Preface

From *Algebra: For the Enthusiastic Beginner* (Draft version, July 2024)
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This book is divided into two volumes. The Preface covers both volumes.

Algebra is the golden key to nearly every field in math, as well as numerous fields in science and beyond. Without it, many subjects are inaccessible. With it, countless doors are opened. It's like being able to dribble if you want to play basketball, or knowing grammar if you want to be a writer. In the short term, you'll find that many subjects in school rely on algebra. In the long term, the same goes for a wide variety of careers that you may eventually want to pursue, whether they involve math, science, engineering, computer science, statistics, economics, finance, and so on. Pretty much every quantitative field uses algebra!

Algebra, loosely defined, involves working with letters, instead of only numbers. One particular aspect of algebra involves solving equations for x (or whatever letter you want to solve for). But there are plenty of other aspects too, such as working with functions and formulas, and proving general theorems. We'll encounter many examples of these and other uses of algebra.

You'll get a lot of practice with solving problems throughout this book. There's really no way to separate problem solving from a subject itself, since the only way to truly learn something is to solve problems on it. Therefore, pervading the whole book is an overarching theme of problem solving.

This is the kind of book where everything is built up from scratch. You won't be asked to just accept things. You'll see where they come from. Therefore, because of all the time spent on the logical development of the material, *this book should not be thought of as a quick reference*. That is not its purpose. Its purpose is to show you the "why" behind things. The concepts are built up in a coherent way, so if you skip around and just read things here and there, you'll miss out on the logical flow.

Two volumes

This book is divided into two volumes, for no reason other than the practical consideration that a single volume would be too much of an unwieldy brick. However, the two volumes are meant to be thought of as a single book. The page numbers and chapter numbers run straight through, which means that the second volume begins with Chapter 7 on page 425. And there are many cross-references back and forth between the two volumes.

The Table of Contents gives the breakdown between the two volumes. The first volume ends (appendices aside) with Chapter 6 (*Coordinate systems and lines*), and the second volume begins with Chapter 7 (*General functions*). This perhaps isn't the most ideal place for the break, but it was chosen so that the volumes have roughly the same length. There are two appendices at the end of the first volume, and one at the end of the second. The following four items are repeated in the two volumes: (1) Table of Contents, (2) Preface, (3) Glossary of Notation, and (4) Index. All four of these cover both volumes.

Algebra is often divided into Algebra 1 and Algebra 2, although the line between these two categories is a bit fuzzy. This book contains both Algebra 1 and Algebra 2, along with some more advanced topics. However, the division into the two volumes doesn't correspond cleanly to the division between Algebra 1 and Algebra 2. The first volume is mainly Algebra 1, and the second volume is mainly Algebra 2, but this distinction is largely a coincidence and not exact. Additionally, the later chapters in Volume 2 give a gentle introduction to a few advanced topics, which are most certainly *not* part of Algebra 2.

Volume 1 contains a number of cross-references to items in Volume 2. The purpose of these is usually to mention topics that we'll eventually cover. These topics aren't necessary for an understanding of the material in Volume 1. So if you're reading Volume 1, there's no need to have Volume 2 on hand.

However, if you're reading Volume 2, then you *will* want to have Volume 1 on hand, because the cross-references back to Volume 1 generally involve topics that you need to know for Volume 2. It is therefore strongly recommended that you obtain at least the Kindle version (best viewed on your computer with the Kindle app) of Volume 1, which is very inexpensive.

Prerequisites

What are the prerequisites for this algebra book? (By "book" we'll always mean both volumes together.) That is, what do you need to know beforehand? Well, multiplying numbers is one thing; you need to know your multiplication tables. Or more precisely, you don't actually *have* to know them, but if you don't, many steps throughout the book will take an annoyingly long time. So it's good to have the products of numbers at your fingertips. (At least small numbers. For large ones you can just use a calculator.) Some familiarity with fractions (adding, multiplying, simplifying, converting to decimals) will also be helpful, although we'll cover fractions in Chapter 1. Likewise for exponents, which we'll cover in Chapter 2.

The sections on fractions and exponents start from scratch and are self-contained, so even if you haven't seen these things before, you could theoretically learn them here. However, if this applies to you, it might make sense to pause after working through the first two chapters, to get a little more practice on those topics with another book, so that you're *really* comfortable with them. Then come back to this book and pick it up with Chapter 3. This is where we start working in full force with letters. Most of the basics of algebra are presented in Chapters 3 and 4, so these chapters lay the foundation for the rest of the book.

