Chapter 8

Quadratic equations

From *Algebra: For the Enthusiastic Beginner* (Draft version, July 2024)

In Section 5.7 we solved some quadratic equations in cases where the numbers were chosen nicely, so that it was easy enough to factor the polynomials by trial and error. In this chapter we'll learn how to solve quadratic equations with a more general method (the quadratic formula), which works even if the numbers are messy.

The quadratic formula is based on a technique called *completing the square*. We'll introduce this in Section 8.1 and use it to solve some equations, and then in Section 8.2 we'll show how it helps in making plots. In Section 8.3 we'll use the technique to derive the *quadratic formula*, which we'll then apply to many examples and exercises. A critical part of the quadratic formula is the *discriminant*, which we'll discuss in Section 8.4, where we'll learn how it can be used to find the maximum or minimum of a function. The discriminant leads us into the realm of *imaginary numbers* and *complex numbers*, which are the subjects of Section 8.5.

8.1 Completing the square

The quadratic equations we solved in Section 5.7 were ones where we could guess what the factoring was. As you recall from Section 4.2.2, factoring into binomials involves finding two numbers whose product and sum (or perhaps a sum of the form $x + 2y$, etc.) take on particular values.

But what about equations where the numbers aren't chosen nicely, and guessing won't get the job done? For these cases, we need to use the quadratic formula. And to understand where this formula comes from, we first need to learn the technique of *completing the square*. In the end, the quadratic formula is simply a general expression (that is, one involving letters) for the result you get when you complete the square. So let's see what this technique entails. We'll introduce it by looking at four quadratic equations, each one slightly more involved than the previous. The coefficients in these equations will involve numbers. We'll apply the technique to letters in Section 8.3.

1. Consider the equation,

$$x^2 = 4. \tag{8.1}$$

How can we solve this for x? Well, we need to undo the squaring operation, so we should take the square root of both sides of the equation. This gives

$$x = \pm\sqrt{4} \quad \Longrightarrow \quad x = \pm 2. \tag{8.2}$$

Note the "±" sign. Both −2 and 2 are solutions, so it would be incorrect to say only that $x = 2$.

> Always remember to include the "±" when taking a square root! Forgetting it is a common error.

2. Here's another equation:

$$(x-3)^2 = 4. \tag{8.3}$$

How can we solve this one? What you *don't* want to do is expand the square on the lefthand side to obtain $x^2 - 6x + 9 = 4$. Actually, for this problem this wouldn't be a terrible idea, because subtracting 4 from both sides would leave you with the polynomial $x^2 - 6x + 5$, which can be factored quickly into $(x-5)(x-1)$, yielding the solutions of 5 and 1. But in general, expanding the square isn't what you want to do, because unless the numbers are chosen nicely, you won't be able to factor the polynomial by trial and error.

What you *do* want to do with Eq. (8.3) is again take the square root of both sides, as we did in Eq. (8.2). This gives

$$x - 3 = \pm\sqrt{4} \quad \Longrightarrow \quad x - 3 = \pm 2 \quad \Longrightarrow \quad x = 3 \pm 2 \quad \Longrightarrow \quad x = 5 \text{ or } 1. \tag{8.4}$$

The only difference between this equation and Eq. (8.2) is that after taking the square root, we're now left with $x - 3$ on the lefthand side, instead of just x. But that's not much of an issue. We simply need to add 3 to both sides to isolate (solve for) x. And then we need to evaluate the two answers stemming from the "±" sign.

Again, don't forget the "±" sign! A quadratic equation has two solutions in general, and you will obtain only one solution if you forget the "±." However, in the special case where the righthand side of Eq. (8.3) is zero, there is in fact only one solution, because you end up with $x = 3 \pm 0$.

3. Here's a third equation:

$$x^2 - 6x + 9 = 4. \tag{8.5}$$

How do we solve this? As mentioned above, you could subtract 4 from both sides and then factor the polynomial. But that method works here only because the numbers were chosen nicely. The better strategy is to note that the lefthand side is a perfect square. That is, it is the square of the binomial $x - 3$. So we can rewrite the equation as $(x-3)^2 = 4$, which we already solved above.

In short, you *don't* want to expand a square if you're given one, but you *do* want to write an expression as a square if you can. That is, you want to "un-expand" it. You can then simply take the square root and follow the steps in Eq. (8.4).

4. Here's our fourth and final equation, one that illustrates the "completing the square" technique we've been building up to:

$$x^2 - 6x + 5 = 0. \tag{8.6}$$

How can we solve this? Again, we could quickly factor it into $(x-5)(x-1) = 0$, but let's pretend that the numbers weren't chosen so nicely.

We solved all of the above cases by using the fact that the lefthand side was a square. But the lefthand side of Eq. (8.6) *isn't* a square. However, we can *make* it be a square by turning the 5 into a 9. Of course, we can't just arbitrarily erase the 5 and write down a 9. But what we *can* do is add 4 to both sides. This turns the 5 into a 9 (and also puts a 4 on the righthand side). So we have

$$(x^2 - 6x + 5) + 4 = 4 \quad\Longrightarrow\quad x^2 - 6x + 9 = 4 \quad\Longrightarrow\quad (x-3)^2 = 4, \tag{8.7}$$

which we already solved above. Adding 4 to both sides here is just another example of performing the same (useful) operation on both sides of an equation, as we did all throughout Chapter 5.

The process of adding 4 to both sides in Eq. (8.7) is called "completing the square." The $x^2 - 6x$ terms on the lefthand side are part of a square, and the missing piece is a +9. So by creating a +9 (by adding 4 to both sides) to yield $x^2 - 6x + 9$, we completed the square. This process could also reasonably be called "forming the square," or "creating the square," or something like that.

> "Completing the square" means modifying the constant term (by performing the same operation on both sides of the equation), so that when it is added to the x^2 and x terms, the result is a square.

As we mentioned a number of times in Chapter 5, there is an infinite number of common operations we can perform on both sides of any given equation. But nearly all of them are useless. For example, we could add 17 to both sides of Eq. (8.6) to obtain $x^2 - 6x + 22 = 17$. This is an entirely true equation, but it doesn't help us at all. Adding 4 to both sides is the special operation that gives us something useful on the lefthand side, namely a perfect square.

The general form of the square of a binomial (assuming the coefficient of x^2 is 1) is $(x+m)^2 = x^2 + (2m)x + m^2$; the above equations had $m = -3$. Therefore,

> When completing the square (assuming the coefficient of x^2 is 1), the constant term (the m^2) is always obtained by taking half of the coefficient of x (half of $2m$ is m) and squaring it.

