Chapter 3

Fourier analysis

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This file contains the Fourier-analysis chapter of a potential book on Waves, designed for college sophomores.

Fourier analysis is the study of how general functions can be decomposed into trigonometric or exponential functions with definite frequencies. There are two types of Fourier expansions:

- *Fourier series*: If a (reasonably well-behaved) function is periodic, then it can be written as a *discrete sum* of trigonometric or exponential functions with specific frequencies.

- *Fourier transform*: A general function that isn't necessarily periodic (but that is still reasonably well-behaved) can be written as a *continuous integral* of trigonometric or exponential functions with a continuum of possible frequencies.

The reason why Fourier analysis is so important in physics is that many (although certainly not all) of the differential equations that govern physical systems are *linear*, which implies that the sum of two solutions is again a solution. Therefore, since Fourier analysis tells us that any function can be written in terms of sinusoidal functions, we can limit our attention to these functions when solving the differential equations. And then we can build up any other function from these special ones. This is a very helpful strategy, because it is invariably easier to deal with sinusoidal functions than general ones.

The outline of this chapter is as follows. We start off in Section 3.1 with Fourier trigonometric series and look at how any periodic function can be written as a discrete sum of sine and cosine functions. Then, since anything that can be written in terms of trig functions can also be written in terms of exponentials, we show in Section 3.2 how any periodic function can be written as a discrete sum of exponentials. In Section 3.3, we move on to Fourier transforms and show how an arbitrary (not necessarily periodic) function can be written as a continuous integral of trig functions or exponentials. Some specific functions come up often when Fourier analysis is applied to physics, so we discuss a few of these in Section 3.4. One very common but somewhat odd function is the *delta function*, and this is the subject of Section 3.5.

Section 3.6 deals with an interesting property of Fourier series near discontinuities called the *Gibbs phenomenon*. This isn't so critical for applications to physics, but it's a very interesting mathematical phenomenon. In Section 3.7 we discuss the conditions under which a Fourier series actually converges to the function it is supposed to describe. Again, this discussion is more just for mathematical interest, because the functions we deal with in

physics are invariably well-enough behaved to prevent any issues with convergence. Finally, in Section 3.8 we look at the relation between Fourier series and Fourier transforms. Using the tools we develop in the chapter, we end up being able to derive Fourier's theorem (which says that any periodic function can be written as a discrete sum of sine and cosine functions) from scratch, whereas we simply had to accept this on faith in Section 3.1.

To sum up, Sections 3.1 through 3.5 are very important for physics, while Sections 3.6 through 3.8 are more just for your amusement.

3.1   Fourier trigonometric series

Fourier's theorem states that any (reasonably well-behaved) function can be written in terms of trigonometric or exponential functions. We'll eventually prove this theorem in Section 3.8.3, but for now we'll accept it without proof, so that we don't get caught up in all the details right at the start. But what we *will* do is derive what the coefficients of the sinusoidal functions must be, under the assumption that any function can in fact be written in terms of them.

Consider a function $f(x)$ that is periodic on the interval $0 \leq x \leq L$. An example is shown in Fig. 1. Fourier's theorem works even if $f(x)$ isn't continuous, although an interesting thing happens at the discontinuities, which we'll talk about in Section 3.6. Other conventions for the interval are $-L \leq x \leq L$, or $0 \leq x \leq 1$, or $-\pi \leq x \leq \pi$, etc. There are many different conventions, but they all lead to the same general result in the end. If we assume $0 \leq x \leq L$ periodicity, then Fourier's theorem states that $f(x)$ can be written as

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi nx}{L}\right) + b_n \sin\left(\frac{2\pi nx}{L}\right) \right] \tag{1}$$

where the a_n and b_n coefficients take on certain values that we will calculate below. This expression is the *Fourier trigonometric series* for the function $f(x)$. We could alternatively not separate out the a_0 term, and instead let the sum run from $n = 0$ to ∞, because $\cos(0) = 1$ and $\sin(0) = 0$. But the normal convention is to isolate the a_0 term.
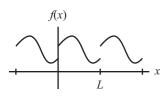
With the 2π included in the arguments of the trig functions, the $n = 1$ terms have period L, the $n = 2$ terms have period $L/2$, and so on. So for any integer n, an integral number of oscillations fit into the period L. The expression in Eq. (1) therefore has a period of (at most) L, which is a necessary requirement, of course, for it to equal the original periodic function $f(x)$. The period can be shorter than L if, say, only the even n's have nonzero coefficients (in which case the period is $L/2$). But it can't be longer than L; the function repeats at least as often as with period L.

We're actually making two statements in Eq. (1). The first statement is that any periodic function *can* be written this way. This is by no means obvious, and it is the part of the theorem that we're accepting here. The second statement is that the a_n and b_n coefficients take on particular values, *assuming* that the function $f(x)$ can be written this way. It's reasonably straightforward to determine what these values are, in terms of $f(x)$, and we'll do this below. But we'll first need to discuss the concept of *orthogonal functions*.

Orthogonal functions

For given values of n and m, consider the integral,

$$\int_0^L \sin\left(\frac{2\pi nx}{L}\right) \cos\left(\frac{2\pi mx}{L}\right) dx. \tag{2}$$

$f(x)$

L

x

Figure 1

Using the trig sum formulas, this can be written as

$$\frac{1}{2} \int_0^L \left[\sin\left((n+m)\frac{2\pi x}{L} \right) + \sin\left((n-m)\frac{2\pi x}{L} \right) \right] dx. \tag{3}$$

But this equals zero, because both of the terms in the integrand undergo an integral number of complete oscillations over the interval from 0 to L, which means that the total area under the curve is zero. (The one exception is the $(n-m)$ term if $n=m$, but in that case the sine is identically zero anyway.) Alternatively, you can simply evaluate the integrals; the results involve cosines whose values are the same at the endpoints of integration. (But again, in the case where $n=m$, the $(n-m)$ term must be treated separately.)

The same kind of reasoning shows that the integral,

$$\int_0^L \cos\left(\frac{2\pi n x}{L} \right) \cos\left(\frac{2\pi m x}{L} \right) dx = \frac{1}{2} \int_0^L \left[\cos\left((n+m)\frac{2\pi x}{L} \right) + \cos\left((n-m)\frac{2\pi x}{L} \right) \right] dx, \tag{4}$$

equals zero except in the special case where $n=m$. If $n=m$, the $(n-m)$ term is identically 1, so the integral equals $L/2$. (Technically $n=-m$ also yields a nonzero integral, but we're concerned only with positive n and m.) Likewise, the integral,

$$\int_0^L \sin\left(\frac{2\pi n x}{L} \right) \sin\left(\frac{2\pi m x}{L} \right) dx = \frac{1}{2} \int_0^L \left[-\cos\left((n+m)\frac{2\pi x}{L} \right) + \cos\left((n-m)\frac{2\pi x}{L} \right) \right] dx, \tag{5}$$

equals zero except in the special case where $n=m$, in which case the integral equals $L/2$.

To sum up, we have demonstrated the following facts:

$$\begin{aligned}
\int_0^L \sin\left(\frac{2\pi n x}{L} \right) \cos\left(\frac{2\pi m x}{L} \right) dx &= 0, \\
\int_0^L \cos\left(\frac{2\pi n x}{L} \right) \cos\left(\frac{2\pi m x}{L} \right) dx &= \frac{L}{2}\delta_{nm}, \\
\int_0^L \sin\left(\frac{2\pi n x}{L} \right) \sin\left(\frac{2\pi m x}{L} \right) dx &= \frac{L}{2}\delta_{nm},
\end{aligned} \tag{6}$$

where δ_{nm} (known as the Kronecker delta) is defined to be 1 if $n=m$, and zero otherwise. In short, the integral of the product of any two of these trig functions is zero unless the two functions are actually the same function. The fancy way of saying this is that all of these functions are *orthogonal*.

We normally use the word "orthogonal" when we talk about vectors. Two vectors are orthogonal if their inner product (also called the dot product) is zero, where the inner product is defined to be the sum of the products of corresponding components of the vectors. For example, in three dimensions we have

$$(a_x, a_y, a_z) \cdot (b_x, b_y, b_z) = a_x b_x + a_y b_y + a_z b_z. \tag{7}$$

For functions, if we define the inner product of the above sine and cosine functions to be the integral of their product, then we can say that two functions are orthogonal if their inner product is zero, just as with vectors. So our definition of the inner product of two functions, $f(x)$ and $g(x)$, is

$$\text{Inner product} \equiv \int f(x)g(x)\,dx. \tag{8}$$

This definition depends on the limits of integration, which can be chosen arbitrarily (we're taking them to be 0 and L). This definition is more than just cute terminology. The inner

product between two functions defined in this way is actually *exactly the same thing* as the inner product between two vectors, for the following reason.

Let's break up the interval $0 \le x \le L$ into a thousand tiny intervals and look at the thousand values of a given function at these points. If we list out these values next to each other, then we've basically formed a thousand-component vector. If we want to calculate the inner product of two such functions, then a good approximation to the continuous integral in Eq. (8) is the discrete sum of the thousand products of the values of the two functions at corresponding points. (We then need to multiply by $dx = L/1000$, but that won't be important here.) But this discrete sum is exactly what you would get if you formed the inner product of the two thousand-component vectors representing the values of the two functions.

Breaking the interval $0 \le x \le L$ into a million, or a billion, etc., tiny intervals would give an even better approximation to the integral. So in short, a function can be thought of as an infinite-component vector, and the inner product is simply the standard inner product of these infinite-component vectors (times the tiny interval dx).

Calculating the coefficients

Having discussed the orthogonality of functions, we can now calculate the a_n and b_n coefficients in Eq. (1). The a_0 case is special, so let's look at that first.

- CALCULATING a_0: Consider the integral, $\int_0^L f(x)\,dx$. Assuming that $f(x)$ can be written in the form given in Eq. (1) (and we are indeed assuming that this aspect of Fourier's theorem is true), then all the sines and cosines undergo an integral number of oscillations over the interval $0 \le x \le L$, so they integrate to zero. The only surviving term is the a_0 one, and it yields an integral of $a_0 L$. Therefore,

$$\int_0^L f(x)\,dx = a_0 L \quad \Longrightarrow \quad \boxed{a_0 = \frac{1}{L}\int_0^L f(x)\,dx} \tag{9}$$

  $a_0 L$ is simply the area under the curve.

- CALCULATING a_n: Now consider the integral,

$$\int_0^L f(x)\cos\left(\frac{2\pi m x}{L}\right)\,dx, \tag{10}$$

  and again assume that $f(x)$ can be written in the form given in Eq. (1). Using the first of the orthogonality relations in Eq. (6), we see that all of the sine terms in $f(x)$ integrate to zero when multiplied by $\cos(2\pi m x/L)$. Similarly, the second of the orthogonality relations in Eq. (6) tells us that the cosine terms integrate to zero for all values of n except in the special case where $n = m$, in which case the integral equals $a_m(L/2)$. Lastly, the a_0 term integrates to zero, because $\cos(2\pi m x/L)$ undergoes an integral number of oscillations. So we end up with

$$\int_0^L f(x)\cos\left(\frac{2\pi m x}{L}\right)\,dx = a_m\frac{L}{2} \quad \Longrightarrow \quad \boxed{a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2\pi n x}{L}\right)\,dx} \tag{11}$$
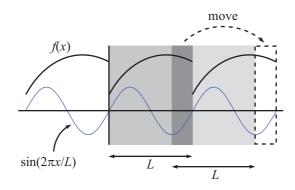
  where we have switched the m label to n, simply because it looks nicer.

- CALCULATING b_n: Similar reasoning shows that

$$\int_0^L f(x)\sin\left(\frac{2\pi m x}{L}\right)\,dx = b_m\frac{L}{2} \quad \Longrightarrow \quad \boxed{b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2\pi n x}{L}\right)\,dx} \tag{12}$$

The limits of integration in the above expressions don't actually have to be 0 and L. They can be any two values of x that differ by L. For example, $-L/4$ and $3L/4$ will work just as well. Any interval of length L will yield the same a_n and b_n coefficients, because both $f(x)$ and the trig functions are periodic. This can be seen in Fig. 2, where two intervals of length L are shown. (As mentioned above, it's fine if $f(x)$ isn't continuous.) These intervals are shaded with slightly different shades of gray, and the darkest region is common to both intervals. The two intervals yield the same integral, because for the interval on the right, we can imagine taking the darkly shaded region and moving it a distance L to the right, which causes the right interval to simply be the left interval shifted by L. Basically, no matter where we put an interval of length L, each point in the period of $f(x)$ and in the period of the trig function gets represented exactly once.

Figure 2

REMARK: However, if we actually *shift the origin* (that is, define another point to be $x = 0$), then the a_n and b_n coefficients change. Shifting the origin doesn't shift the function $f(x)$, but it does shift the sine and cosine curves horizontally, because by definition we're using the functions $\sin(2\pi mx/L)$ and $\cos(2\pi mx/L)$ with no phases. So it matters where we pick the origin. For example, let $f(x)$ be the alternating step function in the first plot in Fig. 3. In the second plot, where we have shifted the origin, the step function remains in the same position on the page, but the sine function is shifted to the right. The integral (over any interval of length L, such as the shaded one) of the product of $f(x)$ and the sine curve in the first plot is not equal to the integral of the product of $g(x)$ and the shifted (by $L/4$) sine curve in the second plot. It is nonzero in the first plot, but zero in the second plot (both of these facts follow from even/odd-ness). We're using g instead of f here to describe the given "curve" in the second plot, because g is technically a different function of x. If the horizontal shift is c (which is $L/4$ here), then g is related to f by $g(x) = f(x+c)$. So the fact that, say, the b_n coefficients don't agree is the statement that the integrals over the shaded regions in the two plots in Fig. 3 don't agree. And this in turn is the statement that

Figure 3

$$\int_0^L f(x) \sin\left(\frac{2\pi mx}{L}\right) dx \neq \int_0^L f(x+c) \sin\left(\frac{2\pi mx}{L}\right) dx. \tag{13}$$

If $f(x)$ has sufficient symmetry, then it is advantageous to shift the origin so that it (or technically the new function $g(x)$) is an even or odd function of x. If $g(x)$ is an even function of x, then only the a_n's survive (the cosine terms), because the b_n's equal the integral of an even function (namely $g(x)$) times an odd function (namely $\sin(2\pi mx/L)$) and therefore vanish. Similarly, if $g(x)$ is an odd function of x, then only the b_n's survive (the sine terms). In particular, the odd function $f(x)$ in the first plot in Fig. 3 has only the b_n terms, whereas the even function $g(x)$ in the second plot has only the a_n terms. But in general, a random choice of the origin will lead to a Fourier series involving both a_n and b_n terms. ♣
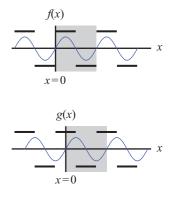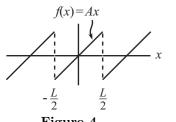
$f(x) = Ax$

$-\dfrac{L}{2}$ $\dfrac{L}{2}$

Figure 4

Example (Sawtooth function):   Find the Fourier series for the periodic function shown in Fig. 4. It is defined by $f(x) = Ax$ for $-L/2 < x < L/2$, and it has period L.

Solution: Since $f(x)$ is an odd function of x, only the b_n coefficients in Eq. (1) are nonzero. Eq. (12) gives them as

$$b_n = \frac{2}{L} \int_{-L/2}^{L/2} f(x) \sin\left(\frac{2\pi n x}{L}\right) dx = \frac{2}{L} \int_{-L/2}^{L/2} Ax \sin\left(\frac{2\pi n x}{L}\right) dx. \tag{14}$$

Integrating by parts (or just looking up the integral in a table) gives the general result,

$$\begin{aligned}
\int x \sin(rx)\, dx &= x\left(-\frac{1}{r}\cos(rx)\right) dx - \int -\frac{1}{r}\cos(rx)\, dx \\
&= -\frac{x}{r}\cos(rx) + \frac{1}{r^2}\sin(rx).
\end{aligned} \tag{15}$$

Letting $r \equiv 2\pi n/L$ yields

$$\begin{aligned}
b_n &= \frac{2A}{L}\left[-x\left(\frac{L}{2\pi n}\right)\cos\left(\frac{2\pi n x}{L}\right)\Big|_{-L/2}^{L/2} + \left(\frac{L}{2\pi n}\right)^2 \sin\left(\frac{2\pi n x}{L}\right)\Big|_{-L/2}^{L/2}\right] \\
&= \left(-\frac{AL}{2\pi n}\cos(\pi n) - \frac{AL}{2\pi n}\cos(-\pi n)\right) + 0 \\
&= -\frac{AL}{\pi n}\cos(\pi n) \\
&= (-1)^{n+1}\frac{AL}{\pi n}.
\end{aligned} \tag{16}$$

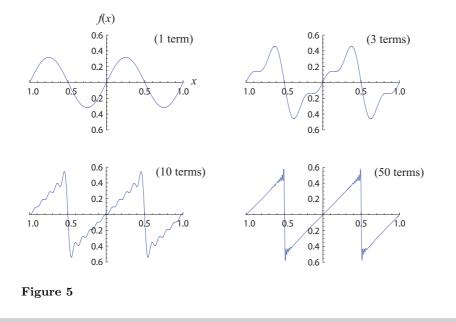Eq. (1) therefore gives the Fourier trig series for $f(x)$ as

$$\begin{aligned}
f(x) &= \frac{AL}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\sin\left(\frac{2\pi n x}{L}\right) \\
&= \frac{AL}{\pi}\left[\sin\left(\frac{2\pi x}{L}\right) - \frac{1}{2}\sin\left(\frac{4\pi x}{L}\right) + \frac{1}{3}\sin\left(\frac{6\pi x}{L}\right) - \cdots\right].
\end{aligned} \tag{17}$$

The larger the number of terms that are included in this series, the better the approximation to the periodic Ax function. The partial-series plots for 1, 3, 10, and 50 terms are shown in Fig. 5. We have arbitrarily chosen A and L to be 1. The 50-term plot gives a very good approximation to the sawtooth function, although it appears to overshoot the value at the discontinuity (and it also has some wiggles in it). This overshoot is an inevitable effect at a discontinuity, known as the *Gibbs phenomenon*. We'll talk about this in Section 3.6.