Having mentioned the above prerequisites, the most important one actually isn't a matter of background and preparation, but rather how you approach the book. The critical requirement for successful use of this book is a desire to *read it slowly, pause often and ponder things, and give all of the exercises your best effort.* That's the "Enthusiastic Beginner" part of the title!

Contents

The chapters can be grouped as follows, often coming in pairs:

CHAPTERS 1 AND 2: These chapters provide the background you'll need for algebra. So they could well be described by the term "pre-algebra." Chapter 1 (*Operations, fractions*) covers the four basic operations (addition, subtraction, multiplication, division), negative numbers, the order of operations, parentheses, fractions, and a few other topics. Chapter 2 (*Laws of algebra, exponents*) begins with the three basic laws of arithmetic and algebra (commutative, associative, distributive), and then moves on to exponents and prime factorization, with applications to the least common multiple (LCM) and greatest common factor (GCF).

CHAPTERS 3 AND 4: This is where we dive into algebra and learn how to work with letters. Chapter 3 (*Expansions, FOIL*) introduces the all-important FOIL method, which is used when multiplying certain combinations of letters. We'll generalize this method in subsequent sections and eventually arrive at the binomial expansion. Chapter 4 (*Factoring, applications*) continues onward to factoring, and then presents a large number of problems for you to practice on.

CHAPTER 5: In this chapter (*Solving for x*) you'll learn how to solve equations for a variable, often called x. This is a topic that often comes to mind when people think of algebra. It is indeed a critical aspect of algebra, and you'll get plenty of practice with it. We'll deal with increasingly complicated equations and work our way up to the important task of translating word problems into mathematical equations. We'll also discuss systems of equations (ones involving more than one variable), along with some simple quadratic equations. We'll deal with more general quadratic equations in Chapter 8.

CHAPTERS 6 AND 7: These two chapters cover functions. Chapter 6 (*Coordinate systems and lines*) deals with the simplest kind of function – a straight line. We'll start off with a discussion of coordinate systems and how to plot points. We'll then investigate many different properties of lines. In Chapter 7 (*General functions*) we'll move on to general functions, looking closely at quadratic ones (involving x^2). We'll discuss a number of properties of general functions, and then finish up with inverse functions.

Note: Chapter 6 is the last chapter in Volume 1 (appendices aside), and Chapter 7 is the first chapter in Volume 2.

CHAPTER 8: In this chapter (*Quadratic equations*) we'll introduce the technique of "completing the square." We'll use this to derive the quadratic formula, which provides the general solution to any quadratic equation. We'll apply the formula to many problems, including the task of finding the maximum or minimum of a function. This application makes use of the discriminant of the quadratic formula. The discriminant then leads us into the realm of imaginary numbers.

CHAPTERS 9 AND 10: These chapters introduce some more advanced (but fairly standard) topics. Chapter 9 (*Logs and exponentials*) covers the log function and its properties, and we'll see how it relates to the exponential function. We'll discuss Euler's number e, which is one of the most important numbers in mathematics. This leads to the nice application of compound interest. Chapter 10 (*Means, sequences, and series*) begins with the arithmetic and geometric means, and then relates them with the Arithmetic-Mean/Geometric-Mean (AM-GM) inequality. We'll then extend the concept of means to sequences, which lead to series (a series is the sum of a sequence). We'll derive formulas for the various types of series.

CHAPTERS 11 AND 12: These chapters cover a number of even more advanced topics, which are *not* part of the standard algebra curriculum. These chapters are optional, so you can consider them to be appendices if you want. Chapter 11 (*Miscellaneous topics*) covers an assortment of topics, including proof by induction, general sum formulas, areas and volumes, and a useful approximation. Chapter 12 (*Slopes of general functions*) examines slopes of general functions and presents two methods for finding them. This topic normally appears in a calculus book, but since by this point we've developed everything we need in order to understand general slopes, it would be a shame not to take advantage of the opportunity!

APPENDICES: Appendices A and B are located at the end of Volume 1, and Appendix C is at the end of Volume 2. Appendix A (*Geometry tools*) introduces a number of geometry results that come in handy at various points throughout the book. Topics include properties of triangles (in particular similar triangles), areas, volumes, and the Pythagorean theorem. Appendix B (*Benefits of using letters*) discusses the many advantages of using letters (that is, doing algebra) instead of only numbers (doing arithmetic). Appendix C (*Binomial coefficients*) goes into more detail on the binomial expansion, which we encountered in Chapter 3. A very interesting byproduct of the binomial expansion is the infinite series for Euler's number e. Following the appendices, a Glossary of Notation is located at the end of each volume.