For example, if a quadratic equation involves the terms $x^2 + 10x$, we want to take half of 10 to obtain 5, and then square 5 to obtain 25. The desired square is then $x^2 + 10x + 25$, which is $(x + 5)^2$. If instead we have the terms $x^2 - 14x$, then since half of -14 is -7, and since $(-7)^2 = 49$, the desired square is $x^2 - 14x + 49$, which is $(x - 7)^2$. Note that the sign of the middle $(2m)x$ term (the $10x$ or $-14x$ here) doesn't affect the m^2 value (the 25 or 49), because the minus sign goes away when we square it. The m^2 term is always positive. So for both $x^2 - 14x$ and $x^2 + 14x$, the desired square is $x^2 - 14x + 49$.

Geometrically, completing the square corresponds to... well, completing the square! The above case of $x^2 + 10x$ is shown in Fig. 8.1. The total shaded area (light and dark) is $x^2 + 10x$. And then when we add on the missing unshaded square with area 5^2, we obtain the area of the entire square (with side length $x + 5$), which is $(x + 5)^2 = x^2 + 10x + 25$.

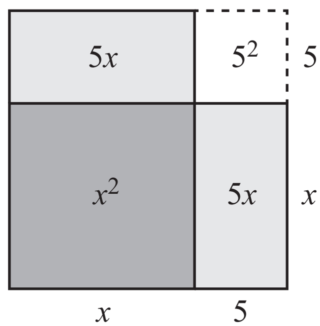

Figure 8.1: Completing the square geometrically, which might look familiar from the front cover!

If the coefficients of a quadratic equation happen to be chosen nicely, there's certainly nothing wrong with factoring it by trial and error. But the point here is that completing the square is a no-fail systematic method that doesn't involve any guessing. You're guaranteed to be successful (algebra mistakes aside!).

Example 8.1 Armed with our completing-the-square technique, let's solve this equation:

$$x^2 - 12x - 5 = 0. \tag{8.8}$$

Solution The numbers in this equation were purposely chosen so that there is no chance of guessing the factorization. So let's complete the square. Following the above $2m \to m^2$ process, half of -12 is -6, and $(-6)^2 = 36$. So we want a 36 to appear on the lefthand side. In order to change the -5 to a 36, we need to add 41. So we'll add 41 to both sides. As always, we must do the same (useful) thing to both sides. We obtain

$$\begin{aligned} x^2 - 12x - 5 + 41 = 41 \quad &\Longrightarrow \quad x^2 - 12x + 36 = 41 \quad \Longrightarrow \quad (x - 6)^2 = 41 \\ &\Longrightarrow \quad x - 6 = \pm\sqrt{41} \quad \Longrightarrow \quad x = 6 \pm \sqrt{41}. \end{aligned} \tag{8.9}$$

Since $\sqrt{41} \approx 6.4$, the two solutions are approximately $6+6.4 = 12.4$ and $6-6.4 = -0.4$.

Again, don't forget the ± sign. Note that you could just as well write the two solutions as $\pm\sqrt{41}+6$. But the $6\pm\sqrt{41}$ form looks a little nicer.

Remark: From our discussion of factoring in Section 4.3, the two solutions in Eq. (8.9) tell us that the factorization of the original $x^2-12x-5$ polynomial is

$$\Big(x-\big(6+\sqrt{41}\big)\Big)\Big(x-\big(6-\sqrt{41}\big)\Big). \tag{8.10}$$

As mentioned above, you're probably not going produce this factorization by guessing the $6\pm\sqrt{41}$ numbers! So our completing-the-square method is definitely the way to go.

Let's check that the product in Eq. (8.10) does indeed equal $x^2-12x-5$. With the help of the difference-of-squares result in Eq. (3.23), applying FOIL to the product correctly yields

$$\begin{aligned} & x^2-\Big(\big(6+\cancel{\sqrt{41}}\big)+\big(6-\cancel{\sqrt{41}}\big)\Big)x+\big(6+\sqrt{41}\big)\big(6-\sqrt{41}\big) \\ = {} & x^2-12x+\big(6^2-\sqrt{41}^{\,2}\big) \\ = {} & x^2-12x-5. \quad ♣ \end{aligned} \tag{8.11}$$

In the above solution, we added 41 to both sides of the equation because 41 is the number you want to add to -5 to produce the desired 36. Another method is to temporarily ignore the -5 and simply add 36 to both sides. After doing this, you can then add 5 to both sides to get rid of the -5 on the lefthand side (since you want only the 36 there). This method gives

$$x^2-12x+36-5=36 \quad\Longrightarrow\quad x^2-12x+36=41, \tag{8.12}$$

in agreement with the second equation in Eq. (8.9). You're going to end up with the same equation either way, of course. In both cases you're adding 41 to both sides, either as 41 all at once, or as $36+5$. The advantage of the method in Eq. (8.12) is that you don't need to figure out what number you need to add to -5 to obtain 36 (not that this is a huge task). But again, both methods get the job done. Pick whichever one you prefer. The second one makes it a little easier to keep things organized when working with letters, but the methods are the same in the end.

In all of the cases we've dealt with so far in this section, the coefficient of the x^2 term has been 1. What if we have an equation like $2x^2-12x+8=0$? The 2 coefficient here doesn't present much of a problem, because we can simply divide the whole equation through by 2 to obtain $x^2-6x+4=0$. We're now back in familiar territory with a coefficient of 1 in the x^2 term. So:

> For equations where the coefficient of x^2 isn't 1, the first (quick) step is to divide through by whatever that coefficient is.

Summary of steps

The completing-the-square process for solving an arbitrary quadratic equation involves a specific sequence of steps, which are listed below. For concreteness here, we'll consider the equation,

$$2x^2 - 12x + 8 = 0. \tag{8.13}$$

The quadratic formula we'll derive in Section 8.3 is simply the result of applying the following steps to letters instead of numbers.

1. Divide the equation through by the coefficient of x^2 (which is 2 here). This yields $x^2 - 6x + 4 = 0$.

2. Take half of the coefficient of x (half of -6 is -3) and square it ($(-3)^2 = 9$). This is the key step in completing the square.

3. Add the result (9 here) to both sides, and then subtract the original constant term (4 here) from both sides. Alternatively, you can do this in one step by adding the difference of the two numbers ($9 - 4 = 5$ here) to both sides. Either way, the result is $x^2 - 6x + 9 = 5$. (This is almost the same as the middle equation in Eq. (8.7), but the 5 here will make the present solutions a little messier.)

4. Write the lefthand side as a square, which was the goal. This gives $(x - 3)^2 = 5$.

5. Take the square root of both sides, remembering the "±." This yields $x - 3 = \pm\sqrt{5}$.

6. Solve for x, which involves adding 3 to both sides. The solutions for x are therefore $x = 3 \pm \sqrt{5}$. The numerical values are approximately 5.24 and 0.76.

7. And always a good final step: Plug your solutions back into the given equation to verify that they are indeed solutions.