Interestingly, if we set $x = L/4$ in Eq. (17), we obtain the cool result,

$$A \cdot \frac{L}{4} = \frac{AL}{\pi}\left(1 + 0 - \frac{1}{3} + 0 + \frac{1}{5}\cdots\right) \implies \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots. \tag{18}$$

Nice expressions for π like this often pop out of Fourier series.

Figure 5

3.2   Fourier exponential series

Any function that can be written in terms of sines and cosines can also be written in terms of exponentials. So assuming that we can write any periodic function in the form given in Eq. (1), we can also write it as

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i2\pi nx/L} \tag{19}$$

where the C_n coefficients are given by

$$C_m = \frac{1}{L} \int_0^L f(x) e^{-i2\pi mx/L} dx \tag{20}$$

as we will show below. This hold for all n, including $n = 0$. Eq. (19) is the *Fourier exponential series* for the function $f(x)$. The sum runs over all the integers here, whereas it runs over only the positive integers in Eq. (1), because the negative integers there give redundant sines and cosines.

We can obtain Eq. (20) in a manner similar to the way we obtained the a_n and b_n coefficients via the orthogonality relations in Eq. (6). The analogous orthogonality relation here is

$$\int_0^L e^{i2\pi nx/L} e^{-i2\pi mx/L} dx = \frac{L}{i2\pi(n-m)} e^{i2\pi(n-m)x/L} \Big|_0^L$$
$$= 0, \quad \text{unless } m = n, \tag{21}$$

because the value of the exponential is 1 at both limits. If $m = n$, then the integral is simply

$\int_0^L 1 \cdot dx = L$. So in the δ_{nm} notation, we have[1]

$$\int_0^L e^{i2\pi nx/L} e^{-i2\pi mx/L} dx = L\delta_{nm}.$$ (22)

Just like with the various sine and cosine functions in the previous section, the different exponential functions are orthogonal.[2] Therefore, when we plug the $f(x)$ from Eq. (19) into Eq. (20), the only term that survives from the expansion of $f(x)$ is the one where the n value equals m. And in that case the integral is simply $C_m L$. So dividing the integral by L gives C_m, as Eq. (20) states. As with a_0 in the case of the trig series, Eq. (20) tells us that $C_0 L$ is the area under the curve.

Example (Sawtooth function again): Calculate the C_n's for the periodic Ax function in the previous example.

Solution: Eq. (20) gives (switching from n to m)

$$C_n = \frac{1}{L} \int_{-L/2}^{L/2} Ax e^{-i2\pi nx/L} dx.$$ (23)

The integral of the general form $\int x e^{-rx} dx$ can be found by integration by parts (or looking it up in a table). The result is

$$\int x e^{-rx} dx = -\frac{x}{r} e^{-rx} - \frac{1}{r^2} e^{-rx}.$$ (24)

This is valid unless $r = 0$, in which case the integral equals $\int x \, dx = x^2/2$. For the specific integral in Eq. (23) we have $r = i2\pi n/L$, so we obtain (for $n \neq 0$)

$$C_n = -\frac{A}{L} \left(\frac{xL}{i2\pi n} e^{-i2\pi nx/L} \Big|_{-L/2}^{L/2} + \left(\frac{L}{i2\pi n} \right)^2 e^{-i2\pi nx/L} \Big|_{-L/2}^{L/2} \right).$$ (25)

The second of these terms yields zero because the limits produce equal terms. The first term yields

$$C_n = -\frac{A}{L} \cdot \frac{(L/2)L}{i2\pi n} \left(e^{-i\pi n} + e^{i\pi n} \right).$$ (26)

The sum of the exponentials is simply $2(-1)^n$. So we have (getting the i out of the denominator)

$$C_n = (-1)^n \frac{iAL}{2\pi n} \qquad \text{(for } n \neq 0\text{)}.$$ (27)

If $n = 0$, then the integral yields $C_0 = (A/L)(x^2/2)\big|_{-L/2}^{L/2} = 0$. Basically, the area under the curve is zero since Ax is an odd function. Putting everything together gives the Fourier exponential series,

$$f(x) = \sum_{n \neq 0} (-1)^n \frac{iAL}{2\pi n} e^{i2\pi nx/L}.$$ (28)

This sum runs over all the integers (positive and negative) with the exception of 0.

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[1]The "L" in front of the integral in this equation is twice the "$L/2$" that appears in Eq. (6). This is no surprise, because if we use $e^{i\theta} = \cos\theta + i\sin\theta$ to write the integral in this equation in terms of sines and cosines, we end up with two $\sin \cdot \cos$ terms that integrate to zero, plus a $\cos \cdot \cos$ and a $\sin \cdot \sin$ term, each of which integrates to $(L/2)\delta_{nm}$.

[2]For complex functions, the inner product is defined to be the integral of the product of the functions, where one of them has been complex conjugated. This complication didn't arise with the trig functions in Eq. (6) because they were real. But this definition is needed here, because without the complex conjugation, the inner product of a function with itself would be zero, which wouldn't be analogous to regular vectors.

As a double check on this, we can write the exponential in terms of sines and cosines:

$$f(x) = \sum_{n \neq 0} (-1)^n \frac{iAL}{2\pi n} \left(\cos\left(\frac{2\pi nx}{L}\right) + i \sin\left(\frac{2\pi nx}{L}\right) \right). \tag{29}$$

Since $(-1)^n/n$ is an odd function of n, and since $\cos(2\pi nx/L)$ is an even function of n, the cosine terms sum to zero. Also, since $\sin(2\pi nx/L)$ is an odd function of n, we can restrict the sum to the positive integers, and then double the result. Using $i^2(-1)^n = (-1)^{n+1}$, we obtain

$$f(x) = \frac{AL}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \sin\left(\frac{2\pi nx}{L}\right), \tag{30}$$

in agreement with the result in Eq. (17).

Along the lines of this double-check we just performed, another way to calculate the C_n's that avoids doing the integral in Eq. (20) is to extract them from the trig coefficients, a_n and b_n, if we happen to have already calculated those (which we have, in the case of the periodic Ax function). Due to the relations,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \text{and} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}, \tag{31}$$

we can replace the trig functions in Eq. (1) with exponentials. We then quickly see that the C_n coefficients can be obtained from the a_n and b_n coefficients via (getting the i's out of the denominators)

$$C_n = \frac{a_n - ib_n}{2}, \qquad \text{and} \qquad C_{-n} = \frac{a_n + ib_n}{2}, \tag{32}$$

and $C_0 = a_0$. The n's in these relations take on only positive values. For the periodic Ax function, we know that the a_n's are zero, and that the b_n's are given in Eq. (16) as $(-1)^{n+1}AL/\pi n$. So we obtain

$$C_n = -i(-1)^{n+1} \frac{AL}{2\pi n}, \qquad \text{and} \qquad C_{-n} = i(-1)^{n+1} \frac{AL}{2\pi n}. \tag{33}$$

If we define m to be $-n$ in the second expression (so that m runs over the negative integers), and then replace the arbitrary letter m with n, we see that both of these expressions can be transformed into a single formula,

$$C_n = (-1)^n \frac{iAL}{2\pi n}, \tag{34}$$

which holds for all integers (positive and negative), except 0. This result agrees with Eq. (27).

Conversely, if we've already calculated the C_n's, and if we want to determine the a_n's and b_n's in the trig series without doing any integrals, then we can use $e^{-z} = \cos z + i \sin z$ to write the exponentials in Eq. (19) in terms of sines and cosines. The result will be a trig series in the form of Eq. (1), with a_n and b_n given by

$$a_n = C_n + C_{-n}, \qquad \text{and} \qquad b_n = i(C_n - C_{-n}), \tag{35}$$

and $a_0 = C_0$. Of course, we can also obtain these relations by simply inverting the relations in Eq. (32). We essentially used these relations in the double-check we performed in the example above.

Real or imaginary, even or odd

If a function $f(x)$ is real (which is generally the case in classical physics), then the nth and $(-n)$th terms in Eq. (19) must be complex conjugates. Because the exponential parts are already complex conjugates of each other, this implies that

$$C_{-n} = C_n^* \qquad \text{(if } f(x) \text{ is real.} \tag{36}$$

For the (real) periodic Ax function we discussed above, the C_n's in Eq. (27) do indeed satisfy $C_{-n} = C_n^*$. In the opposite case where a function is purely imaginary, we must have $C_{-n} = -C_n^*$.

Concerning even/odd-ness, a function $f(x)$ is in general neither an even nor an odd function of x. But if one of these special cases holds, then we can say something about the C_n coefficients.

- If $f(x)$ is an *even* function of x, then there are only cosine terms in the trig series in Eq. (1), so the relative plus sign in the first equation in Eq. (31) implies that

$$C_n = C_{-n}. \tag{37}$$

  If additionally $f(x)$ is real, then the $C_{-n} = C_n^*$ requirement implies that the C_n's are purely real.

- If $f(x)$ is an *odd* function of x, then there are only sine terms in the trig series in Eq. (1), so the relative minus sign in the second equation in Eq. (31) implies that

$$C_n = -C_{-n}. \tag{38}$$

  This is the case for the C_n's we found in Eq. (27). If additionally $f(x)$ is real, then the $C_{-n} = C_n^*$ requirement implies that the C_n's are purely imaginary, which is also the case for the C_n's in Eq. (27).

We'll talk more about these real/imaginary and even/odd issues when we discuss Fourier transforms.

3.3   Fourier transforms

The Fourier trigonometric series given by Eq. (1) and Eqs. (9,11,12), or equivalently the Fourier exponential series given by Eqs. (19,20), work for periodic functions. But what if a function isn't periodic? Can we still write the function as a series involving trigonometric or exponential terms? It turns out that indeed we can, but the series is now a continuous integral instead of a discrete sum. Let's see how this comes about.

The key point in deriving the continuous integral is that we can consider a *nonperiodic* function to be a *periodic* function on the interval $-L/2 \leq x \leq L/2$, with $L \to \infty$. This might sound like a cheat, but it is a perfectly legitimate thing to do. Given this strategy, we can take advantage of all the results we derived earlier for periodic functions, with the only remaining task being to see what happens to these results in the $L \to \infty$ limit. We'll work with the exponential series here, but we could just as well work with the trigonometric series.

Our starting point will be Eqs. (19) and (20), which we'll copy here:

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i2\pi nx/L}, \qquad \text{where} \qquad C_n = \frac{1}{L} \int_{-L/2}^{L/2} f(x) e^{-i2\pi nx/L} dx, \tag{39}$$

where we have taken the limits to be $-L/2$ and $L/2$. Let's define $k_n \equiv 2\pi n/L$. The difference between successive k_n values is $dk_n = 2\pi(dn)/L = 2\pi/L$, because dn is simply 1. Since we are interested in the $L \to \infty$ limit, this dk_n is very small, so k_n is essentially a continuous variable. In terms of k_n, the expression for $f(x)$ in Eq. (39) becomes (where we have taken the liberty of multiplying by $dn = 1$)

$$
\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} C_n e^{ik_n x}(dn) \\
&= \sum_{n=-\infty}^{\infty} C_n e^{ik_n x}\left(\frac{L}{2\pi}\,dk_n\right).
\end{aligned}
\tag{40}
$$

But since dk_n is very small, this sum is essentially an integral:

$$
\begin{aligned}
f(x) &= \int_{-\infty}^{\infty}\left(C_n\frac{L}{2\pi}\right)e^{k_n x}dk_n \\
&\equiv \int_{-\infty}^{\infty} C(k_n)e^{ik_n x}dk_n,
\end{aligned}
\tag{41}
$$

where $C(k_n) \equiv (L/2\pi)C_n$. We can use the expression for C_n in Eq. (39) to write $C(k_n)$ as (in the $L \to \infty$ limit)

$$
\begin{aligned}
C(k_n) \equiv \frac{L}{2\pi}\,C_n &= \frac{L}{2\pi}\cdot\frac{1}{L}\int_{-\infty}^{\infty} f(x)e^{-ik_n x}dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ik_n x}dx.
\end{aligned}
\tag{42}
$$

We might as well drop the index n, because if we specify k, there is no need to mention n. We can always find n from $k \equiv 2\pi n/L$ if we want. Eqs. (41) and (42) can then be written as

$$
\boxed{f(x) = \int_{-\infty}^{\infty} C(k)e^{ikx}dk} \qquad \text{where} \qquad \boxed{C(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx}
\tag{43}
$$

$C(k)$ is known as the *Fourier transform* of $f(x)$, and vice versa. We'll talk below about how we can write things in terms of trigonometric functions instead of exponentials. But the term "Fourier transform" is generally taken to refer to the exponential decomposition of a function.

Similar to the case with Fourier series, the first relation in Eq. (43) tells us that $C(k)$ indicates how much of the function $f(x)$ is made up of e^{ikx}. And conversely, the second relation in Eq. (43) tells us that $f(x)$ indicates how much of the function $C(k)$ is made up of e^{-ikx}. Of course, except in special cases (see Section 3.5), essentially *none* of $f(x)$ is made up of e^{ikx} for one particular value of k, because k is a continuous variable, so there is zero chance of having *exactly* a particular value of k. But the useful thing we can say is that if dk is small, then essentially $C(k)\,dk$ of the function $f(x)$ is made up of e^{ikx} terms in the range from k to $k + dk$. The smaller dk is, the smaller the value of $C(k)\,dk$ is (but see Section 3.5 for an exception to this).

REMARKS:

1. These expressions in Eq. (43) are symmetric except for a minus sign in the exponent (it's a matter of convention as to which one has the minus sign), and also a 2π. If you want, you can define things so that each expression has a $1/\sqrt{2\pi}$, by simply replacing $C(k)$ with a $B(k)$ function defined by $B(k) \equiv \sqrt{2\pi}\,C(k)$. In any case, the product of the factors in front of the integrals must be $1/2\pi$.

2. With our convention that one expression has a 2π and the other doesn't, there is an ambiguity when we say, "$f(x)$ and $C(k)$ are Fourier transforms of each other." Better terminology would be to say that $C(k)$ is the Fourier transform of $f(x)$, while $f(x)$ is the "reverse" Fourier transform of $C(k)$. The "reverse" just depends on where the 2π is. If x measures a position and k measures a wavenumber, the common convention is the one in Eq. (43).

   Making both expressions have a "$1/\sqrt{2\pi}$" in them would lead to less of a need for the word "reverse," although there would still be a sign convention in the exponents which would have to be arbitrarily chosen. However, the motivation for having the first equation in Eq. (43) stay the way it is, is that the $C(k)$ there tells us how much of $f(x)$ is made up of e^{ikx}, which is a more natural thing to be concerned with than how much of $f(x)$ is made up of $e^{ikx}/\sqrt{2\pi}$. The latter is analogous to expanding the Fourier series in Eq. (1) in terms of $1/\sqrt{2\pi}$ times the trig functions, which isn't the most natural thing to do. The price to pay for the convention in Eq. (43) is the asymmetry, but that's the way it goes.

3. Note that both expressions in Eq. (43) involve integrals, whereas in the less symmetric Fourier-series results in Eq. (39), one expression involves an integral and one involves a discrete sum. The integral in the first equation in Eq. (43) is analogous to the Fourier-series sum in the first equation in Eq. (39), but unfortunately this integral doesn't have a nice concise name like "Fourier series." The name "Fourier-transform expansion" is probably the most sensible one. At any rate, the Fourier transform itself, $C(k)$, is analogous to the Fourier-series *coefficients* C_n in Eq. (39). The transform doesn't refer to the whole integral in the first equation in Eq. (43). ♣

Real or imaginary, even or odd

Anything that we can do with exponentials we can also do with trigonometric functions. So let's see what Fourier transforms look like in terms of these. Because cosines and sines are even and odd functions, respectively, the best way to go about this is to look at the even/odd-ness of the function at hand. Any function $f(x)$ can be written uniquely as the sum of an even function and an odd function. These even and odd parts (call them $f_e(x)$ and $f_o(x)$) are given by

$$f_e(x) = \frac{f(x) + f(-x)}{2} \qquad \text{and} \qquad f_o(x) = \frac{f(x) - f(-x)}{2}. \qquad (44)$$

You can quickly verify that these are indeed even and odd functions. And $f(x) = f_e(x) + f_o(x)$, as desired.