Problem solving

This book contains roughly 100 examples and 350 exercises. The "examples" are problems whose solutions are included in the text. The "exercises" are problems whose solutions are set aside at the end of each chapter, so that you can solve them on your own. Examples have wide lines separating them from the main text. Exercises have double thin lines.

It is absolutely critical that you spend time solving the exercises. Even if the text and examples make complete sense, it is still important to engage in problem solving, because that's the only way you're going to find out if you *actually* understand something. We've all had times when we've nodded our head while listening to directions or an explanation, thinking we understood. But then when we were asked to actively *do* something, we realized we didn't know as much as we thought.

How many of the exercises should you do? The answer is: *All* of them. If you think you understand things very well and you'll just be bored with a given exercise, then it won't take that long to solve it, so not much is lost. If, on the other hand, it takes you a while to figure it out (and there's nothing wrong with that!), then that means you really did need to do it. So don't rush through the book. Take it slow and ponder the examples and exercises until they thoroughly sink in. Reading gives you a baseline of knowledge, but to get past that and reach the level where you can actually *use* your knowledge, you have to engage in problem solving.

> Reading teaches you *how* to do something. Problem solving teaches you how to *do* something.

Some topics might come quickly to you, others not so much. Struggling with problems is a critical part of learning. Just because "No pain, no gain" is a cliche, that doesn't mean it's not true! This saying is often quite relevant when learning a new subject, so be careful not to look at a solution to an exercise too soon. There's nothing wrong with putting a problem aside for a while and coming back to it later. Indeed, that is probably the best way to learn. If you head to the solution at the first sign of not being able to solve a problem, then you have wasted it. If you do need to look at a solution, cover it up and read one word at a time. When you reach a point where you've gotten a hint, set the solution aside and try it again on your own. For the exercises you're able to completely solve on your own, be sure to still take a look at the solution I've given at the end of the chapter, because there are often multiple solutions and extra comments included.

A problem-solving fact that often gets overlooked is that you need to know more than the correct way(s) to do a problem; you also need to be familiar with many *incorrect* ways of doing it. Otherwise, when you come upon a new problem, there may be a number of decent-looking approaches to take, and you won't be able to immediately weed out the poor ones. Struggling a bit with a problem invariably leads you down some wrong paths, and this is an essential part of learning. To understand something, you not only have to know what's right about the right things; you also have to know what's wrong about the wrong things. Learning takes a serious amount of effort, many wrong turns, and a lot of sweat. Alas, there are no shortcuts.

Theorems and general results are sometimes included in the exercises so that you can have the satisfaction and fun of deriving various classic results on your own. I don't want to derive them solely in the text and steal all the fun! In the same regard, by all means feel free to treat some of the examples as exercises and solve them on your own. Since the only real difference between an example and an exercise is the location of the solution (the exercise solutions are located far enough away at the end of each chapter so that you don't peek at them), all you need to do to turn an example into an exercise is cover up the solution and try to solve it yourself. If you initially don't think you can solve a particular example, but then after a sentence or two of the solution you say, "Wait, I can do this!" then please do!

There are no unsolved problems in the book. Every problem (example or exercise) has an included solution, either adjacent or at the end of the chapter. Consistent with this, the book is intended more for self-study than as the sole textbook for a class, since there isn't a supply of unsolved problems that can be used for homework assignments. However, the book can work well in a class if used in tandem with another resource.

Limericks

There are many limericks scattered throughout the book. Their purpose is to lighten things up and also perhaps to provide an educational benefit by making things stick in your brain a little better. Limericks are short poems that have five lines with a rhyming scheme of AABBA. That is, the 1st, 2nd, and 5th lines rhyme, as do the 3rd and 4th. The rhythm consists of triplets, which means that an accented syllable is followed by two unaccented syllables. So if you repeatedly count 1, 2, 3, 1, 2, 3, 1, 2, 3, while accenting the 1's, that's the rhythm.

The accented syllables (the 1's) are called "downbeats" in music terminology. As an example of a triplet rhythm, if you listen to Billy Joel's song "Piano Man," you'll find yourself counting along with a 1, 2, 3, 1, 2, 3 rhythm. In contrast, the majority of songs are written with a 1, 2, 3, 4, 1, 2, 3, 4 rhythm. A classic example of this is The Beatles' "Hey Jude," where you'll find yourself counting to 4 over and over again. But limericks have a triplet rhythm, so you'll never get to 4.

The 1st, 2nd, and 5th lines of a limerick have three downbeat syllables, along with a silent downbeat (a pause) at the end. So including the silent one, there are four downbeats in each of these three lines. The 3rd and 4th lines have two downbeats each. There are often some "pickup" (unaccented) syllables at the beginnings of lines, before the first downbeat (an accented 1).