With regard to the last step here, there is an alternative method for checking your answers that is usually quicker than plugging them back into the given equation. As we saw in Section 4.3.2, if r_1 and r_2 are the two roots (solutions) of $x^2 - 6x + 4$, then the factorization takes the form of $x^2 - 6x + 4 = (x - r_1)(x - r_2)$. Multiplying this out gives $x^2 - (r_1 + r_2)x + r_1 r_2$. The middle term here says that the $-(r_1 + r_2)$ coefficient of x is the negative of the sum $r_1 + r_2$. (Equivalently, the sum $r_1 + r_2$ is the negative of the coefficient.) And the last term says that the constant term equals the product $r_1 r_2$. The sum and product of the roots are therefore given by (assuming that the coefficient of x^2 is 1):

$$\boxed{\begin{aligned} \text{Sum: } & r_1 + r_2 = -(\text{coefficient of } x) \\ \text{Product: } & r_1 r_2 = \text{constant term} \end{aligned}} \qquad (\text{if the coefficient of } x^2 \text{ is 1}) \tag{8.14}$$

These two relations are very quick to check – much quicker than plugging each of the roots into $x^2 - 6x + 4$ and showing that the results are zero. This quickness is due to the

$3 \pm \sqrt{5}$ form of the roots, or in general the $m \pm \sqrt{n}$ form. (We're assuming the coefficients in the quadratic equation are nice rational numbers, that is, they don't contain any square roots themselves.) With this $m \pm \sqrt{n}$ form, you don't need to worry about the sum and product being messy quantities involving square roots. The square roots will always go away, for the following reason.

For the above $3 \pm \sqrt{5}$ roots of the $x^2 - 6x + 4 = 0$ equation, the sum of $3 + \sqrt{5}$ and $3 - \sqrt{5}$ is simply 6. This is correctly the negative of the -6 coefficient of x, as Eq. (8.14) states. The square roots cancel in the sum, due to the "±" in the roots. Furthermore, as we observed at the end of Example 8.1, the product of the roots yields a nice difference of squares (which will always be the case, again due to the "±" in the roots), namely $3^2 - (\sqrt{5})^2 = 9 - 5 = 4$. This is correctly the constant term in the equation, as Eq. (8.14) states.

> The square roots will always cancel in the sum of the roots. Additionally, the product will always generate a nice difference of squares, which gets rid of the square roots there too. So checking the roots via Eq. (8.14) (or Eq. (8.15) below) is usually quick.

> The product and sum might appear
> To be messy, but please have no fear.
> They turn out quite fine
> Since the plus/minus sign
> Guarantees the square roots disappear.

If you have a quadratic equation of the general form $ax^2 + bx + c = 0$, where the coefficient of x^2 isn't 1, you can divide through by a to obtain $x^2 + (b/a)x + c/a = 0$. Eq. (8.14) then becomes

$$\boxed{\begin{aligned} \text{Sum:}\quad r_1 + r_2 &= -\frac{b}{a} \\ \text{Product:}\quad r_1 r_2 &= \frac{c}{a} \end{aligned}} \tag{8.15}$$

Exercise 8.1 Solve each of the following quadratic equations by (1) factoring (except in (d)), and (2) completing the square.

(a) $x^2 - 6x - 112 = 0$

(b) $3x^2 + 24x - 99 = 0$

(c) $2x^2 + 20x + 37.5 = 0$

(d) $x^2 - 8x - 5 = 0$ (don't try to factor this one)

(e) $x^2 - (2a)x = 0$

Exercise 8.2 We solved the $(x-3)^2 = 4$ equation in Eq. (8.3) by taking the square root of both sides (while remembering the ±); see Eq. (8.4). Solve the equation in a different way, by putting the 4 on the lefthand side and then using the difference-of-squares result in Eq. (3.24).

Exercise 8.3 We actually already completed the square back in Exercises 6.5 and 7.3, when we rewrote quadratic expressions in a way that contained a squared term of the form $[x - (\text{something})]^2$. Use that method to solve for the m and n that make the equation $(x+m)^2 = n$ be equivalent to the equation $x^2 - 6x - 112 = 0$. You will technically need to solve a system of equations for m and n, but it's quick.

Exercise 8.4 If you are asked to solve the quadratic equation $(x-r)(x-s) = 0$, there is no need to do any work, because the equation is already factored. You can simply say that the solutions are $x = r$ and $x = s$. The task of this exercise is to solve the equation in a much longer way, by basically going around in circles:

Expand the product $(x-r)(x-s)$ with FOIL, and then complete the square and solve for x. *Hint*: Show that the righthand side of your completed-square equation can be rewritten in the form of $\big((r-s)/2\big)^2$.

REMARK: (This remark isn't too important. It's included here just in case you were wondering if there are other ways to complete the square.) Consider the equation,

$$x^2 - 14x + 9 = 0. \tag{8.16}$$

To complete the square the way we've been doing it, we note that half of -14 is -7, and the square of this is 49. So we want to produce a 49 on the lefthand side. Adding 40 to both sides gives $x^2 - 14x + 49 = 40 \implies (x-7)^2 = 40$. We completed the square here by fiddling with the constant term, changing it from 9 to 49.

There is also another way to create a square. We can fiddle with the x term (the $-14x$) instead. Looking at the $x^2 + 9$ terms in the given equation, it would be nice if we had a $6x$ between them, because $x^2 + 6x + 9$ is a square, namely $(x+3)^2$. We can turn the given $-14x$ into $6x$ by adding $20x$ to both sides. This produces the equation $x^2 + 6x + 9 = 20x \implies (x+3)^2 = 20x$. (Adding $8x$ to both sides also produces a square, since it yields a $-6x$ on the lefthand side, which is the middle term in $(x-3)^2$.)

However, although $(x+3)^2 = 20x$ is a true equation with a square on the lefthand side, it is useless because taking the square root gives $x + 3 = \pm\sqrt{20x}$, and hence $x = -3 \pm \sqrt{20x}$. This equation involves both an x and a $\sqrt{x}$, and because of these two different powers, there is no simple way to solve for x. All we've done is solve for x in terms of its square root, whereas our goal was to solve for x in terms of just regular numbers.

The moral of this is that when completing the square, you want to modify the *constant* term (the regular number) and not the x term. This way, the thing that you add to both sides of the equation will be a regular number, and so all of the x's will remain on the lefthand side. Completing the square by modifying the x term introduces an unwanted x term (which becomes a $\sqrt{x}$ in the end) on the righthand side.

There is one more option for completing the square: You can fiddle with the x^2 term, without touching the other two terms. This method actually *does* work, but it's a bit of a mess, so we won't pursue it. ♣

8.2 Functions and plots

We plotted a few quadratic functions in Section 7.2. Let's do some more plotting here, with an emphasis on completing the square. Consider the quadratic function $f(x) = 2x^2 - 12x + 8$. What does its plot look like? We could just plug the function into Desmos and view the plot that way, as we did with many functions in the previous chapter. But if we don't have a computer handy, the easiest way to visualize the plot of a quadratic function is to complete the square. Doing this will make many features immediately evident, as we observed in our discussion of Fig. 7.2, where the $f(x) = (x - 1)^2/2 - 5$ function was already in completed-square form.