REMARK: These functions are unique, for the following reason. Assume that there is another pair of even and odd functions whose sum is $f(x)$. Then they must take the form of $f_e(x) + g(x)$ and $f_o(x) - g(x)$ for some function $g(x)$. The first of these relations says that $g(x)$ must be even, but the second says that $g(x)$ must be odd. The only function that is both even and odd is the zero function. So the new pair of functions must actually be the same as the pair we already had. ♣

In terms of $f_e(x)$ and $f_o(x)$, the $C(k)$ in the second equation in Eq. (43) can be written as

$$\begin{aligned} C(k) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Big(f_e(x) + f_o(x) \Big) \Big(\cos(-kx) + i\sin(-kx) \Big) dx. \end{aligned} \qquad (45)$$

If we multiply out the integrand, we obtain four terms. Two of them (the $f_e \cos$ and $f_o \sin$ ones) are even functions of x. And two of them (the $f_e \sin$ and $f_o \cos$ ones) are odd functions of x. The odd ones integrate to zero, so we are left with

$$C(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_e(x) \cos(kx) dx - \frac{i}{2\pi} \int_{-\infty}^{\infty} f_o(x) \sin(kx) dx. \qquad (46)$$

Up to this point we've been considering the even/odd-ness of $f(x)$ as a function of x. But let's now consider the even/odd-ness of $C(k)$ as a function of k. The first term in Eq. (46) is an even function of k (because k appears only in the cosine function), and the second is odd (because k appears only in the sine function). So we can read off the even and odd parts of $C(k)$:

$$\boxed{C_{\mathrm{e}}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_{\mathrm{e}}(x) \cos(kx)\, dx} \qquad \text{and} \qquad \boxed{C_{\mathrm{o}}(k) = -\frac{i}{2\pi} \int_{-\infty}^{\infty} f_{\mathrm{o}}(x) \sin(kx)\, dx} \quad (47)$$

Likewise, the inverse relations are quickly obtained by letting $C(k) = C_{\mathrm{e}}(k) + C_{\mathrm{o}}(k)$ in the expression for $f(x)$ in Eq. (43). The result is

$$f(x) = \int_{-\infty}^{\infty} C_{\mathrm{e}}(k) \cos(kx)\, dk + i \int_{-\infty}^{\infty} C_{\mathrm{o}}(k) \sin(kx)\, dk, \qquad (48)$$

and so

$$\boxed{f_{\mathrm{e}}(x) = \int_{-\infty}^{\infty} C_{\mathrm{e}}(k) \cos(kx)\, dk} \qquad \text{and} \qquad \boxed{f_{\mathrm{o}}(x) = i \int_{-\infty}^{\infty} C_{\mathrm{o}}(k) \sin(kx)\, dk} \qquad (49)$$

Similar to the case with the exponential decomposition, $C_{\mathrm{e}}(k)$ tells us how much of the function $f(x)$ is made up of $\cos(kx)$, and $iC_{\mathrm{o}}(k)$ tells us how much of the function $f(x)$ is made up of $\sin(kx)$. In view of Eq. (48), the functions $C_{\mathrm{e}}(k)$ and $iC_{\mathrm{o}}(k)$ can reasonably be called the "Fourier trig transforms" of $f(x)$.

Note that if we replace the $\cos(kx)$ in the first equations Eqs. (47) and (49) by e^{ikx}, the integral is unaffected, because the sine part of the exponential is an odd function (of either x or k, as appropriate) and therefore doesn't contribute anything to the integral. These equations therefore tell us that $C_{\mathrm{e}}(k)$ is the Fourier transform of $f_{\mathrm{e}}(x)$. And likewise $iC_{\mathrm{o}}(k)$ is the Fourier transform of $f_{\mathrm{o}}(x)$. We see that the pair of functions $f_{\mathrm{e}}(x)$ and $C_{\mathrm{e}}(k)$ is completely unrelated to the pair $f_{\mathrm{o}}(x)$ and $C_{\mathrm{o}}(k)$, as far as Fourier transforms are concerned. The two pairs don't "talk" to each other.
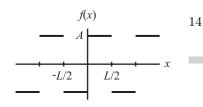
Eq. (46) tells us that if $f(x)$ is real (which is generally the case in classical physics), then the real part of $C(k)$ is even, and the imaginary part is odd. A concise way of saying these two things is that $C(k)$ and $C(-k)$ are complex conjugates. That is, $C(-k) = C(k)^*$. Likewise, Eq. (48) tells us that if $C(k)$ is real, then $f(-x) = f(x)^*$. (Related statements hold if $f(x)$ or $C(k)$ is imaginary.)

For the special cases of purely even/odd and real/imaginary functions, the following facts follow from Eq. (47):

- If $f(x)$ is even and real, then $C(k)$ is even and real.

- If $f(x)$ is even and imaginary, then $C(k)$ is even and imaginary.

- If $f(x)$ is odd and real, then $C(k)$ is odd and imaginary.

- If $f(x)$ is odd and imaginary, then $C(k)$ is odd and real.

The converses are likewise all true, by Eq. (49).

Let's now do some examples where we find the Fourier trig series and Fourier (trig) transform of two related functions.

Figure 6

Example (Periodic odd step function): Calculate the Fourier trig series for the periodic odd step function shown in Fig. 6. The function takes on the values $\pm A$.

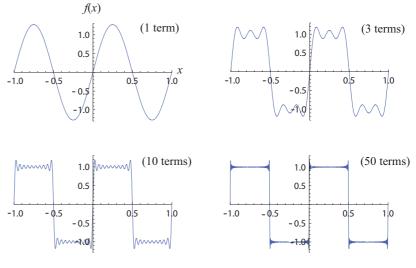**Solution:** The function is odd, so only the sin terms survive in Eq. (1). Eq. (12) gives the b_n coefficients as

$$b_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\sin\left(\frac{2\pi nx}{L}\right)dx = 2\cdot\frac{2}{L}\int_{0}^{L/2} A\sin\left(\frac{2\pi nx}{L}\right)dx$$

$$= -\frac{4A}{L}\left(\frac{L}{2\pi n}\right)\cos\left(\frac{2\pi nx}{L}\right)\Big|_0^{L/2} = \frac{2A}{\pi n}\big(1 - \cos(\pi n)\big). \tag{50}$$

This equals $4A/\pi n$ is n is odd, and zero if n is even. So we have

$$f(x) = \sum_{n=1,\text{odd}}^{\infty} \frac{4A}{\pi n}\sin\left(\frac{2\pi nx}{L}\right)$$

$$= \frac{4A}{\pi}\left(\sin\left(\frac{2\pi x}{L}\right) + \frac{1}{3}\sin\left(\frac{6\pi x}{L}\right) + \frac{1}{5}\sin\left(\frac{10\pi x}{L}\right) + \cdots\right). \tag{51}$$

Some partial plots are shown in Fig. 7. The plot with 50 terms yields a very good approximation to the step function, although there is an overshoot at the discontinuities (and also some fast wiggles). We'll talk about this in Section 3.6.

Figure 7

For the fun of it, we can plug $x = L/4$ into Eq. (51) to obtain

$$A = \frac{4A}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right) \implies \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots, \tag{52}$$

which is the same result we found in Eq. (18).

Figure 8

Example (Non-periodic odd step function): Calculate the Fourier trig transform of the non-periodic odd step function shown in Fig. 8. The value is $\pm A$ inside the region $-L/2 < x < L/2$, and zero outside. Do this in two ways:

(a) Find the Fourier series for the periodic "stretched" odd step function shown in Fig. 9, with period NL, and then take the $N \to \infty$ limit.
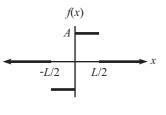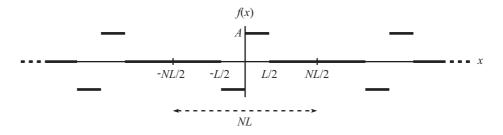
Figure 9

(b) Calculate $C(k)$ by doing the integral in Eq. (43). The result will look different from the result in part (a), but you can demonstrate that the two are equivalent.

Solution:

(a) The function in Fig. 9 is odd, so only the sin terms survive in Eq. (1). The length of the interval is NL, so Eq. (12) gives the b_n coefficients as

$$b_n = \frac{2}{NL} \int_{-NL/2}^{NL/2} f(x) \sin\left(\frac{2\pi nx}{NL}\right) dx. \tag{53}$$

But $f(x)$ is nonzero only in the interval $-L/2 < x < L/2$. We can consider just the $0 < x < L/2$ half and then multiply by 2. So we have

$$b_n = 2 \cdot \frac{2}{NL} \int_0^{L/2} A \sin\left(\frac{2\pi nx}{NL}\right) dx \quad = \quad -\frac{4A}{NL}\left(\frac{NL}{2\pi n}\right)\cos\left(\frac{2\pi nx}{NL}\right)\Big|_0^{L/2}$$

$$= \quad \frac{2A}{\pi n}\left(1 - \cos\left(\frac{\pi n}{N}\right)\right). \tag{54}$$

If $N = 1$, this agrees with the result in Eq. (50) in the previous example, as it should. The Fourier series for the function is therefore

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nx}{NL}\right) = \sum_{n=1}^{\infty} \frac{2A}{\pi n}\left(1 - \cos\left(\frac{\pi n}{N}\right)\right)\sin\left(\frac{2\pi nx}{NL}\right). \tag{55}$$

We will now find the Fourier trig transform by taking the $N \to \infty$ limit. Define z_n by $z_n \equiv n/N$. The changes in z_n and n are related by $dn = N\,dz_n$. dn is simply 1, of course, so we can multiple the result in Eq. (55) by dn without changing anything. If we then replace dn by $N\,dz_n$, and also get rid of the n's in favor of z_n's, we obtain

$$f(x) = \sum_{n=1}^{\infty} \frac{2A}{\pi(Nz_n)}\left(1 - \cos(\pi z_n)\right)\sin\left(\frac{2\pi z_n x}{L}\right) N\,dz_n \tag{56}$$

In the $N \to \infty$ limit, dz_n is very small, so we can change this sum to an integral. And we can drop the n subscript on z, because the value of z determines the associated value of n (if we happen to care what it is). So we arrive at the desired Fourier trig transform:

$$f(x) = \frac{2A}{\pi} \int_0^{\infty} \frac{1}{z}\left(1 - \cos(\pi z)\right)\sin\left(\frac{2\pi zx}{L}\right) dz \qquad \text{(for non-periodic } f\text{).} \tag{57}$$

If we want to put this in the standard Fourier-transform form where the integral runs from $-\infty$ to ∞, we just need to make the coefficient be A/π. The integrand is an even function of z, so using a lower limit of $-\infty$ simply doubles the integral.

The Fourier trig *transform* in Eq. (57) for the *non-periodic* odd step function should be compared with the Fourier *series* for the *periodic* odd step function in the previous

example. If we use the form of b_n in Eq. (50) (that is, without evaluating it for even and odd n), the series looks like

$$f(x) = \frac{2A}{\pi} \sum_{n=0}^{\infty} \frac{1}{n}\Big(1 - \cos(\pi n)\Big) \sin\left(\frac{2\pi n x}{L}\right) \qquad \text{(for periodic } f\text{)} \qquad (58)$$

The only difference between this equation and Eq. (57) is that n can take on only integer values here, whereas z is a continuous variable in Eq. (57). It is quite fascinating how the switch to a continuous variable changes the function from a periodic one to a function that is zero everywhere except in the region $-L/2 < x < L/2$.

REMARK: What if we try to approximate the integral in Eq. (57) by discretizing z into small units and calculating a discrete sum? For example, if we take the "bin" size to be $\Delta z = 0.2$, then the resulting $f(x)$ appears to have the desired value of zero outside the $-L/2 < x < L/2$ region, as shown in Fig. 10 (we've chosen $L = 1$, and we've truncated the discrete sum at $z = 200$, which is plenty large to get a good approximation). However, the zoomed-out view in Fig. 11 shows that the function actually repeats at $x = \pm 5$. And if we zoomed out farther, we would find that it repeats at ± 10 and ± 15, etc. In retrospect this is clear, for two reasons.

First, this discrete sum is exactly the sum in Eq. (56) (after canceling the N's with $dz_n = 0.2$, which corresponds to $N = 5$. So it's no surprise that we end up reproducing the original "stretched" periodic function with $N = 5$. Second, the lowest-frequency sine function in the discrete sum is the $z = 0.2$ one, which is $\sin\big(2\pi(0.2)x/L\big) = \sin(2\pi x/5L)$. This has a period of $5L$. And the periods of all the other sine functions are smaller than this; they take the form of $5L/m$, where m is an integer. So the sum of all the sines in the discrete sum will certainly repeat after $5L$ (and not any sooner).

This argument makes it clear that no matter how we try to approximate the Fourier-transform integral in Eq. (57) by a discrete sum, it will inevitably eventually repeat. Even if we pick the bin size to be a very small number like $\Delta z = 0.001$, the function will still repeat at some point (at $x = \pm 1000L$ in this case). We need a *truly* continuous integral in order for the function to never repeat. We'll talk more about these approximation issues when we look at the Gaussian function in Section 3.4.1. ♣

(b) Eq. (43) gives

$$
\begin{aligned}
C(k) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \\
&= \frac{1}{2\pi} \int_{-L/2}^{0} (-A) e^{-ikx} dx + \frac{1}{2\pi} \int_{0}^{L/2} (A) e^{-ikx} dx \\
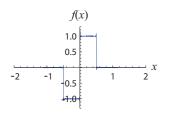&= \frac{1}{2\pi} \cdot \frac{A}{(-ik)} \left(-e^{-ikx}\Big|_{-L/2}^{0} + e^{-ikx}\Big|_{0}^{L/2} \right) \\
&= \frac{iA}{2\pi k}\Big(-(1 - e^{ikL/2}) + (e^{-ikL/2} - 1)\Big) \\
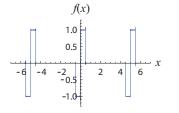&= \frac{-iA}{\pi k}\left(1 - \cos\left(\frac{kL}{2}\right)\right). \qquad\qquad (59)
\end{aligned}
$$

So the expression for $f(x)$ in Eq. (43) becomes

$$
\begin{aligned}
f(x) &= \int_{-\infty}^{\infty} C(k) e^{ikx} dk \\
&= \frac{-iA}{\pi} \int_{-\infty}^{\infty} \frac{1}{k}\left(1 - \cos\left(\frac{kL}{2}\right)\right) e^{ikx} dk. \qquad\qquad (60)
\end{aligned}
$$

This looks different from the expression for $f(x)$ in Eq. (57), but it had better be the same thing, just in a different form. We can rewrite Eq. (60) as follows. Since the $1/k$ factor is an odd function of k, and since $\cos(kL/2)$ is even, only the odd part of e^{ikx}

Figure 10

Figure 11

(the $\sin(kx)$ part) survives in the integral. The integrand is then an even function of k, so we can have the integral run from 0 to ∞ and then multiply by 2. We obtain

$$
\begin{aligned}
f(x) &= \frac{-iA}{\pi} \cdot 2 \int_0^\infty \frac{1}{k}\left(1 - \cos\left(\frac{kL}{2}\right)\right)\left(i\sin(kx)\right) dk \\
&= \frac{2A}{\pi} \int_0^\infty \frac{1}{k}\left(1 - \cos\left(\frac{kL}{2}\right)\right)\sin(kx)\,dk \quad\quad (61)
\end{aligned}
$$

This still looks a little different from the expression for $f(x)$ in Eq. (57), but if we define $k \equiv 2\pi z/L \implies dk = (2\pi/L)dz$, then Eq. (61) turns into Eq. (57), as desired. So both forms generate the same function $f(x)$, as they must.

REMARK: The "z" notation in Eq. (57) has the advantage of matching up with the "n" notation in the Fourier series in Eq. (58). But the "k" notation in Eqs. (60) and (61) is more widely used because it doesn't have all the π's floating around. The "n" notation in the Fourier series in Eq. (58) was reasonable to use, because it made sense to count the number of oscillations that fit into one period of the function. But when dealing with the Fourier transform of a non-periodic function, there is no natural length scale of the function, so it doesn't make sense to count the number of oscillations, so in turn there is no need for the π's. ♣

3.4 Special functions

The are a number of functions whose Fourier transforms come up often in the study of waves. Let's look at a few of the more common ones.