This is all best understood with an example. In the limerick below, I've written the downbeat syllables (the accented 1's) in bold, and I've indicated the pause (and additional counting) at the end of the 1st and 2nd lines. The bold "1" is the downbeat of the pause.

3 1 2 3 1 2 3 1 2 3 **1** 2
When **read**ing a **lim**'rick, you'll **see** *(pause)*
3 1 2 3 1 2 3 1 2 3 **1** 2
You **don't** need a **fan**cy de**gree**. *(pause)*
3 1 2 3 1
It's **trip**lets ga**lore** –
2 3 1 2 3 1
You won't **ev**en hit **4**,
2 3 1 2 3 1 2 3 1
Since you're **al**ways just **count**ing to **3**!

The apostrophe in "lim'rick" is a slight cheat here. But without it there would be too many syllables between "lim" and "see," which would be a definite no-no. The triplet rhythm should never be violated in limericks. So "cheat" if you must when writing one.

Note that the first three lines in the above limerick all have one pickup (unaccented) syllable at the beginning, namely "When," "You," and "It's." The last two lines have two pickup syllables, namely "You won't" and "Since you're."

It is permissible to have additional unaccented syllables at the ends of lines, as well as the beginnings, as long as they don't mess up the rhythm. In the following limerick, there is only one "unused" syllable (the downbeat of the pause) at the end of the 1st and 2nd lines.

3 1 2 3 1 2 3 1 2 3 **1**
Is **al**gebra **in**compre**hens**ible? *(pause)*
2 3 1 2 3 1 2 3 1 2 3 **1**
Not at **all**, if your **strat**egy's **sens**ible: *(pause)*
2 3 1 2 3 1 2
Take it **slow**, don't go **fast**er,
3 1 2 3 1 2
And **soon** you will **mast**er
3 1 2 3 1 2 3 1 2 3
A **sub**ject that's **quite** indi**spens**able!

Of course, if you're going to add extra unaccented syllables at the ends of lines, you need to make sure they rhyme. In the above limerick, the 1st, 2nd, and 5th lines all end with an ensible-sounding rhyme.

On a first reading of a limerick, it might not be obvious where a certain downbeat is. But usually it's fairly clear. For example, in the first limerick above, if you started the 2nd line with the downbeat on the "You," the next downbeat (three syllables later) would be on the "a," and then the next one on the "de," which would sound awfully strange (as you can verify!). So you'd try again with the downbeat on the "don't," and you'd find that it sounds *much* better.

Personally, I don't like to have any pauses between the 3rd, 4th, and 5th lines, although this isn't a strict requirement of limericks. For example, in the first limerick above, the "Since" at the start of the 5th line could be missing. However, limericks are easier to read if there are no pauses in the 3rd-to-4th-to-5th lines, because then you don't have to guess where the downbeat is. In the first limerick, as long as you put a downbeat on the "trip" part of "triplets" in the 3rd line, everything from there on is guaranteed to be correct, provided that there are no pauses. You're basically forced to read the limerick correctly if every number in the 1, 2, 3, 1, 2, 3, . . . chain in the last three lines corresponds to a syllable, and not a pause.

All in all, there are 116 limericks in this book (including the two above). I hope you enjoy them!

Odds and ends

In the first four chapters, I'll mainly use the letters a, b, etc., instead of x, y, etc. The reason for this is a purely personal opinion (and quite trivial): I think that the rounded a and b letters look less harsh and scary than x and y. That said, when solving for variables (in Chapter 5, for example), I'll generally use x and y. This is again a personal opinion: To me, x and y often represent variables that you're trying to solve for, while a and b represent quantities that appear in an expression, which you can make be whatever you want. When working with functions, x is usually the letter of choice.

When rounding numbers, I'll generally use an equals sign "=" even though it should actually be an approximate sign "≈." So I'll write things like $3/7 = 0.43$, when in fact it should be $3/7 \approx 0.43$. Other conventions are stated in Chapter 1 and in the Glossary of Notation.

Throughout the book, I'll bounce back and forth between "you" and "we" in statements like "You can plug x into this equation," or "We can divide by a to obtain. . . ." Technically "we" means both you (the reader) and me (the writer), while "you" means only you (the reader). But the usages are pretty much equivalent, and the two words are generally interchangeable (*someone* is dividing by the letter a; it really doesn't matter who). There isn't much rhyme or reason to which word I'll use on a given occasion. Sometimes I'll also use "we" to mean "I," as in "In the next section we'll show how to. . . ." This usage probably originated in books or papers with multiple authors.