In our present discussion of plotting functions, we won't be setting the above $2x^2 - 12x + 8$ function equal to zero, as we did in Eq. (8.13). We don't have an *equation* now. We just have a *function*. If we want to complete the square, we therefore can't just divide through by 2 (the first of the seven steps on page 481). Although it's fine to divide both sides of an *equation* by 2 when we're trying to solve it, there's no equation now. There's just a *function* that we want to write as a square (plus a constant). Dividing by 2 would change the function, which we don't want to do.

To complete the square, we want to *factor out* the 2, instead of dividing by it. More precisely, we want to factor it out of the terms involving x's. The constant term 8 can just sit there for now (although see below for an alternative method). So we have

$$f(x) = 2(x^2 - 6x) + 8. \tag{8.17}$$

We can now complete the square inside the parentheses. As usual, we take half of -6 and square it to obtain $(-3)^2 = 9$. We then add that on inside the parentheses, to produce the perfect square.

But here's the critical point: Since we added a 9 inside the parentheses, it gets multiplied by the 2 out front. So we've actually added $2 \cdot 9 = 18$ to the function. We must therefore also subtract 18, because we don't want to change the function. Adding 18 isn't allowed, but adding *zero* in the form of $18 - 18$ is perfectly allowed (and quite useful). Adding 0 in the form of $N - N$ (for a wisely chosen N) is analogous to multiplying a fraction by 1 in the form of N/N (for a wisely chosen N) when getting a common denominator, as we noted back in Section 1.13.

So we now have

$$f(x) = 2(x^2 - 6x + 9) - 2\cdot 9 + 8 = 2(x-3)^2 - 10. \tag{8.18}$$

We have successfully completed the square. It's fine to have a constant like -10 sitting there, as long as the terms involving x are generated by an $(x-m)^2$-type square. The $(x-3)^2$ square in Eq. (8.18) happens to be multiplied by 2, but that's fine.

For a quadratic *equation*, you can divide both sides by the coefficient of x^2 (to make that coefficient be 1), and then add the same number to both sides (to complete the square). In contrast, for a quadratic *function*, you need to *factor out* the coefficient of x^2, and then add *zero* to the function in the form of $N - N$ (to complete the square).

A slight variation on the strategy in Eqs. (8.17) and (8.18) is to factor the 2 out of all three terms in the function, instead of just the first two. This gives $f(x) = 2(x^2 - 6x + 4)$. Completing the square inside the parentheses involves adding 5 in order to obtain the desired 9. But then we must also subtract 5 inside the parentheses, so that we don't change the function. This gives

$$f(x) = 2\big(x^2 - 6x + 4 + (5-5)\big) = 2\big((x^2 - 6x + 9) - 5\big) = 2(x-3)^2 - 10, \tag{8.19}$$

in agreement with Eq. (8.18). Yet another method is to add and subtract a 9 inside the parentheses:

$$f(x) = 2\big(x^2 - 6x + (9-9) + 4\big) = 2\big((x^2 - 6x + 9) - 5\big) = 2(x-3)^2 - 10. \tag{8.20}$$

It's all the same in the end. Pick whichever completing-the-square method in Eqs. (8.18)–(8.20) you prefer. In all cases, just make sure you're adding zero in the form of $N - N$.

As mentioned above, completing the square makes it easier to visualize a function. In the original $f(x) = 2x^2 - 12x + 8$ form of our function, it's hard to tell what the plot looks like. But in the completed-square $f(x) = 2(x-3)^2 - 10$ form in Eqs. (8.18)–(8.20), a number of things are immediately evident, if you happen to not have a computer or graphing calculator handy. We'll list these items below, but first note the following difference between equations and functions, with regard to the fact that equations are things you *solve*, whereas functions are things you *plot*:

For a quadratic *equation*, the goal (when trying to *solve* it) is to get a squared binomial alone on one side, and a constant term alone on the other side; see, for example, the $(x-6)^2 = 41$ equation in Eq. (8.9). In contrast, for a quadratic *function*, the goal (when trying to *visualize* it) is to write it as a square plus a constant term; see the $2(x-3)^2 - 10$ *function* in Eq. (8.18). There is just a single function now; there aren't two sides of an equation.

Let's now list out the benefits of writing a function in the completed-square form. We already noted most of these things in the discussion of Eq. (7.3) in Section 7.2. But in that section, we were *given* an already completed square. We now know how to *produce* it.

- Since a square is always positive (or zero), the $2(x-3)^2$ part of $f(x)$ is always positive (or zero). So $f(x)$ is always greater than or equal to -10.

- The minimum value (namely -10) of $f(x)$ occurs when $x-3=0 \implies x=3$. We know from Section 7.2 that the plot of a quadratic function is a parabola, so by completing the square, we know that $f(x)$ is an upward-opening parabola with its minimum located at the point $(3,-10)$. This is shown in Fig. 8.2.

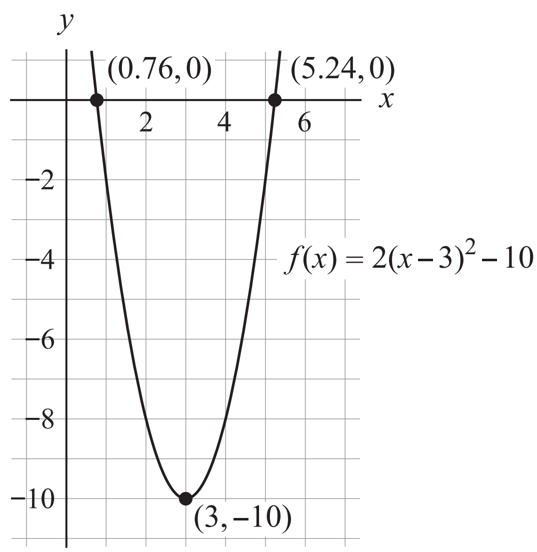

Figure 8.2: The plot of the function $f(x) = 2(x-3)^2 - 10$.

- The factor of 2 in $2(x-3)^2$ determines how fast the parabola increases as you move away from the minimum. This feature isn't as obvious as the location of the minimum.

- The roots of the equation $f(x) = 0$ are given by the points where the function crosses the x-axis, because all points on the x-axis have $y = 0$ (equivalently, $f(x) = 0$). Setting the completed-square form of $f(x)$ equal to zero gives $2(x-3)^2 - 10 = 0 \implies (x-3)^2 = 5$. As we found in step 6 on page 481, the roots are $x = 3 \pm \sqrt{5}$ (approximately 5.24 and 0.76).