3.4.1 Gaussian

What is the Fourier transform of the Gaussian function, $f(x) = Ae^{-ax^2}$, shown in Fig. 12? (For the purposes of plotting a definite function, we have chosen $A = a = 1$.) If we plug this $f(x)$ into the second equation in Eq. (43), we obtain

$$
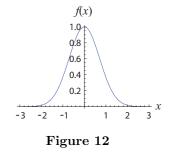C(k) = \frac{1}{2\pi} \int_{-\infty}^\infty Ae^{-ax^2}e^{-ikx}\,dx. \quad\quad (62)
$$

In order to calculate this integral, we'll need to complete the square in the exponent. This gives

$$
C(k) = \frac{1}{2\pi} \int_{-\infty}^\infty Ae^{-a(x+ik/2a)^2}e^{-k^2/4a}\,dx. \quad\quad (63)
$$

It turns out that we can just ignore the constant (as far as x goes) term, $ik/2a$, in the exponent. This follows from a theorem in complex analysis involving the poles (divergences) of a function, but we won't get into that here. With the $ik/2a$ removed, we now have

$$
C(k) = \frac{e^{-k^2/4a}}{2\pi} \int_{-\infty}^\infty Ae^{-ax^2}\,dx. \qu\quad (64)
$$

Although a closed-form expression for the *indefinite* integral $\int e^{-ax^2}\,dx$ doesn't exist, it is fortunately possible to calculate the *definite* integral, $\int_{-\infty}^\infty e^{-ax^2}\,dx$. We can do this by using

f(x)

Figure 12

a slick trick, as follows. Let the desired integral be labeled as $I \equiv \int_\infty^\infty e^{-ax^2}\,dx$. Then we have (we'll explain the steps below)

$$
\begin{aligned}
I^2 &= \left(\int_{-\infty}^{\infty} e^{-ax^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-ay^2}\,dy\right) \\
&= \int_{-\infty}^{\infty}\int_{\infty}^{\infty} e^{-ax^2} e^{-ay^2}\,dx\,dy \\
&= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-ar^2} r\,dr\,d\theta \\
&= 2\pi \int_{0}^{\infty} re^{-ar^2}\,dr \\
&= -2\pi \frac{e^{-ar^2}}{2a}\Bigg|_{0}^{\infty} \\
&= \frac{\pi}{a}\,.
\end{aligned}
\tag{65}
$$

The second line follows from the fact that if you evaluate the double integral by doing the dx integral first, and then the dy one, you'll simply obtain the product of the two integrals in the first line. And the third line arises from changing the Cartesian coordinates to polar coordinates. (Note that if the limits in the original integral weren't $\pm\infty$, then the upper limit on r would depend on θ, and then the θ integral wouldn't be doable.)

Taking the square root of the result in Eq. (65) gives the nice result for the definite integral,[3]

$$
\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}\,.
\tag{66}
$$

As a check on this, if a is small, then the integral is large, which makes sense. Plugging Eq. (66) into Eq. (64) give the desired Fourier transform,

$$
C(k) = \frac{e^{-k^2/4a}}{2\pi}\cdot A\sqrt{\frac{\pi}{a}} \quad\Longrightarrow\quad \boxed{C(k) = \frac{Ae^{-k^2/4a}}{2\sqrt{\pi a}}}
\tag{67}
$$

Interestingly, this is a Gaussian function of k. So the Fourier transform of the Gaussian function of x, $f(x) = Ae^{-ax^2}$, is a Gaussian function of k, $C(k) = (A/2\sqrt{\pi a})e^{-k^2/4a}$. So in some sense, a Gaussian function is the "nicest" function, at least as far as Fourier transforms are concerned. Gaussians remain Gaussians. Note the inverse dependences of $C(k)$ and $f(x)$ on a. For small a, $f(x)$ is very wide, whereas $C(k)$ is very tall (because of the a in the denominator) and sharply peaked (because of the a in the denominator of the exponent).

We can double check the result for $C(k)$ in Eq. (67) by going in the other direction. That is, we can plug $C(k)$ into the first equation in Eq. (43) and check that the result is the original Guassian function, $f(x)$. Indeed,

$$
\begin{aligned}
f(x) &= \int_{-\infty}^{\infty} C(k)e^{ikx}\,dk \\
&= \int_{-\infty}^{\infty} \frac{Ae^{-k^2/4a}}{2\sqrt{\pi a}}\,e^{ikx}\,dk
\end{aligned}
$$

---

[3]As a bonus, you can efficiently calculate integrals of the form $\int_{-\infty}^{\infty} x^{2n} e^{-ax^2}\,dx$ by repeatedly differentiating this expression with respect to a.

$$= \frac{A}{2\sqrt{\pi a}} \int_{-\infty}^{\infty} e^{-(k-2iax)^2/4a} e^{-ax^2} dk$$

$$= \frac{Ae^{-ax^2}}{2\sqrt{\pi a}} \int_{-\infty}^{\infty} e^{-k^2/4a} dk$$

$$= \frac{Ae^{-ax^2}}{2\sqrt{\pi a}} \sqrt{\frac{\pi}{1/4a}}$$

$$= Ae^{-ax^2}, \tag{68}$$

as desired. In obtaining the fourth line above, we (justifiably) ignored the constant (as far as k goes) term, $-2iax$, in the exponent. In obtaining the fifth line, we used Eq. (66) with $1/4a$ in place of a.
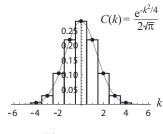
Making approximations to $f(x) = \int_{-\infty}^{\infty} C(k)e^{ikx} dk$

As we discussed shortly after Eq. (43), the first relation in Eq. (43) tells us that the Fourier transform $C(k)$ indicates how much of the function $f(x)$ is made up of e^{ikx}. More precisely, it tells us that $C(k)\,dk$ of the function $f(x)$ is made up of e^{ikx} terms in the range from k to $k + dk$. Since $f(x)$ is even in the present Gaussian case, we know that only the $\cos kx$ part of e^{ikx} will be relevant, but we'll keep writing the whole e^{ikx} anyway.
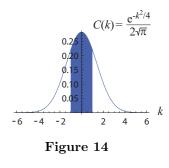
As a rough test to see if the Gaussian function $C(k)$ in Eq. (67) does what it is supposed to do (that is, produce the function $f(x)$ when plugged into the integral $\int_{-\infty}^{\infty} C(k)e^{ikx} dk$), let's make some approximations to this integral. There are two basic things we can do.[4]
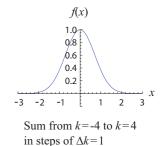
- We can approximate the integral by performing a sum over discrete values of k. That is, we can break up the continuous integral into discrete "bins." For example, if pick the "bin size" to be $\Delta k = 1$, then this corresponds to making the approximation where all k values in the range $-0.5 < k < 0.5$ have the value of $C(k)$ at $k = 0$. And all k values in the range $0.5 < k < 1.5$ have the value of $C(k)$ at $k = 1$. And so on. This is shown in Fig. 13, for the values $A = a = 1$.

- The second way we can approximate the integral in Eq. (43) is to keep the integral a continuous one, but have the limits of integration be $k = \pm\ell$ instead of $\pm\infty$. We can then gradually make ℓ larger until we obtain the $\ell \to \infty$ limit. For example, If $\ell = 1$, then this means that we're integrating only over the shaded region shown in Fig. 14.

Let's see in detail what happens when we make these approximations. Consider the first kind, where we perform a sum over discrete values of k. With $A = a = 1$, we have $f(x) = e^{-x^2}$ and $C(k) = e^{-k^2/4}/2\sqrt{\pi}$. This function $C(k)$ is very small for $|k| \geq 4$, so we can truncate the sum at $k = \pm 4$ with negligible error. With $\Delta k = 1$, the discrete sum of the nine terms from $k = -4$ to $k = 4$ (corresponding to the nine bins shown in Fig. 13) yields an approximation to the $f(x)$ in Eq. (43) that is shown in Fig. 15. This provides a remarkably good approximation to the actual Gaussian function $f(x) = e^{-x^2}$, even though our discretization of k was a fairly coarse one. If we superimposed the true Gaussian on this plot, it would be hard to tell that it was actually a different curve. This is due to the high level of smoothness of the Gaussian function. For other general functions, the agreement invariably isn't this good. And even for the Gaussian, if we had picked a larger bin size of say, 2, the resulting $f(x)$ would have been noticeably non-Gaussian.
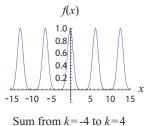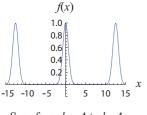
---

[4]Although the following discussion is formulated in terms of the Gaussian function, you can of course apply it to any function. You are encouraged to do so for some of the other functions discussed in this chapter.

$C(k) = \dfrac{e^{-k^2/4}}{2\sqrt{\pi}}$

Figure 13

$C(k) = \dfrac{e^{-k^2/4}}{2\sqrt{\pi}}$

Figure 14

$f(x)$

Sum from $k=-4$ to $k=4$
in steps of $\Delta k = 1$

Figure 15

f(x)

Sum from k=-4 to k=4
in steps of Δk=1

Figure 16

f(x)

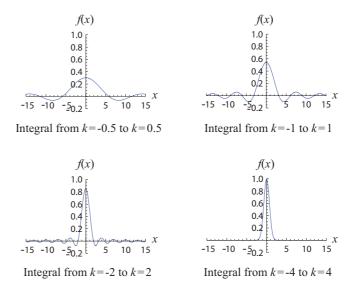Sum from k=-4 to k=4
in steps of Δk=0.5

Figure 17

However, all is not as well as it seems. If we zoom out and let the plot of $f(x)$ run from, say, $x = -15$ to $x = 15$, then we obtain the curve shown in Fig. 16. This is hardly a Gaussian function. It is an (infinite) periodic string of nearly Gaussian functions. In retrospect, this should be no surprise. As we saw above in Fig. 11 and in the accompanying remark, the sum over discrete values of k will inevitably be periodic, because there is a lowest-frequency e^{ikx} component, and all other components have frequencies that are multiples of this. In the present scenario, the $k = 1$ component has the lowest frequency (call is $k_{\min}$), and the period is $2\pi/k_{\min} = 2\pi$. And indeed, the bumps in Fig. 16 occur at multiples of 2π.

What if we make the bin size smaller? Fig. 17 shows the case where k again runs from -4 to 4, but now with a bin size of $\Delta k = 0.5$. So there are 17 terms in the sum, with each term being multiplied by the $\Delta k = 0.5$ weighting factor. The smallest (nonzero) value of k is $k_{\min} = 0.5$. We obtain basically the same shape for the bumps (they're slightly improved, although there wasn't much room for improvement over the $\Delta k = 1$ case), but the period is now $2\pi/k_{\min} = 12\pi$. So the bumps are spread out more. This plot is therefore better than the one in Fig. 16, in the sense that it looks like a Gaussian for a larger range of x. If we kept decreasing the bin size Δk (while still letting the bins run from $k = -4$ to $k = 4$), the bumps would spread out farther and farther, until finally in the $\Delta k \to 0$ limit they would be infinitely far apart. In other words, there would be no repetition at all, and we would have an exact Gaussian function.

Consider now the second of the above approximation strategies, where we truncate the (continuous) integral at $k = \pm\ell$. If we let $\ell = 0.5$, then we obtain the first plot in Fig. 18. This doesn't look much like a Gaussian. In particular, it dips below zero, and the value at $x = 0$ isn't anywhere near the desired value of 1. This is because we simply haven't integrated over enough of the $C(k)$ curve. In the limit of very small ℓ, the resulting $f(x)$ is very small, because we have included only a very small part of the entire integral.

f(x)	f(x)
Integral from k=-0.5 to k=0.5	Integral from k=-1 to k=1
Integral from k=-2 to k=2	Integral from k=-4 to k=4

Figure 18

We can improve things by increasing ℓ to 1. The result is shown in the second plot in Fig. 18. The third plot corresponds to $\ell = 2$, and finally the fourth plot corresponds to $\ell = 4$. This looks very much like a Gaussian. Hardly any of the $C(k)$ curve lies outside $k = \pm 4$, so $\ell = 4$ provides a very good approximation to the full $\ell = \infty$ integral.

Compared with the first approximation method involving discrete sums, the plots generated with this second method have the disadvantage of not looking much like the desired

Gaussian for small values of ℓ. But they have the advantage of not being periodic (because there is no smallest value of k, since k is a now continuous variable). So, what you see is what you get. You don't have to worry about the function repeating at some later point.

3.4.2 Exponential, Lorentzian

What is the Fourier transform of the exponential function, $Ae^{-b|x|}$, shown in Fig. 19? (We have chosen $A = b = 1$ in the figure.) If we plug this $f(x)$ into Eq. (43), we obtain

$$
\begin{aligned}
C(k) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} Ae^{-b|x|} e^{-ikx} dx \\
&= \frac{1}{2\pi} \int_{-\infty}^{0} Ae^{bx} e^{-ikx} dx + \frac{1}{2\pi} \int_{0}^{\infty} Ae^{-bx} e^{-ikx} dx.
\end{aligned}
\tag{69}
$$

$f(x)$

Figure 19

The imaginary terms in the exponents don't change the way these integrals are done (the fundamental theorem of calculus still holds), so we obtain

$$
\begin{aligned}
C(k) &= \frac{Ae^{(b-ik)x}}{2\pi(b-ik)} \Big|_{-\infty}^{0} + \frac{Ae^{(-b-ik)x}}{2\pi(-b-ik)} \Big|_{0}^{\infty} \\
&= \frac{A}{2\pi} \left(\frac{1}{b-ik} + \frac{1}{b+ik} \right) \\
&= \boxed{\frac{Ab}{\pi(b^2+k^2)}}
\end{aligned}
\tag{70}
$$

A function of the general form $c_1/(c_2 + c_3 k^2)$ is called a *Lorentzian* function of k. From the plot of $C(k)$ shown in Fig. 20 (with $A = b = 1$), we see that a Lorentzian looks somewhat like a Gaussian. (The main difference is that it goes to zero like $1/k^2$ instead of exponentially.) This isn't too much of a surprise, because the exponential function in Fig. 19 looks vaguely similar to the Gaussian function in Fig. 12, in that they both are peaked at $x = 0$ and then taper off to zero.
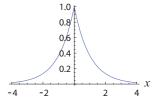

$C(k)$

Figure 20

As in the case of the Gaussian, the $f(x)$ and $C(k)$ functions here have inverse dependences on b. For small b, $f(x)$ is very wide, whereas $C(k)$ is sharply peaked and very tall. $C(k)$ is very tall because the value at $k = 0$ is $A/\pi b$, which is large if b is small. And it is sharply peaked in a relative sense because it decreases to half-max at $k = b$, which is small if b is small. Furthermore, it is sharply peaked in an absolute sense because for any given value of k, the value of $C(k)$ is less than $Ab/\pi k^2$, which goes to zero as $b \to 0$.[5]

If you want to go in the other direction, it is possible to plug the Lorentzian $C(k)$ into the first equation in Eq. (43) and show that the result is in fact $f(x)$. However, the integral requires doing a contour integration, so we'll just accept here that it does in fact work out.

3.4.3 Square wave, sinc

What is the Fourier transform of the "square wave" function shown in Fig. 21? This square wave is defined by $f(x) = A$ for $-a \le x \le a$, and $f(x) = 0$ otherwise. If we plug this $f(x)$ into Eq. (43), we obtain

$$
C(k) = \frac{1}{2\pi} \int_{-a}^{a} Ae^{-ikx} dx
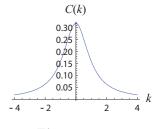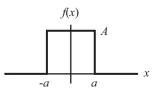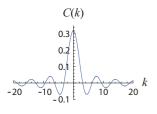$$

$f(x)$

A

$-a$ a x

Figure 21

---

[5]More precisely, for a given k, if you imagine decreasing b, starting with a value larger than k, then $C(k)$ increases to a maximum value of $A/2\pi k$ at $b = k$, and then decreases to zero as $b \to 0$.

$$= \frac{1}{2\pi} \cdot \frac{Ae^{-ikx}}{-ik}\Big|_{-a}^{a}$$

$$= \frac{A}{2\pi} \cdot \frac{e^{-ika} - e^{ika}}{-ik}$$

$$= \boxed{\frac{A\sin(ka)}{\pi k}} \tag{71}$$

Figure 22

This is called a *sinc* function of k. sinc(z) is defined generically to be $\sin(z)/z$. A plot of $C(k)$ is shown in Fig. 22 (with $A = a = 1$). It looks vaguely similar to a Gaussian and a Lorentzian near $k = 0$, but then for larger values of k it oscillates around zero, unlike the Gaussian and Lorentzian curves.

As in the Lorentzian case, it is possible to go in the other direction and plug $C(k)$ into the first equation in Eq. (43) and show that the result is $f(x)$. But again it requires a contour integration, so we'll just accept that it works out.

The dependence of the $C(k)$ sinc function on a isn't as obvious as with the Gaussian and Lorentzian functions. But using $\sin\epsilon \approx \epsilon$ for small ϵ, we see that $C(k) \approx Aa/\pi$ for k values near zero. So if a is large (that is, $f(x)$ is very wide), then $C(k)$ is very tall near $k = 0$. This is the same behavior that the Gaussian and Lorentzian functions exhibit.

