it follows that $a^2 + b^2 = c^2$.

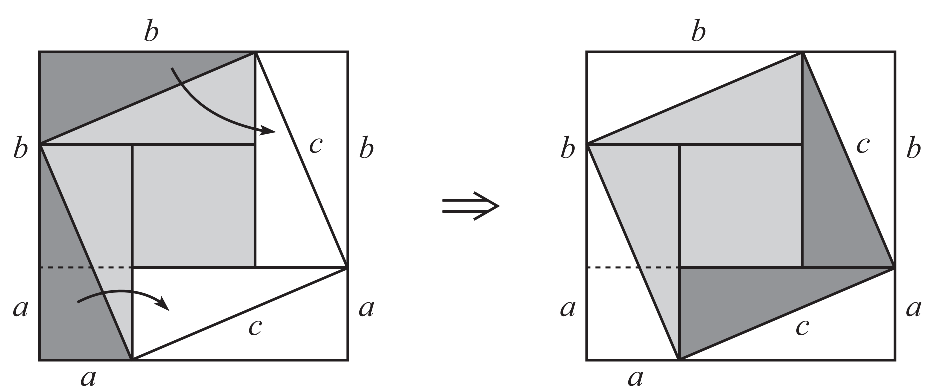


**Figure A.51:** Showing how to move two of the triangles.

This proof and the preceding one show that sometimes a proof doesn't require any words (even though we did use some). A simple picture by itself can do the trick!

> The proof she gave somehow succeeded
> (A bit odd, given how it proceeded).
> But the picture she drew
> Soon convinced us it's true
> That in some cases words are not needed!

12. For convenience in Fig. A.52, let $\angle A$ be labeled as $\alpha$, as shown. Then $\angle B = 90° - \alpha$ because $\angle ACB = 90°$, and the three angles in triangle $ABC$ must add up to 180°.

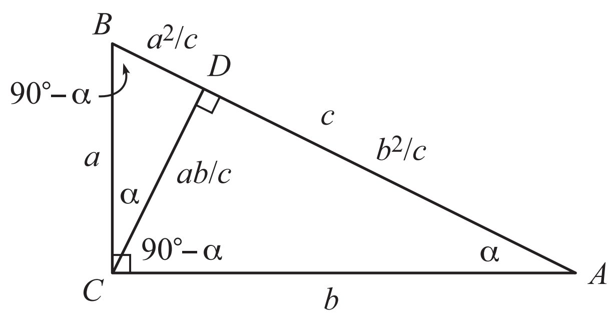


**Figure A.52:** Showing that triangles $ACD$ and $CBD$ are similar to triangle $ABC$, by finding all the angles.

Likewise, in right triangle $ACD$, we have $\angle ACD = 90° - \alpha$ because $\angle ADC = 90°$, and the three angles in triangle $ACD$ must add up to 180°.

Finally, due to the original right angle at $C$, we have

$$\angle DCB = 90° - \angle DCA = 90° - (90° - \alpha) = \alpha. \qquad \text{(A.66)}$$

You can also deduce this angle $\alpha$ by demanding that the angles in triangle $CBD$ add up to 180°.

We therefore see that all three of the triangles $ABC$, $ACD$, and $CBD$ have the same three angles of 90°, $\alpha$, and $90° - \alpha$. Hence they are all similar, as we wanted to show.

We can now use the similarity of the triangles (see Section A.3) to write down some useful ratios. In the overall triangle $ABC$, the short leg $a$ is $a/c$ times the hypotenuse $c$, and the long leg $b$ is $b/c$ times the hypotenuse $c$. Since the other two smaller right triangles are similar to the overall one, they must have these same ratios. That is, in each triangle the short leg is $a/c$ times the hypotenuse, and the long leg is $b/c$ times the hypotenuse.

So in the smallest triangle, the short leg $BD$ is $a/c$ times the hypotenuse, which is $CB = a$. So $BD = (a/c)a = a^2/c$, as shown above in Fig. A.52.

Likewise, in the medium triangle, the long leg $AD$ is $b/c$ times the hypotenuse, which is $AC = b$. So $AD = (b/c)b = b^2/c$, as shown.

We can now use the fact that $BD + AD$ equals the hypotenuse $c$ of the overall triangle. This yields

$$BD + AD = c \quad \Longrightarrow \quad \frac{a^2}{c} + \frac{b^2}{c} = c \quad \Longrightarrow \quad a^2 + b^2 = c^2, \tag{A.67}$$

as desired.

REMARK: We can also find the length of the altitude $CD$ in a number of different ways. Using the ratios of the sides we found above, we can say that in the smallest triangle, the long leg $CD$ is $b/c$ times the hypotenuse, which is $CB = a$. So $CD = (b/c)a = ab/c$, as shown above in Fig. A.52.

Alternatively, in the medium triangle, the short leg $CD$ is $a/c$ times the hypotenuse, which is $AC = b$. So $CD = (a/c)b = ab/c$.

Alternatively again, we can find $CD$ by writing down two valid (base)(height)/2 expressions for the area of the overall triangle. We can consider $b$ to be the base and $a$ to be the height. Or we can consider $c$ to be the base and $CD$ to be the height. Equating the two resulting expressions for the area gives

$$\frac{ba}{2} = \frac{c(CD)}{2} \quad \Longrightarrow \quad \frac{ab}{c} = CD. \qquad \clubsuit \tag{A.68}$$