  We therefore know three points that the parabola passes through: the minimum at $(3,-10)$, and the two points $(0.76, 0)$ and $(5.24, 0)$ on the x-axis. If you draw a rough parabolic-ish curve passing through these three points, you're going to get a good idea of what the actual function looks like. It won't be exact, of course, because there's no way you'll get the curvature exactly correct – you'll inevitably draw the bottom of the parabola a little too pointy or too boxy. But it will be roughly correct.

In terms of general letters, the above bullet points say that for a general quadratic function in completed-square form,

$$f(x) = A(x - B)^2 + C, \tag{8.21}$$

the following things are true. (We're using uppercase letters A, B, C here so that they aren't confused with the lowercase letters a, b, c below in Section 8.3. But the A, B, C here *do* correspond to the a, b, c in Eq. (7.3).)

1. C is the value of the minimum of the function, assuming A is positive so that we have an upward-opening parabola. If A is negative, as it is in, say $-2(x-3)^2 - 10$, then the parabola opens *downward*. So $C = -10$ is now the *maximum* value instead of the minimum, because $-2(x-3)^2$ is always negative (or zero).

2. B is the x location of the minimum or maximum.

3. The size of A affects how fast $f(x)$ changes as you move away from the max/min located at the point (B, C). Said in another way, the larger the A, the thinner the parabola.

4. The roots of $f(x) = 0$ are given by

$$\begin{aligned} A(x-B)^2 + C = 0 \quad\Longrightarrow\quad (x-B)^2 = -C/A \quad&\Longrightarrow\quad x - B = \pm\sqrt{-C/A} \\ &\Longrightarrow\quad x = B \pm \sqrt{-C/A}. \end{aligned} \tag{8.22}$$

   If the fraction C/A is negative, so that $-C/A$ is positive, then the roots are real. (See Section 8.5 below for a discussion of real and imaginary numbers. Real numbers are just the regular numbers we've been working with so far in this book.) If the roots are real, then the function crosses the x-axis at the x values of $B \pm \sqrt{-C/A}$. These roots are located at equal distances of $\sqrt{-C/A}$ from the max/min at $x = B$, due to the $\pm\sqrt{-C/A}$ in Eq. (8.22). This is evident in Fig. 8.2, where we see that the parabola is symmetric with respect to its bottom point.

   If instead C/A is *positive*, so that $-C/A$ is *negative*, then the roots aren't real, which means that the function does *not* cross the x-axis; see Fig. 8.3. There are no (real) x values for which the function equals zero. You can check this result by using Desmos to plot some concrete functions where C/A is positive. (So you'll want to pick A and C both positive, or both negative.) We'll discuss this case further in Sections 8.4 and 8.5.

   The sign of A determines whether the parabola opens upward or downward, and the sign of C/A determines whether the parabola crosses the x-axis. (It always crosses y-axis, of course, since you're free to plug $x = 0$ into the function.)

To summarize, completing the square of a quadratic function (by adding zero in the from of $N - N$, for a wisely chosen N) helps you visualize the parabolic plot by determining the coordinates (B, C) of the max or min. Additionally, if you set the function equal to zero (so you now have an *equation*), the roots correspond to the points where the function crosses the x-axis.

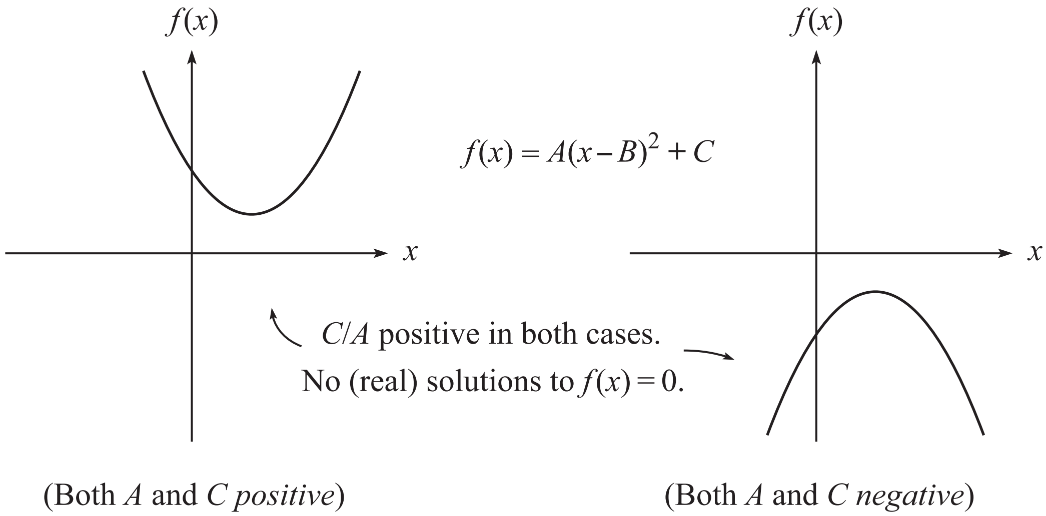

Figure 8.3: Quadratic functions that don't cross the x-axis.

The max of the function lies where? The students all pointed, "Right there!" They found it so quick, Was it some sort of trick? No, they simply completed the square!

Exercise 8.5 For each of the following functions, complete the square to find the (x, y) coordinates of the maximum or minimum. Then set the function equal to zero to find the roots (where the plot crosses the x-axis). Then make a rough sketch of the plot. You can check your plot in Desmos.

(a) $f(x) = 2x^2 - 20x + 32$

(b) $f(x) = -3x^2 + 18x - 15$

8.3 The quadratic formula

We'll now derive the *quadratic formula*, which gives the solutions to the general quadratic equation,

$$ax^2 + bx + c = 0. \tag{8.23}$$

In this equation, a, b, and c can take on arbitrary values, and our goal is to solve for x. The strategy for solving this general quadratic equation is *exactly* the same as our completing-the-square strategy in Section 8.1, which is summarized in the list of steps on page 481.

There's nothing new here. The only difference is that we'll be working with letters instead of numbers, so we'll have to do more algebra and the expressions will be longer. But it all comes down to following the steps on page 481. So let's march through them:

1. Dividing Eq. (8.23) through by a gives

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0. \tag{8.24}$$

2. Half of b/a is $b/2a$, and the square of this is $b^2/4a^2$.

3. Adding $b^2/4a^2$ to both sides, and then subtracting c/a from both sides, gives

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}. \tag{8.25}$$

4. Writing the lefthand side as a square yields

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}. \tag{8.26}$$

5. Taking the square root of both sides gives

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}. \tag{8.27}$$

6. Solving for x yields

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}. \tag{8.28}$$

7. The last step of checking the roots (the solutions) is left for Exercise 8.7.

Eq. (8.28) gives the desired roots of Eq. (8.23), so technically we're done. However, we can make the result look a little nicer if we add the two fractions under the square root by getting a common denominator. Multiplying the c/a term by $4a/4a$ yields

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}. \tag{8.29}$$

Since we have a common denominator here, we can add the two fractions and write the result in the slightly more compact form,

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}} \qquad \text{(Quadratic formula)} \tag{8.30}$$

This is the desired quadratic formula. In words, it says:

x equals negative b plus or minus the square root of b squared minus $4ac$, over $2a$.