However, the "sharply peaked" characteristic seems to be missing from $C(k)$. Fig. 23 shows plots of $C(k)$ for the reasonably large a values of 10 and 40. We're still taking A to be 1, and we've chosen to cut off the vertical axis at 2 and not show the peak value of Aa/π at $k = 0$. The envelope of the oscillatory sine curve is simply the function $A/\pi k$, and this is independent of a. As a increases, the oscillations become quicker, but the envelope doesn't fall off any faster. However, for the purposes of the Fourier transform, the important property of $C(k)$ is its integral, and if $C(k)$ oscillates very quickly, then it essentially averages out to zero. So for all practical purposes, $C(k)$ is basically the zero function except right near the origin. So in that sense it is sharply peaked.
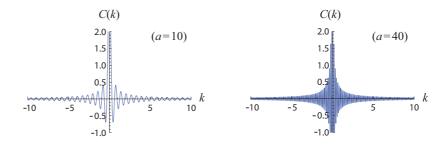

Figure 23

REMARK: The wild oscillatory behavior of the $C(k)$ sinc function in Fig. 23, which is absent in the $C(k)$ Gaussian and Lorentzian functions, is due to the discontinuity in the $f(x)$ square wave. A more complicated combination of e^{ikx} functions are required to generate the discontinuity, especially if the function stays at a constant value of A for a very large interval before abruptly dropping to zero.

So, the lower degree of "niceness" of the sinc function, compared with the Gaussian and Lorentzian functions, can be traced to the discontinuity in the *value* of the $f(x)$ square wave. Similarly, a Lorentzian isn't quite as nice as a Gaussian. It may be a matter of opinion, but the Lorentzian in Fig. 20 isn't quite as smooth as the Gaussian in Fig. 12; it's a little less rounded. This lower degree of niceness of the Lorentzian can be traced to the discontinuity in the *first derivative* of the $f(x) = Ae^{-b|x|}$ exponential function. In a sense, the Gaussian function is the nicest and

smoothest of all functions, and this is why its Fourier transform ends up being as nice and as smooth as possible. In other words, it is another Gaussian function. ♣

3.5 The delta function

3.5.1 Definition

Consider the tall and thin square-wave function shown in Fig. 24. The height and width are related so that the area under the "curve" is 1. In the $a \to 0$ limit, this function has an interesting combination of properties: It is zero everywhere except at the origin, but it still has area 1 because the area is independent of a. A function with these properties is called a *delta function* and is denoted by $\delta(x)$. (It is sometimes called a "Dirac" delta function, to distinguish it from the "Kronecker" delta, δ_{nm}, that came up in Sections 3.1 and 3.2.) So we have the following definition of a delta function:

- *A delta function, $\delta(x)$, is a function that is zero everywhere except at the origin and has area 1.*

Strictly speaking, it's unclear what sense it makes to talk about the area under a curve, if the curve is nonzero at only one point. But there's no need to think too hard about it, because you can avoid the issue by simply thinking of a delta function as a square wave or a Gaussian (or various other functions, as we'll see below) in the sharply-peaked limit.

Consider what happens when we integrate the product of a delta function $\delta(x)$ with some other arbitrary function $f(x)$. We have (we'll explain the steps below)

$$\int_{-\infty}^{\infty} \delta(x)f(x)\,dx = \int_{-\infty}^{\infty} \delta(x)f(0)\,dx = f(0)\int_{-\infty}^{\infty} \delta(x)\,dx = f(0). \tag{72}$$

The first equality comes from the fact that the delta function is zero everywhere except at the origin, so replacing $f(x)$ with $f(0)$ everywhere else has no effect on the integral; the only relevant value of $f(x)$ is the value at the origin. The second equality comes from pulling the constant value, $f(0)$, outside the integral. And the third equality comes from the fact that the area under the delta function "curve" is 1. More generally, we have
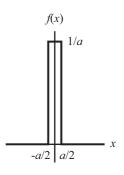
$$\int_{-\infty}^{\infty} \delta(x-x_0)f(x)\,dx = \int_{-\infty}^{\infty} \delta(x-x_0)f(x_0)\,dx = f(x_0)\int_{-\infty}^{\infty} \delta(x-x_0)\,dx = f(x_0). \tag{73}$$

The reasoning is the same as in Eq. (72), except that now the only value of $f(x)$ that matters is $f(x_0)$, because the delta function is zero everywhere except at $x = x_0$ (because this is where the argument of the delta function is zero). This result is interesting. It says that a delta function can be used to "pick out" the value of a function at any point $x = x_0$, by simply integrating the product $\delta(x-x_0)f(x)$.

A note on terminology: A delta function is technically not a function, because there is no way for a function that is nonzero at only one point to have an area of 1. A delta function is actually a *distribution*, which is something that is defined through its integral when multiplied by another function. So a more proper definition of a delta function (we'll still call it that, since that's the accepted convention) is the relation in Eq. (72) (or equivalently Eq. (73)):

- *A delta function, $\delta(x)$, is a distribution that satisfies*

$$\int_{-\infty}^{\infty} \delta(x)f(x)\,dx = f(0). \tag{74}$$

Figure 24

However, having said this, the most convenient way to visualize a delta function is usually to think of it as a very tall and very thin function whose area is 1. This function could be the thin rectangle in Fig. 24, but there are infinitely many other functions that work just as well. For example, we can have a tall and thin triangle with base $2a$ and height $1/a$. Or a tall and thin Gaussian whose parameters are related in a certain way (as we'll see below). And so on.

As an exercise, you can verify the following properties of the delta function:

$$
\begin{aligned}
\delta(x) &= \delta(-x) \\
\delta(ax) &= \frac{\delta(x)}{|a|} \\
\delta(g(x)) &= \frac{\delta(x - x_0)}{|g'(x_0)|} \qquad \text{if} \;\; g(x_0) = 0 \\
\int \delta'(x) f(x)\, dx &= -f'(0).
\end{aligned}
\tag{75}
$$

The second property follows from a change of variables in Eq. (74). The third follows from expanding $g(x)$ in a Taylor series around x_0 and then using the second property. The fourth follows from integration by parts. In the third property, if $g(x)$ has multiple zeros, then $\delta(g(x))$ consists of a delta function at each zero, which means that $\delta(g(x)) = \sum \delta(x - x_i)/|g'(x_i)|$.

3.5.2   The Fourier transform of $f(x) = 1$

It turns out that the delta function plays an important role with regard to Fourier transforms, in that it is the Fourier transform of one of the simplest functions of all, namely the constant function, $f(x) = 1$. Consistent with the general "inverse shape" behavior of $f(x)$ and $C(k)$ that we observed in the examples in Section 3.4, the function $f(x) = 1$ is infinitely spread out, and the delta function is infinitely sharply peaked.

To calculate the Fourier transform of $f(x) = 1$, it seems like all we have to do is plug $f(x) = 1$ into the second equation in Eq. (43). This gives

$$
C(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (1) e^{-ikx} dx.
\tag{76}
$$

This is an interesting integral. If $k = 0$, then we have $\int_{-\infty}^{\infty} (1)(1)\, dx$, which is infinite. And if $k \neq 0$, then the integrand oscillates indefinitely in both the positive and negative directions, so it's unclear what the integral is. If we use large but finite limits, and if e^{-ikx} undergoes an integral number of cycles within these limits, then the integral is zero. But if there is a "leftover" partial cycle, then the integral is nonzero. The integral therefore depends on the limits of integration, and hence is not well defined. Therefore, our first task in calculating the integral is to make sense of it. We can do this as follows.

Instead of having the function be identically 1 for all x, let's have it be essentially equal to 1 for a very large interval and then slowly taper off to zero for very large x. If it tapers off to zero on a length scale of ℓ, then after we calculate the integral in terms of ℓ, we can take the $\ell \to \infty$ limit. In this limit the function equals the original constant function, $f(x) = 1$. There are (infinitely) many ways to make the function taper off to zero. We'll consider a few of these below, and in the process we'll generate various representations of the delta function. We'll take advantage of the results for the special functions we derived in Section 3.4.

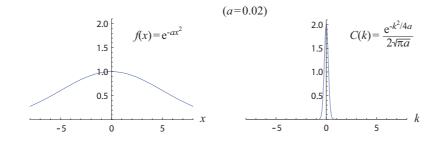### 3.5.3 Representations of the delta function

Gaussian

A standard way to make the $f(x) = 1$ function taper off is to use the Gaussian function, e^{-ax^2}, instead of the constant function, 1. If a is very small, then this Gaussian function is essentially equal to 1 for a large range of x values, and then it eventually tapers off to zero for large x. It goes to zero on a length scale of $\ell \sim 1/\sqrt{a}$. This is true because the function is very close to 1 for, say, $x = 1/(10\sqrt{a})$, where it equals $e^{-1/100} \approx 1 - 1/100$. And it is essentially equal to zero for, say, $x = 5/\sqrt{a}$, where it equals $e^{-25} \approx 0$. In the end, we will take the $a \to 0$ limit, which will make the Gaussian function be essentially equal to 1 for all x. The point is that if we jumped right to the $a = 0$ case, there would be no way to get a handle on the integral. But if we work with a finite value of a, we can actually calculate the integral, and *then* we can take the $a \to 0$ limit.

We calculated the Fourier transform of the Gaussian function $f(x) = Ae^{-ax^2}$ in Eq. (67). With $A = 1$ here, the result is the Gaussian function of k,

$$C(k) = \frac{e^{-k^2/4a}}{2\sqrt{\pi a}} \, . \tag{77}$$

In the $a \to 0$ limit, the a in the exponent makes this Gaussian fall off to zero very quickly (it goes to zero on a length scale of $2\sqrt{a}$). And the a in the denominator makes the value at $k = 0$ very large. The Gaussian is therefore sharply peaked around $k = 0$. So the Fourier transform of a very wide Gaussian is a sharply peaked one, as shown in Fig. 25, for $a = 0.02$. This makes sense, because if $f(x)$ is very wide, then it is made up of only very slowly varying exponential/sinusoidal functions. In other words, $C(k)$ is dominated by k values near zero.

Figure 25

What is the area under the $C(k)$ curve? Using the $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\pi/a}$ result from Eq. (66), we have

$$
\begin{aligned}
\int_{-\infty}^{\infty} C(k)\, dk &= \int_{-\infty}^{\infty} \frac{e^{-k^2/4a}}{2\sqrt{\pi a}} \, dk \\
&= \frac{1}{2\sqrt{\pi a}} \cdot \sqrt{\frac{\pi}{1/4a}} \\
&= 1.
\end{aligned} \tag{78}
$$

So the area under the Fourier transform of the Gaussian function $f(x) = e^{-ax^2}$ equals 1. This is a nice result. And it is independent of a. Even if a is very small, so that the Fourier transform $C(k)$ is very sharply peaked, the area is still 1. We have therefore found the

following representation of the delta function:

$$\delta(k) = \lim_{a \to 0} \frac{e^{-k^2/4a}}{2\sqrt{\pi a}} \tag{79}$$

REMARKS:

1. We should say that the above process of finding the Fourier transform of the constant function $f(x) = 1$ was by no means necessary for obtaining this representation of the delta function. Even if we just pulled Eq. (79) out of a hat, it would still be a representation, because it has the two required properties of being infinitely sharply peaked and having area 1. But we wanted to motivate it by considering it to be the Fourier transform of the constant function, $f(x) = 1$.

2. It is no coincidence that the integral in Eq. (78) equals 1. It is a consequence of the fact that the value of the $f(x)$ Gaussian function equals 1 at $x = 0$. To see why, let's plug the $C(k)$ from Eq. (77) into the first equation in Eq. (43). And let's look at what this equation says when x takes on the particular value of $x = 0$. We obtain

$$f(0) = \int_{-\infty}^{\infty} \frac{e^{-k^2/4a}}{2\sqrt{\pi a}} e^{ik(0)} \, dk \quad \Longrightarrow \quad 1 = \int_{-\infty}^{\infty} \frac{e^{-k^2/4a}}{2\sqrt{\pi a}} \, dk, \tag{80}$$

which is just what Eq. (78) says. This is a general result: if $C(k)$ is the Fourier transform of $f(x)$, then the area under the $C(k)$ curve equals $f(0)$. ♣

Exponential, Lorentzian

Another way to make the $f(x) = 1$ function taper off is to use the exponential function, $e^{-b|x|}$. If b is very small, then this exponential function is essentially equal to 1 for a large range of x values, and then it eventually tapers off to zero for large x. It goes to zero on a length scale of $1/b$.

We calculated the Fourier transform of the exponential function $Ae^{-b|x|}$ in Eq. (70). With $A = 1$ here, the result is the Lorentzian function of k,

$$C(k) = \frac{b}{\pi(b^2 + k^2)} . \tag{81}$$

In the $b \to 0$ limit, this function is very tall, because the value at $k = 0$ is $1/\pi b$. And it is very narrow, because as we mentioned in Section 3.4.2, for any given value of k, the value of $C(k)$ is less than $b/\pi k^2$, which goes to zero as $b \to 0$. The Fourier transform of a wide exponential is therefore a sharply peaked Lorentzian, as shown in Fig. 26.
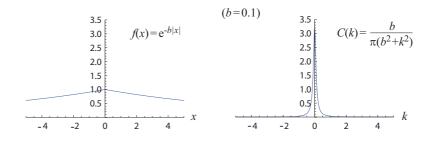

Figure 26

What is the area under the Lorentzian curve? From the above remark containing Eq. (80), we know that the area must equal 1, but let's check this by explicitly evaluating the integral. Using $\int_{-\infty}^{\infty} b\,dk/(b^2+k^2) = \tan^{-1}(k/b)$, we have

$$\int_{-\infty}^{\infty} \frac{b\,dk}{\pi(b^2+k^2)} = \frac{1}{\pi}\left(\tan^{-1}(\infty) - \tan^{-1}(\infty)\right) = \frac{1}{\pi}\left(\frac{\pi}{2} - \frac{-\pi}{2}\right) = 1. \tag{82}$$

Since the area is 1, and since the curve is sharply peaked in the $b \to 0$ limit, the Lorentzian function in Eq. (81) therefore gives another representation of the delta function:

$$\boxed{\delta(k) = \lim_{b\to 0} \frac{b}{\pi(b^2+k^2)}} \tag{83}$$

This makes sense, because the $b \to 0$ limit of the exponential function $e^{-b|x|}$ is essentially the same function as the $a \to 0$ limit of the Gaussian function a^{-ax^2} (they both have $f(x) = 1$ as the limit), and we found above in Eq. (79) that the Fourier transform of a Gaussian is a delta function in the $a \to 0$ limit.

We can look at things the other way, too. We can approximate the $f(x) = 1$ function by a very wide Lorentzian, and then take the Fourier transform of that. The Lorentzian function that approximates $f(x) = 1$ is the $b \to \infty$ limit of $f(x) = b^2/(b^2+x^2)$, because this falls off from 1 when x is of order b. Assuming (correctly) that the Fourier transform process is invertible, we know that the Fourier transform of $f(x) = b^2/(b^2+x^2)$ must be an exponential function. The only question is what the factors are. We could do a contour integration if we wanted to produce the Fourier transform from scratch, but let's do it the easy way, as follows.

The known Fourier-transform relation between $f(x) = e^{-b|x|}$ and $C(k) = b/\pi(b^2+k^2)$, which we derived in Eqs. (69) and (70), is given by Eq. (43) as

$$e^{-b|x|} = \int_{-\infty}^{\infty} \frac{b}{\pi(b^2+k^2)}\, e^{ikx} dk, \qquad \text{where} \qquad \frac{b}{\pi(b^2+k^2)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-b|x|}e^{-ikx} dx. \tag{84}$$

Interchanging the x and k letters in the first of these equations, shifting some factors around, and changing e^{ikx} to e^{-ikx} (which doesn't affect things since the rest of the integrand is an even function of x) gives

$$(b/2)e^{-b|k|} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{b^2}{(b^2+x^2)}\, e^{-ikx} dx. \tag{85}$$

But this is the statement that the Fourier transform of $f(x) = b^2/(b^2+x^2)$ (which is essentially the constant function $f(x) = 1$ if $b \to \infty$) is $C(k) = (b/2)e^{-b|k|}$.[6]

This $C(k) = (b/2)e^{-b|k|}$ exponential function is infinitely sharply-peaked in the $b \to \infty$ limit. And you can quickly show that it has an area of 1 (but again, this follows from the above remark containing Eq. (80)). So we have found another representation of the delta function:

$$\boxed{\delta(k) = \lim_{b\to\infty} (b/2)e^{-b|k|}} \tag{86}$$

Just like with the wide Gaussian and exponential functions, the wide Lorentzian function is essentially equal to the $f(x) = 1$ function, so its Fourier transform should be roughly the same as the above ones. In other words, it should be (and is) a delta function.
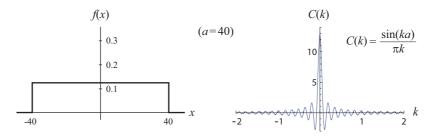
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[6]By "Fourier transform" here, we mean the transform in the direction that has the $1/2\pi$ in front of the integral, since that's what we set out to calculate in Eq. (76).