There are many "Remarks" scattered throughout the text. These begin with "Remark" and end with a shamrock (♣). Their purpose is to say something that is informative but slightly tangential, without disrupting the overall flow of the discussion. In some sense these are "extra" thoughts, although they are invariably helpful in understanding the material.

This book grew out of many enjoyable math lessons with Jack Vance, who has kept me on my toes and inspired me to put together this material for other enthusiastic learners. I am also grateful to the following friends and colleagues for their input: Jacob Barandes, Jade Chin, Dionne Clarke, Anna Klales, Theresa Morin Hall, Alexia Schulz, Bill Schulz, Harrison Tinger, and Mariah Tinger.

Despite careful editing, there is essentially zero probability that this book is error free. If anything looks amiss, please check www.davidmorin.physics.fas.harvard.edu for typos, updates, additional material, etc. And please let me know if you discover something that isn't already posted. Suggestions are always welcome.

Cover references

If you're curious about all the references to algebra in the wonderful cover illustration (front, back, and spine) by Maki Naro, here they are!

- Cake: The geometric representation of $(a + b)^3$. See Fig. 3.11.
- Picnic blanket: Completing the square. See Fig. 8.1.
- $x = y$: Algebra often involves solving for x. See Chapter 5. In the reverse order, $y = x$ is the equation for a line. See Section 6.4.
- Bicyclists and bird: A classic algebra problem. See Exercise 10.40.
- Trains: Another classic algebra problem. See Section 5.5.2.
- Key leaning against tree: Algebra is the golden key to many subjects!
- Square roots of 1's: See Exercise 8.20.
- Kite: A square containing four rectangles can be used to prove the Arithmetic-Mean/Geometric-Mean inequality. See Exercise 10.9.
- Fibonacci numbers: See Exercise 10.31.

Summary of important points

The following points were all mentioned above, but it's good to emphasize them again:

1. THIS BOOK IS NOT A QUICK REFERENCE. It is meant to be read *slowly*. It isn't for learning quickly, but rather for learning thoroughly. If your goal is just to look up a particular result, other books will surely serve this purpose better. But if your goal is to understand where a certain result comes from, you're in the right place. This book is designed to be read straight through. If you skip around, you'll miss out on the logical flow. So take a seat and settle in. Have a good wall to stare at as you ponder, and have a pencil in hand ready for action.

2. THIS BOOK CAN BE WORDY AT TIMES. Many of the explanations could be said more concisely, but I wanted to err on the side of belaboring the point and saying everything that someone might say if they were explaining things to you in person. This includes discussing wrong turns, variations, pitfalls, generalizations, and so on. While these technically might not be needed for the specific goal at hand, they are immensely helpful in the long run.

3. THERE ARE THREE BASIC STEPS TO MASTERING MATERIAL:

   (a) *Read*: By reading, you become familiar with the material and get introduced to the important concepts.

   (b) *Think*: You need to pause after each sentence or paragraph to make sure you actually understand what you've read. Reading a math book isn't like reading a novel!

   (c) *Do*: Solving problems is the only way to make sure you truly master the material.

   All three of these steps are necessary. This is an instance where 2 out of 3 actually *is* bad!

4. DO ALL OF THE EXERCISES. It is critical that you make a good effort on all of the exercises. Some sections have lots of them (Sections 3.3 and 8.3, for example), although others might have only one. If you need to look at the solution to an exercise, look only at enough of it to get a hint, and then set the solution aside and finish it yourself. (Repeat these two hint/solve steps as necessary.) After you're done, be sure to look at the entire solution I've given, because there are often additional remarks and alternative solutions.

5. DO PROBLEMS IN ANOTHER BOOK AS WELL. Although this book contains many exercises (more than 350 of them), it's always good to do more. So ideally you'll have another book on hand that contains problems (with solutions, or at least answers). The more you do, the better.

6. THIS IS A SELF-STUDY OR COMPANION BOOK. Since every problem in the book has an included solution, there isn't a supply of unsolved problems that can be used for homework assignments. This book therefore isn't intended to be the sole textbook for a class. But it can work well in tandem with another book. And it is perfect for self-study, given all the solved problems.

7. THIS IS NOT A MATH COMPETITION BOOK. Instead, it is a book for anyone who wants to (1) learn algebra, and (2) put in the necessary time to proceed slowly and carefully through the material. The main goal is to teach the foundations of algebra (although there are certainly a few extra nuggets thrown in!). This isn't a book for math "hotshots," although it's quite possible you'll feel like a hotshot by the end!