You'll find yourself repeating this mantra many times throughout this chapter. And also throughout life, if you end up doing anything related to math!

REMARKS:

1. As we mentioned, there is *nothing new* in the above derivation. All we did was repeat the completing-the-square process that we performed many times in Section 8.1. But since we used letters this time, we did things once and for all. From now on we can just systematically plug into Eq. (8.30) whatever numbers we're given.

> Though students might find enigmatic
> This formula labeled "Quadratic,"
> We simply repeated
> The squares we completed,
> With letters, to be systematic.

2. Having derived the quadratic formula, the completing-the-square method for solving quadratic equations won't be so relevant from here on. All of the content of the completing-the-square method is built into the quadratic formula, so you can simply use that when solving problems. If you complete the square, you're essentially rederiving the quadratic formula, which isn't the most efficient thing to do. The nice thing about formulas/theorems is that once you prove them, you can just use the results. You don't have to keep redoing all the work.

3. However, the completing-the-square method is still useful for getting a general idea of what the plot of a function looks like. As we saw in Section 8.2, completing the square enables us to locate the maximum or minimum of a quadratic function (and also to find the roots). This is more information than just the roots in Eq. (8.30). When you put the max/min together with the two roots, the resulting three points give you a good idea of what the parabola looks like.

4. When solving a quadratic equation, if the coefficients look reasonably nice, the best strategy is probably to spend 15 seconds trying to factor the polynomial by trial and error, as we did in Section 5.7. But if this method doesn't immediately work, you can switch to the quadratic formula. And certainly if the numbers look messy, you can start right off with the formula.

   The nice thing about the quadratic formula is that it's guaranteed to work, no matter what the coefficients are, which definitely can't be said about the trial-and-error factoring method. The quadratic formula involves a very mechanical procedure; you just plug numbers in, and the answers pop out. If the coefficients are messy, there's nothing fundamentally more difficult about the formula in that case. It just takes a little longer to do the calculation.

5. Eq. (8.30) is valid for all values of a, b, and c except $a = 0$, because it's illegal to divide by zero. However, if $a = 0$ then the original quadratic equation $ax^2 + bx + c = 0$ reduces to the linear equation $bx + c = 0$. So the solution is simply $x = -c/b$, and there's no need to use the quadratic formula anyway. In Exercise 8.8 you can check that Eq. (8.30) gives the correct results for the $b = 0$ and $c = 0$ cases.

6. Although Eq. (8.30) works for all nonzero values of a, including negative values, it's good to get in the habit of writing your quadratic equations in a form with a positive a (by multiplying through by -1, or equivalently putting all the terms on the other side of the equation). Otherwise it's easy to make a mistake by forgetting the minus sign in the $2a$ term in the denominator.

7. The number and nature of the roots in Eq. (8.30) depend on the quantity $b^2 - 4ac$ under the square root (called the *discriminant*). In particular, this quantity may be positive, negative, or zero. For now, we'll deal mainly with cases where it is positive. We'll save the general discussion of all three possibilities for Sections 8.4 and 8.5. The negative-discriminant case is covered in Section 8.5, where we'll learn how to take the square root of a negative number, which at the moment doesn't seem possible, because as far as we know now, the square of any number is positive (or zero).

8. The quadratic formula can also be obtained via the factoring method used in Exercise 8.2. Instead of taking the square root of both sides of the equation in Step 5 above, we can rewrite Eq. (8.26) in the difference-of-squares form,

$$\left(x + \frac{b}{2a}\right)^2 - \left(\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}\right)^2 = 0. \tag{8.31}$$

   Factoring the lefthand side via Eq. (3.24) then produces the same roots as in Eq. (8.28), as you can check. But note that however you want to solve Eq. (8.26) (taking the square root, or factoring), you still need to complete the square to arrive at Eq. (8.26) in the first place.

9. As with quadratic equations, a formula also exists for the roots of cubic equations (ones involving x^3). But the formula is much messier than Eq. (8.30), so we won't get into that here. Likewise, a formula exists for the roots of quartic equations (ones involving x^4). But that formula is even messier than the cubic one (a *lot* messier). So if you ever find yourself complaining about the quadratic formula, just be thankful that you're not dealing with cubic or quartic equations!

   Interestingly, it is possible (but difficult) to prove that an analogous formula does *not* exist for general quintic (involving x^5) or higher-power equations. That is, there is no general algebraic expression (along the lines of Eq. (8.30)) for the solutions. If you want to solve a quintic or higher-power equation, you have to do it numerically with a computer. You can determine the solutions to whatever accuracy you want (as many decimal places as desired). It's just that higher accuracy means more computer time. ♣

Example 8.2 We have encountered many different ways of solving quadratic equations, namely: (1) factoring, (2) completing the square, (3) using the quadratic formula (which comes from completing the square), and (4) plotting. Find the roots of $x^2 - 2x - 15 = 0$ by using these four different methods.

Solution

FACTORING: From the discussion in Section 4.2.2, we want to find two numbers that multiply to -15 and add to -2. A little fiddling gives -5 and 3. So the factorization is $(x-5)(x+3) = 0$, which means the roots are 5 and -3.

COMPLETING THE SQUARE: Half of -2 is -1, and the square of this is 1. Adding 1 to both sides to complete the square, and then adding 15 to both sides (or just adding 16 all at once) gives

$$x^2 - 2x + 1 = 16 \quad \Longrightarrow \quad (x-1)^2 = 4^2 \quad \Longrightarrow \quad x - 1 = \pm 4. \tag{8.32}$$

So the roots are $x = 1 \pm 4$, which yields 5 and -3.

QUADRATIC FORMULA: With $a = 1$, $b = -2$, and $c = -15$, Eq. (8.30) gives

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-15)}}{2 \cdot 1} = \frac{2 \pm \sqrt{64}}{2} = \frac{2 \pm 8}{2}. \tag{8.33}$$

So the roots are $(2+8)/2 = 5$ and $(2-8)/2 = -3$.

PLOTTING: Fig. 8.4 shows the plot of the function $f(x) = x^2 - 2x - 15$. We see that the parabola crosses the x-axis at $x = -3$ and $x = 5$, so those are the desired roots.

Note that a slight modification of our completing-the-square work in Eq. (8.32) tells us that the function $f(x) = x^2 - 2x - 15$ can be written as $(x-1)^2 - 16$. (This is just a function, not an equation; see Section 8.2.) So the minimum value of $f(x)$ is -16, and it occurs at $x = 1$. This is consistent with the plot.

Of course, from looking only at the plot and not using any other facts we know about quadratic equations, we can't be sure that the roots aren't instead equal to, say, -2.99 and 5.01. We can read off values from a plot only so precisely. But assuming that the numbers are nice, the roots must be -3 and 5.