Square wave, sinc

Another way to make the $f(x) = 1$ function "taper off" is to use a square-wave function that takes on the constant value of 1 out to $x = \pm a$ and then abruptly drops to zero. We calculated the Fourier transform of the square wave in Eq. (71). With $A = 1$ here, the result is the "sinc" function of k,

$$C(k) = \frac{\sin(ka)}{\pi k} \ . \tag{87}$$

Using $\sin \epsilon \approx \epsilon$ for small ϵ, we see that $C(k) \approx a/\pi$ for k values near zero. So if a is large then $C(k)$ is very tall near $k = 0$. The Fourier transform of a very wide square wave (large a) is therefore a very tall sinc function, as shown in Fig. 27. This is the same behavior we saw with the Gaussian, exponential, and Lorentzian functions.

Figure 27

However, as we saw in Fig. 23 in Section 3.4.3, the "sharply peaked" characteristic seems to be missing. The envelope of the oscillatory curve is the function $1/\pi k$, and this is independent of a. As a increases, the oscillations become quicker, but the envelope doesn't fall off any faster. But when it comes to considering $C(k)$ to be a delta function, the important property is its integral. If $C(k)$ oscillates very quickly, then it essentially averages out to zero. So for all practical purposes, $C(k)$ is basically the zero function except right near the origin. So in that sense it is sharply peaked.

Furthermore, we know from the above remark containing Eq. (80) (or a contour integration would also do the trick) that the (signed) area under the $C(k)$ curve equals 1, independent of a. We therefore have yet another representation of the delta function:

$$\boxed{\delta(k) = \lim_{a \to \infty} \frac{\sin(ka)}{\pi k}} \tag{88}$$

We can look at things the other way, too. We can approximate the $f(x) = 1$ function by a very wide sinc function, and then take the Fourier transform of that. The sinc function that approximates $f(x) = 1$ is the $a \to 0$ limit of $\sin(ax)/(ax)$, because this equals 1 at $x = 0$, and it falls off from 1 when x is of order $1/a$. Assuming (correctly) that the Fourier-transform process is invertible, you can show that the above Fourier-transform relation between the square wave and $\sin(ka)/\pi k$ implies (with the 2π in the opposite place; see the step going from Eq. (84) to Eq. (85)) that the Fourier transform of $f(x) = \sin(ax)/(ax)$ is a square-wave function of k that has height $1/2a$ and goes from $-a$ to a. This is a very tall and thin rectangle in the $a \to 0$ limit. And the area is 1. So we have returned full circle to our original representation of the delta function in Fig. 24 (with the a in that figure now $2a$).

Again, just like with the other wide functions we discussed above, a wide sinc function is essentially equal to the $f(x) = 1$ function, so its Fourier transform should be roughly the same as the above ones. In other words, it should be (and is) a delta function.

Integral representation

There is one more representation of the delta function that we shouldn't forget to write down. We've actually been using it repeatedly in this section, so it's nothing new. As we've seen many times, the Fourier transform of the function $f(x) = 1$ is a delta function. So we should write this explicitly:

$$\boxed{\delta(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} dx} \tag{89}$$

where we haven't bothered to write the "1" in the integrand. Of course, this integral doesn't quite make sense the way it is, so it's understood that whenever it comes up, we make the integrand taper off to zero by our method of choice (for example, any of the above methods), and then eventually take the limit where the tapering length scale becomes infinite. More generally, we have

$$\delta(k - k_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(k-k_0)x} dx. \tag{90}$$

Similar generalizations to a $k - k_0$ argument hold for all of the above representations too, of course.

Eq. (90) allows us to generalize the orthogonality relation in Eq. (22), which was

$$\int_0^L e^{i2\pi nx/L} e^{-i2\pi mx/L} dx = L\delta_{nm}. \tag{91}$$

In that case, the inner product was defined as the integral from 0 to L. But if we define a different inner product that involves the integral from $-\infty$ to ∞ (we are free to define it however we want), then Eq. (90) gives us a new orthogonality relation,

$$\int_{-\infty}^{\infty} e^{ik_1 x} e^{-ik_2 x} dx = 2\pi\delta(k_2 - k_1). \tag{92}$$

(Alternatively, we could define the inner product to be $1/2\pi$ times the integral, in which case the 2π wouldn't appear on the righthand side.) So the e^{ikx} functions are still orthogonal; the inner product is zero unless the k values are equal, in which case the inner product is infinite. We see that whether we define the inner product as the integral from 0 to L, or from $-\infty$ to ∞, we end up with a delta function. But in the former case it is the standard "Kronecker" delta, whereas in latter case it is the "Dirac" delta function.
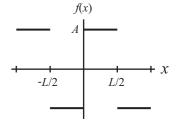
3.6 Gibbs phenomenon

Let's return to the study of Fourier series, as opposed to Fourier transforms. The "Gibbs phenomenon" refers to the manner in which the Fourier series for a periodic function overshoots the values of the function on either side of a discontinuity. This effect is best illustrated (for many reasons, as we will see) by the periodic step function shown in Fig. 28. This step function is defined by

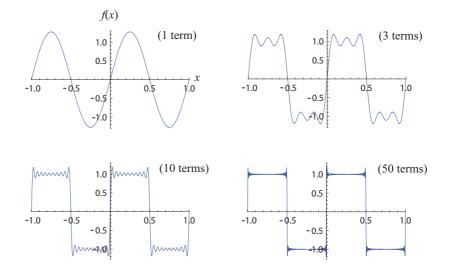$$f(x) = \begin{cases} -A & (-L/2 < x < 0) \\ A & (0 < x < L/2), \end{cases} \tag{93}$$

Figure 28

and has period L. We already found the Fourier series for this function in Eq. (51). It is

$$\begin{aligned} f(x) &= \frac{4A}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin\left(\frac{2\pi nx}{L}\right) \\ &= \frac{4A}{\pi} \left(\sin\left(\frac{2\pi x}{L}\right) + \frac{1}{3}\sin\left(\frac{6\pi x}{L}\right) + \frac{1}{5}\sin\left(\frac{10\pi x}{L}\right) + \cdots \right). \end{aligned} \tag{94}$$
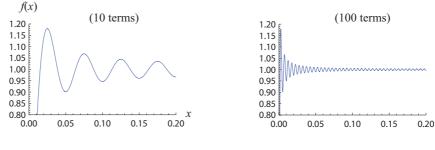
By "n odd" we mean the positive odd integers. If we plot a few partial sums (with $A = L = 1$), we obtain the plots shown in Fig. 29 (this is just a repeat of Fig. 7). The first plot includes only the first term in the series, while the last plot includes up to the 50th term (the $n = 99$ one).

Figure 29

The more terms we include, the more the curve looks like a step function. However, there are two undesirable features. There are wiggles near the discontinuity, and there is an overshoot right next to the discontinuity. As the number of terms grows, the wiggles get pushed closer and closer to the discontinuity, in the sense that the amplitude in a given region decreases as the number of terms, N, in the partial series increases. So in some sense the wiggles go away as N approaches infinity. However, the overshoot unfortunately never goes away. Fig. 30 shows the zoomed-in pictures near the discontinuity for $N = 10$ and $N = 100$. If we included more terms, the overshoot would get pushed closer and closer to the discontinuity, but it would still always be there.

Figure 30

It turns out that as long as the number of terms in the partial series is large, the height of the overshoot is always about 9% of the jump in the function, independent of the (large) number N. This is consistent with the plots in Fig. 30, where the overshoot is about 0.18, which is 9% of the jump from -1 to 1. Furthermore, this 9% result holds for a discontinuity in *any* function, not just a step function. This general result is provable in a page or two,
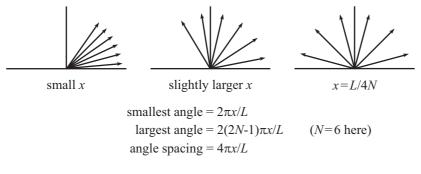
so let's do that here. Our strategy will be to first prove it for the above step function, and then show how the result for any other function follows from the specific result for the step function.

To demonstrate the 9% the result for the step function, our first task is to find the location of the maximum value of the overshoot. Let's assume that we're truncating the series in Eq. (94) at the Nth term (the $n = 2N - 1$ one), and then we'll take the $N \to \infty$ limit. To find the location of the maximum, we can take the derivative of the partial series (which we'll call $f_N(x)$) and set the result equal to zero. This gives

$$0 = \frac{df_N(x)}{dx} = \frac{8A}{L} \left(\cos \left(\frac{2\pi x}{L} \right) + \cos \left(\frac{6\pi x}{L} \right) + \cdots + \cos \left(\frac{2(2N - 1)\pi x}{L} \right) \right). \qquad (95)$$

The smallest solution to this equation is $x = L/4N$. This is a solution because it makes the arguments of the cosines in the first and last terms in Eq. (95) be supplementary, which means that the cosines cancel. And likewise for the second and second-to-last terms. And so on. So all of the terms cancel in pairs. (If N is odd, there is a middle term that is zero.) Furthermore, there is no (positive) solution that is smaller than this one, for the following reason.

For concreteness, let's take $N = 6$. If we start with a very small value of x, then the sum of the cosine terms in Eq. (95) (there are six in the $N = 6$ case) equals the sum of the horizontal components of the vectors in the first diagram in Fig. 31. (The vectors make angles $2\pi n x/L$ with the horizontal axis, where $n = 1, 3, 5, 7, 9, 11$.) This sum is clearly not zero, because the four vectors are "lopsided" in the positive direction.

small x slightly larger x $x = L/4N$

smallest angle = $2\pi x/L$
largest angle = $2(2N-1)\pi x/L$ (N=6 here)
angle spacing = $4\pi x/L$

Figure 31

If we increase x a little, we get the six vectors in the second diagram in Fig. 31. The vectors are still lopsided in the positive direction, so the sum again isn't zero. But if we keep increasing x, we finally get to the situation shown in the third diagram in Fig. 31. The vectors are now symmetric with respect to the y axis, so the sum of the horizontal components is zero. This is the $x = L/4N$ case we found above, where the vectors are supplementary in pairs (setting the sum of the first and Nth angles equal to π gives $x = L/4N$). So $x = L/4N$ is indeed the smallest positive solution to Eq. (95) and is therefore the location of the overshoot peak in Fig. 30.

Having found the location of the maximum value of the overshoot, we can plug $x = L/4N$ into Eq. (94) to find this maximum value. We obtain

$$\begin{aligned} f_N \left(\frac{L}{4N} \right) &= \frac{4A}{\pi} \left(\sin \left(\frac{\pi}{2N} \right) + \frac{1}{3} \sin \left(\frac{3\pi}{2N} \right) + \cdots + \frac{1}{2N - 1} \sin \left(\frac{(2N - 1)\pi}{2N} \right) \right) \\ &= \frac{4A}{\pi} \sum_{m=1}^{N} \frac{1}{2m - 1} \sin \left(\frac{(2m - 1)\pi}{2N} \right). \end{aligned} \qquad (96)$$

In the $N \to \infty$ limit, the arguments of the sines increase essentially continuously from 0 to π. Therefore, because of the $(2m-1)$ factor in the denominator, this sum is essentially the integral of $\sin(y)/y$ from 0 to π, except for some factors we need to get straight. If we multiply Eq. (96) through by 1 in the form of $(\pi/2N)(2N/\pi)$, we obtain

$$f_N\left(\frac{L}{4N}\right) = \frac{\pi}{2N} \cdot \frac{4A}{\pi} \sum_{m=1}^{N} \frac{2N}{(2m-1)\pi} \sin\left(\frac{(2m-1)\pi}{2N}\right). \tag{97}$$

Each term in this sum is weighted by a factor of 1, whereas in the integral $\int_0^\pi \left(\sin(y)/y\right) dy$ each term is weighted by dy. But dy is the difference between successive values of $(2m-1)\pi/2N$, which is $2\pi/2N$. So $dy = \pi/N$. The above sum is therefore N/π times the integral from 0 to π. So in the $N \to \infty$ limit we obtain

$$\begin{aligned} f_N\left(\frac{L}{4N}\right) &= \frac{\pi}{2N} \cdot \frac{4A}{\pi} \left(\frac{N}{\pi} \int_0^\pi \frac{\sin y}{y} \, dy\right) \\ &= \frac{2A}{\pi} \int_0^\pi \frac{\sin y}{y} \, dy. \end{aligned} \tag{98}$$

Alternatively, you can systematically convert the sum in Eq. (96) to the integral in Eq. (98) by defining

$$y \equiv \frac{(2m-1)\pi}{2N} \implies dy = \frac{\pi \, dm}{N}. \tag{99}$$

If you multiply Eq. (96) by dm (which doesn't affect anything, since $dm = 1$) and then change variables from m to y, you will obtain Eq. (98).

The value of the actual step function just to the right of the origin is A, so to get the overshoot, we need to subtract off A from $f(L/4N)$. And then we need to divide by $2A$ (which is the total height of the jump) to obtain the fractional overshoot. The result is

$$\frac{1}{2A}\left(f\left(\frac{L}{4N}\right) - A\right) = \frac{1}{\pi} \int_0^\pi \frac{\sin y}{y} \, dy - \frac{1}{2} \approx 0.0895 \approx 9\%, \tag{100}$$

where we have calculated the integral numerically. For the simple step function, we have therefore demonstrated the desired 9% result.
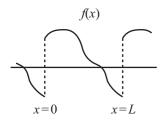
As we mentioned above, this 9% result also holds for any discontinuity in any other function. Having obtained the result for the simple step function, the generalization to other arbitrary functions is surprisingly simple. It proceeds as follows.
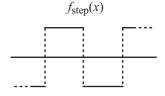
Consider an arbitrary function $f(x)$ with period L and with a discontinuity, as shown in the first plot in Fig. 32. Without loss of generality, we can assume that the discontinuity in question (there may very well be others) is located at the origin. And we can also assume that the discontinuity is vertically symmetric around the origin, that is, it goes from $-A$ to A (or vice versa) for some value of A. This assumption is valid due to the fact that shifting the function vertically simply changes the a_0 value in Eq. (1), which doesn't affect the nature of the overshoot.

Now consider the periodic step function, $f_{\text{step}}(x)$, that jumps from $-A$ to A and also has period L, as shown in the second plot in Fig. 32. And finally consider the function $f_{\text{diff}}(x)$ defined to be the difference between $f(x)$ and $f_{\text{step}}(x)$, as shown in the third plot in Fig. 32. So we have

$$f(x) = f_{\text{step}}(x) + f_{\text{diff}}(x). \tag{101}$$

Since $f_{\text{diff}}(x) = f(x) - f_{\text{step}}(x)$, the plot of $f_{\text{diff}}(x)$ is obtained by simply subtracting or adding A from $f(x)$, depending on whether the step function is positive or negative at x, respectively. The critical point to realize is that by construction, $f_{\text{diff}}(x)$ is a *continuous*
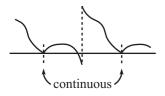

$f(x)$

$x{=}0$ $x{=}L$

$f_{\text{step}}(x)$

$f_{\text{diff}}(x)$

continuous

Figure 32

function at the discontinuity at $x = 0$ (and $x = L$, etc.). Its derivative might not be continuous (and in fact isn't, in the present case), but that won't matter here. Also, $f_{\text{diff}}(x)$ may very well have discontinuities elsewhere (in fact, we introduced one at $x = L/2$), but these aren't relevant as far as the discontinuity at the origin goes.

Since $f_{\text{diff}}(x)$ is continuous at $x = 0$, its Fourier series has no overshoot there (see the first remark below). The Fourier series for $f_{\text{diff}}(x)$ therefore contributes nothing to the overshoot of the Fourier series of $f(x)$ at $x = 0$. Hence, in view of Eq. (101), the overshoot in the Fourier series for $f(x)$ is exactly equal to the overshoot in the Fourier series for $f_{\text{step}}(x)$. The same 9% result therefore holds, as we wanted to show.

REMARKS:

1. We claimed in the previous paragraph that since $f_{\text{diff}}(x)$ is continuous at $x = 0$, its Fourier series has no overshoot there. Said in another way, a kink in $f_{\text{diff}}(x)$ (that is, a discontinuity in its derivative) doesn't cause an overshoot. The can be shown with the same strategy that we used above, where we looked at a specific function (the step function) and then argued why any other function gave the same result.

   For the present task, we'll consider the specific triangular function shown in Fig. 35 below, which we'll discuss in Section 3.7. We'll find that the Fourier series for this triangular function is smooth at $x = 0$. There is no overshoot. Now, any function $f_{\text{diff}}(x)$ with a kink looks (at least locally near the kink) like a triangular function plus possibly a linear function of x, which has the effect of tilting the triangle so that the slopes on either side of the kink aren't equal and opposite. The triangular function has no overshoot, and the linear function certainly doesn't have one, either. So we conclude that $f_{\text{diff}}(x)$ doesn't have an overshoot, as we wanted to show.