Example 8.3 What two numbers add to 9 and multiply to 12? Answer this by writing down the two equations that represent the two given facts, and then solving the system of equations.

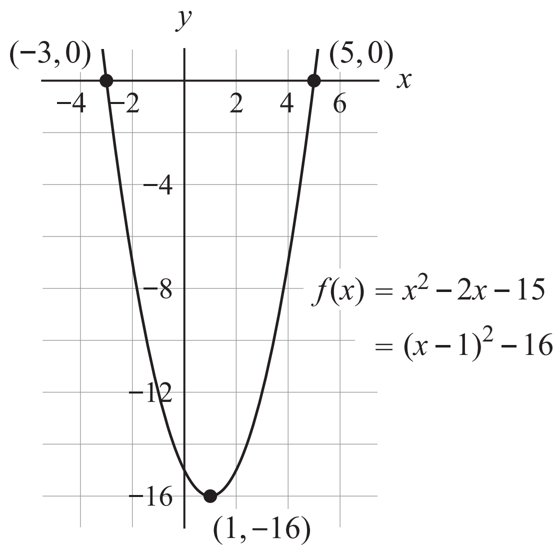

Figure 8.4: The plot of the function $f(x) = x^2 - 2x - 15$, or $(x-1)^2 - 16$.

Solution If the two numbers are x and y, then the given information tells us that

$$x + y = 9 \qquad \text{and} \qquad xy = 12. \tag{8.34}$$

Let's solve this system by eliminating y in favor of x. (Eliminating x in favor of y would work just as well.) Solving for y in the first equation gives $y = 9 - x$. Plugging this into the second equation yields

$$x(9-x) = 12 \quad \Longrightarrow \quad 0 = x^2 - 9x + 12. \tag{8.35}$$

Alternatively, you can solve for y in the second equation and then plug the result into the first. Or, you can solve for y in both equations and then equate the two expressions. You should verify that the various strategies all lead to the same quadratic equation in Eq. (8.35).

Applying the quadratic formula to $x^2 - 9x + 12 = 0$ (with $a = 1$, $b = -9$, and $c = 12$) yields

$$x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(1)(12)}}{2 \cdot 1} = \frac{9 \pm \sqrt{33}}{2}. \tag{8.36}$$

A calculator gives the numerical values as 7.37 and 1.63. These are the two possible answers for x. If $x = 7.37$, then either equation in Eq. (8.34) gives $y = 1.63$. And if $x = 1.63$, then either equation in Eq. (8.34) gives $y = 7.37$. So the desired two numbers are 7.37 and 1.63 (in either order; it doesn't matter which one we label as x or y).

As a check, these two numbers correctly sum to 9 and multiply to 12 (up to rounding errors). And the exact values in Eq. (8.36) do also. They certainly add to 9 since the $\pm\sqrt{33}$ cancels in the sum. And they multiply to $(9^2 - 33)/4 = 12$, because the product yields a nice difference of squares, which will always be the case due to the $\pm$ in the roots.

In retrospect, we could have simply written down the quadratic equation in Eq. (8.35) without doing any system-of-equations work (although the work was only a couple lines anyway). This is true because Eq. (8.14) tells us that the coefficient of x is the negative of the sum of the roots (hence -9), and the constant term is the product (hence 12).

This example is the same type of problem as Example 5.18. However, in that example we were able to find the solutions by factoring (which involves guess-and-check fiddling), whereas in the present example we needed to use the quadratic formula. There's not much chance of guessing the solutions in Eq. (8.36).

Example 8.4 As we saw on a few occasions in Section 5.7.1, sometimes an equation involving denominators doesn't look like a quadratic, but it becomes one after you rewrite it without the denominators. Solve this equation by rewriting it and then using the quadratic formula:

$$\frac{3x-4}{x} = 2 + \frac{3}{2x-3} \tag{8.37}$$

Solution Multiplying the equation through by the common denominator $x(2x-3)$ yields

$$\begin{aligned}
(3x-4)\cdot(2x-3) &= 2\cdot x(2x-3) + 3\cdot x \\
\Longrightarrow \quad 6x^2 - 9x - 8x + 12 &= (4x^2 - 6x) + 3x \\
\Longrightarrow \quad 2x^2 - 14x + 12 &= 0 \\
\Longrightarrow \quad x^2 - 7x + 6 &= 0.
\end{aligned} \tag{8.38}$$

The quadratic formula then gives

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(6)}}{2\cdot 1} = \frac{7 \pm \sqrt{25}}{2} = \frac{7 \pm 5}{2}. \tag{8.39}$$

So the roots are $(7+5)/2 = 6$ and $(7-5)/2 = 1$. Alternatively, you can just note that $x^2 - 7x + 6$ quickly factors into $(x-6)(x-1)$, so the roots are 6 and 1.

Example 8.5 Here is a wonderful application of the quadratic formula. In Fig. 8.5, a rectangle with sides x and 1 is divided into a square with side 1 and a smaller rectangle with sides 1 and $x-1$. What should x be so that the smaller rectangle has the same shape as the larger rectangle? (That is, the two rectangles are similar; they have the same ratio of side lengths.)

Solution Equating the ratio of the long side to the short side in the two rectangles gives

$$\frac{x}{1} = \frac{1}{x-1} \quad \Longrightarrow \quad x^2 - x = 1 \quad \Longrightarrow \quad x^2 - x - 1 = 0. \tag{8.40}$$

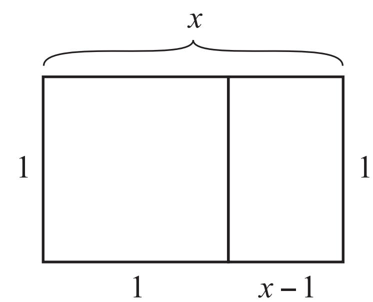

Figure 8.5: The golden rectangle. The smaller rectangle is similar to the larger one.

The quadratic formula gives the solution for x as

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2 \cdot 1} = \boxed{\frac{1+\sqrt{5}}{2}} \equiv \varphi \approx 1.618. \qquad (8.41)$$

We have ignored the $(1 - \sqrt{5})/2 \approx -0.618$ solution because it is negative, but x must be positive since it represents a length.

The number $(1+\sqrt{5})/2 \approx 1.618$ in Eq. (8.41) is known as the *golden ratio* and is often labeled with the Greek letter φ (phi), as we wrote above. The golden ratio has been studied for millennia, since the time of the ancient Greeks. With x taking on the value in Eq. (8.41), the rectangle in Fig. 8.5 is known as the *golden rectangle*. The ratio of the long side to the short side is $\varphi = 1.618$. This shape is often considered to be the most pleasing shape for a rectangle (not too square-ish, not too thin).