2. The reasoning in this section can also be used to prove a quick corollary, namely, that the value of the Fourier series at a discontinuity equals the midpoint of the jump. This follows because it is true for the step function (the value at $x = 0$ is zero, because we have only sines in the series), and it is also true for $f_{\text{diff}}(x)$ (in a trivial sense, because $f_{\text{diff}}(x)$ is continuous and has no jump). Also, any a_0 constant we stripped off from the original function doesn't affect the result, because it increases the endpoints and midpoint of the jump by the same amount.

3. The overshoot (which occurs at $x = L/4N$) gets squeezed arbitrarily close to the discontinuity as N becomes large. So for any given nonzero value of x near the discontinuity, the overshoot is irrelevant as far as the value of $f(x)$ goes. Also, at any given nonzero value of x the amplitude of the wiggles in Fig. 30 goes to zero (we'll show this in the following subsection). Therefore, the Gibbs phenomenon is irrelevant for any given nonzero value of x, in the $N \to \infty$ limit. The only possibly value of x where we might have an issue is $x = 0$ (and $x = L$, etc.). From the previous remark, we know that the value of the series at $x = 0$ is the midpoint of the jump. This may or may not equal the arbitrary value we choose to assign to $f(0)$, but we can live with that. ♣

The shape of the wiggles

The strategy that we used above, where we turned the sum in Eq. (94) (or rather, the partial sum of the first N terms) into the integral in Eq. (98), actually works for any small value of x, not just the $x = L/4N$ location of the overshoot. We can therefore use this strategy to calculate the value of the Fourier series at *any* small value of x, in the large-N limit.[7] In other words, we can find the shape (and in particular the envelope) of the wiggles in Fig. 30.

---

[7]We need x to be small compared with L, so that the sine terms in Eq. (102) below vary essentially continuously as m varies. If this weren't the case, then we wouldn't be able to approximate the sum in Eq. (102) by an integral, which we will need to do.

The procedure follows the one above. First, write the partial sum of the first N terms in Eq. (94) as

$$f_N(x) = \frac{4A}{\pi} \sum_{m=1}^{N} \frac{1}{2m-1} \sin\left(\frac{2(2m-1)\pi x}{L}\right). \tag{102}$$

Then define

$$y \equiv \frac{2(2m-1)\pi x}{L} \quad \implies \quad dy = \frac{4\pi x\, dm}{L}. \tag{103}$$

Then multiply Eq. (102) by dm (which doesn't affect anything, since $dm = 1$) and change variables from m to y. The result is (for large N)

$$f_N(x) \approx \frac{2A}{\pi} \int_0^{(4N-2)\pi x/L} \frac{\sin y}{y}\, dy. \tag{104}$$

As double check on this, if we let $x = L/4N$ then the upper limit of integration is equal to π (in the large-N limit), so this result reduces to the one in Eq. (98).

We were concerned with the $N \to \infty$ limit in the above derivation of the overshoot, but now we're concerned with just large (but not infinite) N. If we actually set $N = \infty$ in Eq. (104), we obtain $(2A/\pi) \int_0^\infty (\sin y)/y \cdot dy$. This had better be equal to A, because that is the value of the step function at x. And indeed it is, because this definite integral equals $\pi/2$ (see the paragraph preceding Eq. (88); it boils down to the fact that $(\sin y)/y$ is the Fourier transform of a square wave). But the point is that for the present purposes, we want to keep N finite, so that we can actually see the x dependence of $f_N(x)$. (So technically the integral approximation to the sum isn't perfect like it was in the $N = \infty$ case. But for large N it's good enough.)
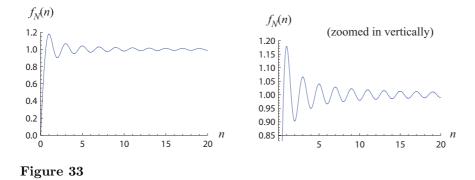
To make the integral in Eq. (104) easier to deal with, let's define n via $x \equiv n(L/4N)$. The parameter n need not be an integer, but if it is, then it labels the nth local maximum or minimum of $f_N(x)$, due to the reasoning we used in Fig. 31. The number n gives the number of half revolutions the vectors in Fig. 31 make as they wrap around in the plane. If n is an integer, then the sum of the horizontal projections (the cosines in Eq. (95)) is zero, and we therefore have a local maximum or minimum of $f_N(x)$.

With this definition of n, the upper limit of the integral in Eq. (104) is essentially equal to $n\pi$ (assuming N is large), so we obtain

$$f_N(n) \approx \frac{2A}{\pi} \int_0^{n\pi} \frac{\sin y}{y}\, dy \qquad \text{(where } x \equiv n(L/4N)\text{)}. \tag{105}$$

Again, n need not be an integer. What does this function of n look like? The integral has to be computed numerically, and we obtain the first plot shown in Fig. 33 (we have chosen $A = 1$). The second plot is a zoomed-in (vertically) version of the first one. Note that there is actually no N dependence in $f_N(n)$. It is simply a function of n. Therefore, the height of the nth bump is independent of N, assuming that N is reasonably large.[8] The N (and L) dependence comes in when we convert from n back to x.

---

[8]If N isn't large, then we can't make the approximation that the upper limit of the integral in Eq. (105) equals $n\pi$. From Eq. (104), the limit actually equals $n\pi(1 - 1/2N)$. But if N isn't large, then the integral approximation to the sum isn't so great anyway.

Figure 33

You can verify in Fig. 33 that the local maxima and minima occur at integer values of n. This curve has the same shape as the two curves in Fig. 30, with the only difference being the horizontal scale. Any actual step-function Fourier series such as the ones in Fig. 30 can be obtained from Fig. 33 by squashing or stretching it horizontally by the appropriate amount. This amount is determined by making the $n = 1$ maximum occur at $x = L/4N$, and then the $n = 2$ minimum occur at $x = 2L/4N$, and so on. The larger N is, the smaller these x values are, so the more squashed the wiggles are. If we increase N by a factor of, say, 10, then the wiggles in the Fourier-series plot get squashed by a factor of 10 horizontally (and are unchanged vertically). We can verify this in Fig. 30. If we look at, say, the fourth maximum, we see that in the $N = 10$ plot it occurs at about $x = 0.175$, and in the $N = 100$ plot it occurs at slightly less than $x = 0.02$ (it's hard to see exactly, but it's believable that it's about 0.0175).

The envelope

What is the envelope of each of the curves in Fig. 30? (We'll be concerned with just the asymptotic behavior, that is, not right near the discontinuity.) To answer this, we must first find the envelope of the $f_N(n)$ curve in Fig. 33. Fig. 34 shows a plot of $(\sin y)/y$. Up to a factor of $2A/\pi$, the function $f_N(n)$ equals the area under the $(\sin y)/y$ curve, out to $y = n\pi$. This area oscillates around its average value (which is $\pi/2$) due to the "up" and "down" bumps of $(\sin y)/y$.

Figure 34

A local maximum of $f_N(n)$ is achieved when $(\sin y)/y$ has just completed an "up" bump, and a local minimum is achieved when $(\sin y)/y$ has just completed a "down" bump. The amplitude of the oscillations of $f_N(n)$ is therefore $2A/\pi$ times half of the area of one of the bumps at $y = n\pi$. The area of a simple sine bump is 2, so half of the area of a bump of $(\sin y)/y$, in a region where y is approximately equal to $n\pi$, is $(1/2) \cdot 2/n\pi = 1/n\pi$. (We're assuming that n is reasonably large, which means that the value of $1/n\pi$ is approximately constant over the span of a bump.) So the amplitude of the oscillations of $f_N(n)$ is

$$\text{Amplitude} \; = \; \frac{2A}{\pi} \cdot \frac{1}{n\pi} = \frac{2A}{n\pi^2} \approx \frac{A}{5n} \; . \tag{106}$$

You can (roughly) verify this by sight in Fig. 33. Remember that n counts the number of maxima and minima, not just the maxima.

To find the envelope of the actual Fourier-series plots in Fig. 30, we need to convert from n back to x. Using $x \equiv n(L/4N) \Longrightarrow n \equiv 4Nx/L$, the amplitude becomes

$$\text{Amplitude} \; = \; \frac{2A}{(4Nx/L)\pi^2} = \frac{AL}{2\pi^2 Nx} \propto \frac{1}{Nx} \; . \tag{107}$$

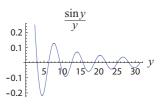So up to some constants, this is a $1/x$ function with a $1/N$ coefficient. The larger N is, the quicker it dies off to zero.

What if $f(x)$ isn't a square wave and instead has a nonzero slope emanating from the discontinuity? The reasoning above (where we wrote $f(x)$ as $f_{\text{step}}(x) + f_{\text{diff}}(x)$) tells us that we still have the same envelope of the form, $AL/2\pi^2 Nx$, with the only difference being that the envelope is now measured with respect to a line with nonzero slope.

3.7   Convergence

Fourier's theorem states that any sufficiently well-behaved function can be written as the series in Eq. (1), where the a_n and b_n coefficients are given by Eqs. (9,11,12). But what does this statement actually mean? Does it mean only that the *value* of the Fourier series at a given x equals the value of the function $f(x)$ at this x? Or does it mean that the two functions on either side of Eq. (1) are actually the exact same function of x? In other words, is it the case that not only the values agree, but all the *derivatives* do also?

It turns out that Fourier's theorem makes only the first of these claims – that the values are equal. It says nothing about the agreement of the derivatives. They might agree, or they might not, depending on the function. Let's give a concrete example to illustrate this.

Consider the periodic triangular function shown in Fig. 35. For ease of notation, we have chosen the period L to be 1. The function is defined by

$$f(x) = \left\{ \begin{array}{ll} x + 1/4 & (-1/2 < x < 0) \\ -x + 1/4 & (0 < x < 1/2). \end{array} \right. \tag{108}$$

The task of Problem [to be added] is to find the Fourier series for this function. The result is
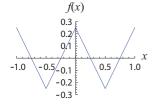
$$\begin{aligned} F(x) &= \frac{2}{\pi^2} \sum_{n \text{ odd}} \frac{\cos(2\pi nx)}{n^2} \\ &= \frac{2}{\pi^2} \left(\cos(2\pi x) + \frac{\cos(6\pi x)}{9} + \frac{\cos(10\pi x)}{25} + \cdots \right). \end{aligned} \tag{109}$$

We're using $F(x)$ to denote the Fourier series, to distinguish it from the actual function $f(x)$. Due to the n^2 factors in the denominators, the partial sums of $F(x)$ converge very quickly to the actual triangular function $f(x)$. This is evident from Fig. 36, which gives a very good approximation to $f(x)$ despite including only five terms in the partial sum.

What if we calculate the derivative of the Fourier series, $F'(x)$, and compare it with the derivative of the actual function, $f'(x)$? The slope of $f(x)$ is either 1 or -1, so $f'(x)$ is simply the step function shown in Fig. 37. The derivative of $F(x)$ is easily calculated by differentiating each term in the infinite sum. The result is

$$\begin{aligned} F'(x) &= -\frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(2\pi nx)}{n} \\ &= -\frac{4}{\pi} \left(\sin(2\pi x) + \frac{\sin(6\pi x)}{3} + \frac{\sin(10\pi x)}{5} + \cdots \right). \end{aligned} \tag{110}$$

This is the (negative of the) Fourier series we already calculated in Eq. (94), with $A = 1$ and $L = 1$. The plots of the partial sums of $F'(x)$ with 10 and 100 terms are shown in Fig. 38. In the infinite-sequence limit, $F'(x)$ does indeed equal the $f'(x)$ step function in Fig. 37. The Gibbs phenomenon discussed in the previous section might make you think otherwise, but the Gibbs overshoot is squeezed infinitely close to the discontinuity in the infinite-sequence limit. So for any given value of x that isn't at the discontinuity, $F'(x)$ approaches the correct value of ± 1 in the infinite-sequence limit.
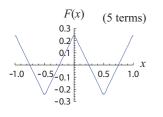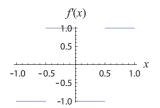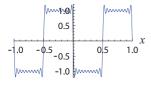

$f(x)$

Figure 35

$F(x)$ (5 terms)

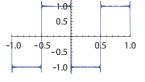**Figure 36**

$f'(x)$

Figure 37

$F'(x)$ (10 terms)

(100 terms)

Figure 38

What if we go a step further and calculate the second derivative of the Fourier series, $F''(x)$, and compare it with the second derivative of the actual function, $f''(x)$? The slope of $f'(x)$ is zero except at the discontinuities, where it is $\pm\infty$, so $f''(x)$ is shown in Fig. 39. The arrows indicate infinite values. These infinite value are actually delta functions (see Problem [to be added]). The derivative of $F'(x)$ is again easily calculated by differentiating each term in the infinite sum. The result is
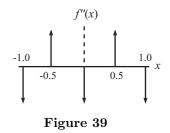
$$
\begin{aligned}
F''(x) &= -8 \sum_{n \text{ odd}} \cos(2\pi nx) \\
&= -8\Big(\cos(2\pi x) + \cos(6\pi x) + \cos(10\pi x) + \cdots \Big).
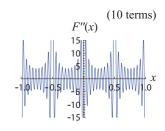\end{aligned} \tag{111}
$$

This can also be obtained by finding the Fourier series coefficients for $f''(x)$ via Eqs. (9,11,12) (see problem [to be added]). Plots of partial sums of $F''(x)$ with 10 and 40 terms are shown in Fig. 40. We now have a problem. These curves don't look anything like the plot of $f''(x)$ in Fig. 39. And if you plotted higher partial sums, you would find that the envelope is the same, with the only difference being the increase in the frequency of the oscillations. The curves definitely don't converge to $f''(x)$, a function that is zero everywhere except at the discontinuities.

Apparently, the Fourier series $F'(x)$ equals the function $f'(x)$ as far as the *value* is concerned, but not as far as the *derivative* is concerned. This statement might sound a little odd, in view of the fact that the second plot in Fig. 38 *seems* to indicate that except at the discontinuities, $F'(x)$ approaches a nice straight line with slope zero. But let's zoom in on a piece of this "line" and see what it looks like up close. Let's look at the middle of a step. The region between $x = 0.2$ and $x = 0.3$ is shown in Fig. 41 for partial sums with 10, 30, and 100 terms. The *value* of the function converges to -1, but slope doesn't converge to zero, because although the amplitude of the oscillations decreases, the frequency increases. So the slope ends up oscillating back and forth with the same amplitude. In short, the plot for the 100-term partial sum is simply an (essentially) scaled down version of the plot for the 10-term partial sum. This can be seen by looking at a further zoomed-in version of the 100-term partial sum, as shown in Fig. 42 (the additional zoom factor is 10 for both axes). This looks just like the plot for the 10-term partial sum. The proportions are the same, so the slope is the same.
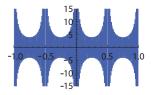
The general condition under which the *value* of the Fourier series $F(x)$ equals the *value* of the original function $f(x)$ (except at isolated discontinuities) is that $f(x)$ be square integrable. That is, the integral (over one period) of the square of $f(x)$ is finite. In the above example, $f(x)$ is square integrable, consistent with the fact that $F(x)$ agrees with $f(x)$. Additionally, $f'(x)$ is square integrable, consistent with the fact that $F'(x)$ agrees with $f'(x)$.
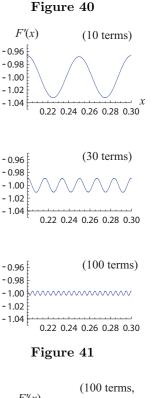
However, $f''(x)$ is *not* square integrable (because it contains delta functions), consistent with the fact that $F''(x)$ does *not* agree with $f''(x)$. A quick way to see why the square of a delta function isn't integrable (in other words, why the integral is infinite) is to consider the delta function to be a thin box with width a and height $1/a$, in the $a \to 0$ limit. The square of this function is a thin box with width a and height $1/a^2$. The area of this box is $a(1/a^2) = 1/a$, which diverges in the $a \to 0$ limit. So $(\delta(x))^2$ isn't integrable. Another even quicker way is to use the main property of a delta function, namely $\int \delta(x)f(x)\,dx = f(0)$. Letting $f(x)$ be a delta function here gives $\int \delta(x)\delta(x)\,dx = \delta(0) = \infty$. Hence, $(\delta(x))^2$ is not integrable.
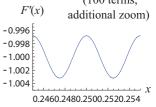
Figure 39

Figure 40

Figure 41

Figure 42

3.8   Relation between transforms and series

The exponential Fourier-series relations in Section 3.2 were

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i2\pi nx/L}} \qquad \text{where} \qquad \boxed{C_n = \frac{1}{L}\int_0^L f(x)e^{-i2\pi nx/L}dx} \qquad (112)$$

(We'll work with exponential series instead of trig series in this section.) And the Fourier-transform relations in Section 3.3 were

$$\boxed{f(x) = \int_{-\infty}^{\infty} C(k)e^{ikx}dk} \qquad \text{where} \qquad \boxed{C(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx} \qquad (113)$$

Eq. (112) says that any (reasonably well-behaved) periodic function can be written as a sum of exponentials (or sines and cosines).[9] But now that we've learned about Fourier transforms, what if we pretend that we don't know anything about Fourier *series* and instead simply try to calculate the Fourier *transform* of a periodic function using Eq. (113)? A periodic function can certainly be viewed as just a "normal" function that we might reasonably want to find the Fourier transform of, so something sensible should come out of this endeavor.
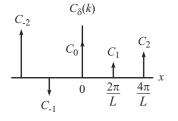
Whatever result we obtain from Eq. (113), it had better somehow reduce to Eq. (112). In particular, there had better not be any frequencies (k values) in the $C(k)$ Fourier-transform expression for $f(x)$ besides the *discrete* ones of the form $2\pi n/L$, because these are the only ones that appear in Eq. (112). So let's see what happens when we take the Fourier transform of a periodic function. Actually, let's first work backwards, starting with Eq. (112), and see what the transform has to be. Then we'll go in the "forward" direction and derive the Fourier series directly by calculating the Fourier transform.

3.8.1   From series to transform

Fourier transforms involve integrals over k, so if we work backwards from the discrete sum in Eq. (112), our goal is to somehow write this sum as an integral. This is where our knowledge of delta functions will help us out. We know that a delta function picks out a particular value of the integration variable, so if we have a sum of delta functions located at different places, then they will turn a continuous integral into a discrete sum. Therefore, assuming that we have a function $f(x)$ that can be written in the form of Eq. (112), we'll propose the following Fourier transform of $f(x)$:

$$\boxed{C_\delta(k) = \sum_{n=-\infty}^{\infty} C_n \delta(k - 2\pi n/L)} \qquad (114)$$

Figure 43

where the C_n's are given by Eq. (112). This $C_\delta(k)$ function is shown schematically in Fig. 43 (it's technically a distribution and not a function, but we won't worry about that). The arrows are the standard way to represent delta functions, with the height being the coefficient of the delta function, which is C_n here. The value of $C_\delta(k)$ at the delta functions is infinite, of course, so remember that the height of the arrow represents the area of the delta function (which is all that matters when doing an integral involving a delta function), and not the (infinite) value of the function.

---

[9]The route we followed in Sections 3.1 and 3.2 was to simply accept this as fact. However, we'll actually prove it from scratch in Section 3.8.3 below.

To determine if the $C_\delta(k)$ function in Eq. (114) is the correct Fourier transform of $f(x)$, we need to plug it into the first equation in Eq. (113) and see if it yields $f(x)$. We find

$$f(x) \overset{?}{=} \int_{-\infty}^{\infty} C_\delta(k)e^{ikx}\,dk = \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} C_n\delta(k - 2\pi n/L) \right) e^{ikx}\,dk. \qquad (115)$$

As k runs from $-\infty$ to ∞, the integrand is zero everywhere except at the delta functions, where k takes the form, $k = 2\pi n/L$. Using the $\int f(k)\delta(k - k_0)\,dk = f(k_0)$ property of the delta function, we see that the integration across each delta function simply yields the value of the rest of the integrand evaluated at $k = 2\pi n/L$. This value is $C_n e^{i2\pi nx/L}$. Therefore, since n runs from $-\infty$ to ∞, Eq. (115) becomes

$$f(x) \overset{?}{=} \sum_{n=-\infty}^{\infty} C_n e^{i2\pi nx/L}. \qquad (116)$$

And from the first equation in Eq. (112), the righthand side here does indeed equal $f(x)$, as desired. So the $C(k)$ in Eq. (114) does in fact make the first equation in Eq. (113) true, and is therefore the correct Fourier transform of $f(x)$.

3.8.2 From transform to series

Given a periodic function $f(x)$, let's now work in the "forward" direction and *derive* the expression for $C_\delta(k)$ in Eq. (114), which we just showed correctly leads to the series in Eq. (112). We'll derive this $C_\delta(k)$ from scratch by starting with the general Fourier-transform expression for $C(k)$ in Eq. (113). We'll pretend that we've never heard about the concept of a Fourier *series* for a periodic function.

Our goal is to show that if $f(x)$ is periodic with period L, then the $C(k)$ given in Eq. (113) reduces to the $C_\delta(k)$ given in Eq. (114), where the C_n's are given by Eq. (112). That is, we want to show that the $C(k)$ in Eq. (113) equals

$$\boxed{C(k) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{L} \int_0^L f(x)e^{-i2\pi nx/L}dx \right) \delta(k - 2\pi n/L)} \qquad (117)$$

To demonstrate this, let's break up the integral for $C(k)$ in Eq. (113) into intervals of length L. This gives

$$C(k) = \frac{1}{2\pi} \left(\cdots \int_{-L}^{0} f(x)e^{-ikx}dx + \int_0^L f(x)e^{-ikx}dx + \int_L^{2L} f(x)e^{-ikx}dx + \cdots \right). \qquad (118)$$

The $f(x)$'s in each of these integrals run through the same set of values, due to the periodicity. And the e^{-ikx} values simply shift by successive powers of e^{-ikL} in each integral. For example, if we define y by $x \equiv y + L$ and substitute this into the $\int_L^{2L}$ integral above, we obtain

$$\int_L^{2L} f(x)e^{-ikx}dx = \int_0^L f(y+L)e^{-ik(y+L)}dy = e^{-ikL} \int_0^L f(y)e^{-iky}dy, \qquad (119)$$

where we have used the $f(y + L) = f(y)$ periodicity. So the $\int_L^{2L}$ integral is simply e^{-ikL} times the $\int_0^L$ integral. Likewise, the $\int_{2L}^{3L}$ integral is e^{-2ikL} times the $\int_0^L$ integral. And so on. Therefore, if we factor the $\int_0^L$ integral out of each integral in Eq. (118), we obtain

$$C(k) = \frac{1}{2\pi} \left(\int_0^L f(x)e^{-ikx}dx \right) \left(\cdots e^{2ikL} + e^{ikL} + 1 + e^{-ikL} + e^{-2ikL} \cdots \right). \qquad (120)$$

We must now evaluate the sum,

$$S(k) \equiv \cdots e^{2ikL} + e^{ikL} + 1 + e^{-ikL} + e^{-2ikL} \cdots . \tag{121}$$

We'll do it quantitatively below, but first let's be qualitative to get an idea of what's going on. $S(k)$ is the sum of unit vectors in the complex plane that keep rotating around indefinitely. So they should average out to zero. The one exception occurs when $e^{ikL} = 1$, that is, when k takes the form of $k = 2\pi n/L$. In this case, all the terms in the (infinite) sum equal 1 (the unit vectors don't rotate at all), so $S(k)$ is infinite. This reasoning is basically correct, except that it doesn't get quantitative about the nature of the infinity (we'll find below that it's delta function). And it's also a little sloppy in the "averaging out to zero" part, because the sum doesn't converge, the way it stands. All the terms have magnitude 1, so it matters where you start and stop the sum.

   Let's now be quantitative. Following the strategy we used many times in Section 3.5, we can get a handle on $S(k)$ by multiplying the terms by successive powers of a number r that is slightly less than 1, starting with the $e^{\pm ikL}$ terms and then working outward in both directions. We'll then take the $r \to 1$ limit to recover the original sum. So our modified sum (call it $S_r(k)$) is

$$S_r(k) = \cdots + r^3 e^{3ikL} + r^2 e^{2ikL} + r e^{ikL} + 1 + r e^{-ikL} + r^2 e^{-2ikL} + r^3 e^{-3ikL} + \cdots . \tag{122}$$

Summing the two geometric series on either side of the "1" turns $S_r(k)$ into

$$S_r(k) = \frac{r e^{ikL}}{1 - r e^{ikL}} + 1 + \frac{r e^{-ikL}}{1 - r e^{-ikL}} . \tag{123}$$

Getting a common denominator and combining these terms yields

$$S_r(k) = \frac{1 - r^2}{1 + r^2 - 2r \cos(kL)} . \tag{124}$$

If $\cos(kL) \neq 1$, then $S_r(k)$ equals zero in the $r \to 1$ limit, because the denominator is nonzero in this limit. So as we saw qualitatively above, if $k \neq 2\pi n/L$, then $S(k) = 0$. However, things are tricker if $\cos(kL) = 1$, that is, if $k = 2\pi n/L$ for some integer n. In this case we obtain $S_r(k) = (1 - r^2)/(1 - r)^2 = (1 + r)/(1 - r)$, which goes to infinity in the $r \to 1$ limit. We can get a handle on this infinity as follows.

   Define κ by $k \equiv 2\pi n/L + \kappa$. We are concerned with very small κ values, because these values correspond to k being very close to $2\pi n/L$. In terms of κ, $S_r(k)$ becomes (using $\cos\theta \approx 1 - \theta^2/2$ to obtain the second line)

$$\begin{aligned} S_r(\kappa) &= \frac{1 - r^2}{1 + r^2 - 2r \cos(\kappa L)} \\ &\approx \frac{1 - r^2}{1 + r^2 - 2r(1 - \kappa^2 L^2/2)} \\ &= \frac{(1 + r)(1 - r)}{(1 - r)^2 + r\kappa^2 L^2} . \end{aligned} \tag{125}$$

If we now let $r \equiv 1 - \epsilon$ (so we're concerned with very small ϵ), then the $(1 + r)$ factor in the numerator is essentially equal to 2. So when we take the $\epsilon \to 0$ limit to obtain the original sum $S(k)$, we get

$$S(\kappa) = \lim_{\epsilon \to 0} S_\epsilon(\kappa) = \lim_{\epsilon \to 0} \frac{2\epsilon}{\epsilon^2 + \kappa^2 L^2} . \tag{126}$$

But from Eq. (83), this limit equals $2\pi\,\delta(\kappa L)$, which from the second equation in Eq. (75) equals $(2\pi/L)\delta(\kappa)$, which in turn equals $(2\pi/L)\delta(k - 2\pi n/L)$, from the definition of κ.[10] $S(k)$ diverges for any k of the form $2n\pi/L$, so we arrive at

$$S(k) = \sum_{n=-\infty}^{\infty} \frac{2\pi}{L}\,\delta(k - 2\pi n/L). \tag{127}$$

It's legal to just add up all the different delta functions, because each one doesn't affect any of the others (because they're zero everywhere except at the divergence). Plugging this $S(k)$ into Eq. (120) then gives

$$C(k) = \frac{1}{L}\left(\int_0^L f(x)e^{-ikx}dx\right)\sum_{n=-\infty}^{\infty}\delta(k - 2\pi n/L). \tag{128}$$

The delta functions mean that only k values of the form $2\pi n/L$ are relevant, so we can replace the k in the exponent with $2\pi n/L$ (after bringing the integral inside the sum). This gives

$$
\begin{aligned}
C(k) &= \sum_{n=-\infty}^{\infty}\left(\frac{1}{L}\int_0^L f(x)e^{-i2\pi nx/L}dx\right)\delta(k - 2\pi n/L), \\
&\equiv \sum_{n=-\infty}^{\infty} C_n\delta(k - 2\pi n/L),
\end{aligned}
\tag{129}
$$

where C_n is defined as in Eq. (112). We have therefore arrived at the desired expression for $C(k)$ in Eq. (117), or equivalently Eq. (114).

And as we saw in Section 3.8.1, if we plug this Fourier *transform*, $C(k)$, into the first equation in Eq. (113) to obtain $f(x)$, we end up with the righthand side of Eq. (116), which is the desired Fourier *series* of $f(x)$. We have therefore demonstrated how the continuous Fourier-transform expansion of a periodic function leads to the discrete Fourier series.

REMARKS:

1. The delta functions in Eq. (129) occur only at k values of the form $2\pi n/L$, so the above derivation explains why all the k values in the Fourier series in Eqs. (1) and (19) are multiples of $2\pi/L$. When we introduced Eq. (1) out of the blue, you may have wondered why no other frequencies were needed to obtain an arbitrary periodic function with period L.

   For example, in the Fourier-series formalism, it's not so obvious why a k value of, say, π/L isn't needed. But in the Fourier-transform formalism, it's clear that $C(\pi/L)$ equals zero, because the terms in the sum in Eq. (120) simply alternate between 1 and -1, and therefore add up to zero. (As usual, this can be made rigorous by tapering off the sum.) And similar reasoning holds for all k values not of the form $2\pi n/L$. Therefore, since non-$2\pi n/L$ values of k don't appear in the Fourier transform of $f(x)$, they don't appear in the Fourier series either.

2. For a periodic function, we found above that every value of $C(k)$ is either zero when $k \neq 2\pi n/L$, or infinite when $k = 2\pi n/L$. There is another fairly quick way to see this, as follows. First, consider a k value of the form, $k = 2\pi R/L$, where R is a rational number, a/b. Then the product $f(x)e^{-ikx}$ is periodic with period bL. If the integral of $f(x)e^{-ikx}$ over this period is *exactly* zero, then the integral from $-\infty$ to ∞ is also zero (again, with the integral tapered off). And if the integral over this period is *not* zero, then the integral from $-\infty$ to ∞ is

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[10]It's no surprise that we obtained the Lozentzian representation of the delta function in Eq. (126). Our "tapering" method involving powers of r is simply a discrete version of an exponential taper. And we know from Section 3.5.3 that an exponential taper leads to a Lorentzian delta function.

infinite, because we're adding up a given nonzero number, whatever it may be, an infinite number of times. Finally, since the rational numbers are dense on the real line, this result should hold for all values of k. So for all k, $C(k)$ is either zero or infinite. As it turns out, $C(k)$ is nearly always zero. ♣

3.8.3   Derivation of Fourier's theorem

We started this chapter by invoking Fourier's theorem. That is, we accepted the fact that any (reasonably well-behaved) periodic function can be written as a Fourier-series sum of sines and cosines, or exponentials; see Eqs. (1) and (19). We then used this fact to derive (by taking the $L \to \infty$ limit) the Fourier-transform relations in Eq. (43). It turns out, however, that armed with our knowledge of delta functions, we can actually derive all of these results from scratch. And the order of the derivation is reversed; we will first derive the Fourier-transform expression for an arbitrary (reasonably well-behaved) function, and then we will use this result to derive the Fourier-series expression for a periodic function. Let's be very systematic and derive things step-by-step:

1. The first step is to use the "tapering" method of Section 3.5 to derive the integral representation of the delta function given in Eq. (90), which we will copy here:

$$\delta(k - k_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(k-k_0)x} dx. \tag{130}$$

   When we derived this in Section 3.5, we formulated things in terms of Fourier transforms, but this language isn't necessary. Even if we had never heard of Fourier transforms, we could have pulled the integral $\int_{-\infty}^{\infty} e^{-ikx} dx$ out of a hat, and then used the tapering method to show that it is a delta function of k. That is, it has area 1 and is zero everywhere except at $k = 0$. We need not know anything about Fourier transforms to derive this fact.

2. Given an arbitrary (not necessarily periodic) function $f(x)$, let's define (out of the blue) a function $C(k)$ by

$$C(k) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)e^{-ikx} dx. \tag{131}$$

   The is a purely definitional statement and has no content. We haven't done anything with actual substance here.

3. Given this definition of $C(k)$, we will now use the expression for the delta function in Eq. (130) to show that

$$f(x) = \int_{-\infty}^{\infty} C(k)e^{ikx} dk. \tag{132}$$

   This is a statement with actual substance. To prove it, we first need to plug the above definition of $C(k)$ into the righthand side to obtain

$$f(x) \stackrel{?}{=} \int_{-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} f(x')e^{-ikx'} dx' \right) e^{ikx} dk, \tag{133}$$

   where we have been careful to label the dummy integration variable from Eq. (131) as x'. Switching the order of integration and rearranging things gives

$$f(x) \stackrel{?}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x') \left(\int_{-\infty}^{\infty} e^{-ik(x'-x)} dk \right) dx'. \tag{134}$$

From Eq. (130), the quantity in parentheses is simply $2\pi\delta(x'-x)$, with different letters. So we obtain

$$f(x) \stackrel{?}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x')\delta(x' - x)\, dx'. \tag{135}$$

And this is indeed true, because the righthand side equals $f(x)$, due to the properties of the delta function. We have therefore successfully derived the Fourier-transform relations in Eq. (43). More precisely, we wrote one of them, Eq. (131), down by definition. And we then derived the other one, Eq. (132).

We should mention again that whenever we deal with integrals of the type $\int_{-\infty}^{\infty} e^{-ikx}dx$, it is understood that we are technically using some sort of tapering method to make sense of them. But we know that the result will always be a delta function, so we don't actually have to go through the tapering procedure each time.

4. Having derived the Fourier-transform relations in Eq. (43) for arbitrary functions, our task is now to derive the Fourier-series relations for periodic functions, Eqs. (19) and (20), which we will copy here:

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i2\pi nx/L}, \tag{136}$$

where the C_n's are given by

$$C_n = \frac{1}{L} \int_0^L f(x)e^{-i2\pi nx/L}dx. \tag{137}$$

(Alternatively, we could use the trig series given in Eq. (3.1).) But we already performed this derivation in Section 3.8.2, so we won't go through it again. The main point is that the periodicity of $f(x)$ produces a collection of delta functions of k, which turns the continuous-integral Fourier-transform expression for $f(x)$ in Eq. (132) into the discrete-sum Fourier-series expression for $f(x)$ in Eq. (136).