> Which rectangle should be extolled?
> The ancient Greeks answered, "Behold!
> The beauty resides
> In the shape that has sides
> In a ratio worthy of gold."

REMARKS:

1. The golden ratio appears in countless places in geometry, science, nature, and also many other disciplines where you might least expect it. It's a matter of opinion, but many people consider the golden ratio to be the third most important and interesting number in mathematics, behind (in no implied order!) $\pi \approx 3.14$ and $e \approx 2.718$. We'll discuss the number e (known as "Euler's number") in Chapter 9, where we'll see what makes it interesting.
2. If we invert (take the reciprocal of) the two sides of the first equation in Eq. (8.40) (so we're now taking the ratio of the short side to the long side), we obtain $1/x = x - 1$.

This says that the inverse (reciprocal) of the golden ratio is obtained by simply subtracting 1. (The same is true for the -0.618 solution we ignored.) And indeed, the inverse of 1.618 is 0.618, which equals $1.618 - 1$.

If you then take the inverse of 0.618, this reverses the original inverse operation and brings you back to 1.618. So if you plug $(1 + \sqrt{5})/2$ into a calculator to obtain 1.618, and if you then keep pressing the "inverse" (x^{-1}) button, the 1 in front of the decimal point will keep disappearing and reappearing.

3. As mentioned above, we ignored the $(1 - \sqrt{5})/2 \approx -0.618$ root because the length x needs to be positive. Quadratic equations always have two roots (unless the $\sqrt{b^2 - 4ac}$ term in Eq. (8.30) is zero), but problems sometimes have only one answer. In such cases, the other root is meaningless. That root solves the quadratic equation in a mathematical sense, but it doesn't solve the original problem, due to a fatal flaw of one kind or another. The flaw here is that $(1 - \sqrt{5})/2$ is negative, whereas the length x must be positive. It's generally clear which answer you want. A positive/negative distinction is often what determines it, but not always.

   To be sure, there are many cases where both roots are relevant. For example, in each of Exercises 8.11 and 8.12 below, both solutions to the quadratic equation are valid answers. A somewhat in-between case was Example 8.3 above. Both solutions for x (namely 7.37 and 1.63) are valid. But whichever one you pick, the y value is the other one. So finding only one root would get the job done in this case. ♣

And now here is a large supply of exercises (15 of them!) that you can practice on. As you will see, and as you already saw in some of the above examples, the quadratic formula opens up a whole new range of problems you can solve!

Exercise 8.6 Solve each of the following quadratic equations by using the four methods presented in Example 8.2:

(a) $x^2 + x - 20 = 0$ (b) $2x^2 - 4x - 6 = 0$

Exercise 8.7 Check that the roots in Eq. (8.30) satisfy the original quadratic equation, $ax^2 + bx + c = 0$. To do this, *don't* plug the roots into the equation; that would be messy. Instead, show that the roots add to $-b/a$ and multiply to c/a, as Eq. (8.15) states.

Exercise 8.8 In the special cases where $b = 0$ or $c = 0$, verify that the two roots in Eq. (8.30) agree with what you would obtain if you instead just solve the simpler equations directly.

Exercise 8.9 If you start with the quadratic equation $ax^2 + bx + c = 0$ and then multiply it through by m to obtain $max^2 + mbx + mc = 0$, this new equation should have the same roots, because you could simply divide by m to get back to the original equation. Verify that Eq. (8.30) correctly gives the same roots.

Exercise 8.10 Consider the system of equations,

$$x + y = -\frac{b}{a} \qquad \text{and} \qquad xy = \frac{c}{a}. \tag{8.42}$$

From Eq. (8.15) we know that x and y are the solutions to $ax^2 + bx + c = 0$. Generate this quadratic equation again by solving for y in terms of x in the first of the above equations, and then plugging the result into the second equation. (This exercise is the general version of Example 8.3.)

Exercise 8.11 Find all numbers with the property that the reciprocal is 1 more than the number.

Exercise 8.12 Find the points where the line $y = 2x + 2$ intersects the parabola $y = x^2 - 2x + 5$.

Exercise 8.13 Solve each of the following equations. *Hint:* Isolate a square root on one side of the equation, and then square both sides.

(a) $x + \sqrt{2x - 1} = 8$ (b) $\sqrt{2x + 3} - \sqrt{x - 2} = 2$

Exercise 8.14 Alice and Bob each have a drinking glass. Alice's glass holds half a cup more than Bob's. They each want to fill up a 5-gallon bucket with glassfuls. (There are 16 cups in a gallon, so there are 80 cups in 5 gallons.) They each fill up their own 5-gallon bucket and then note that it took a combined total of 72 glassfuls. How large is each glass (that is, how many cups does each hold)?

Exercise 8.15 Starting with the quadratic formula in Eq. (8.30), rewrite it by performing various operations (multiplying by $2a$, adding b, squaring, etc.) until you end up with the original equation, $ax^2 + bx + c = 0$. Note that reversing these steps generates a new derivation of the quadratic formula.

Exercise 8.16 Derive the quadratic formula by using the completing-the-square method in Eqs. (8.17) and (8.18) (where the coefficient of x^2 is factored out of the first two terms), instead of dividing the equation through by a, as we did in Eq. (8.24). It suffices to reproduce Eq. (8.26).

Exercise 8.17 The relations in Eq. (8.15) suggest another way to derive the quadratic formula. Since the sum of the roots equals $-b/a$, their average is $-b/2a$. So they must take the form of $-b/2a \pm z$, for some z; they are equally spaced on either side of $-b/2a$. Use the information about the product in Eq. (8.15) to solve for z and thereby generate the quadratic formula.

Exercise 8.18

(a) What are the values of the four fractions below? What is the value if you continue the recipe and add on another level of fractions? And then another level? And another? (Note that you don't need to calculate things from scratch each time. You can use your previous result to quickly obtain the next one, because each fraction appears as a subpart of the next one.) Do your answers approach a familiar number you've seen somewhere earlier in this chapter?

$$\frac{1}{1+1} \qquad \frac{1}{1+\cfrac{1}{1+1}} \qquad \frac{1}{1+\cfrac{1}{1+\cfrac{1}{1+1}}} \qquad \frac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+1}}}} \tag{8.43}$$

(b) Find the value of the fraction if the 1's extend infinitely far down to the right, represented by the diagonal dots below. *Hint:* If you let the desired value of the entire fraction be x, then the value of the fraction inside the box shown also equals x, because it also extends infinitely far down to the right. You can use this fact to produce an equation that x must satisfy.

$$\frac{1}{1+\boxed{\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+1_{\ddots}}}}}} \tag{8.44}$$

Exercise 8.19 The preceding exercise involved *continued fractions*. The present one explains how square roots of numbers can be written as infinite continued fractions. (This exercise doesn't have anything to do with the quadratic formula. It's just an extension of the preceding exercise.)

(a) Show that this equation is true:

$$\sqrt{a^2+b}-a=\frac{b}{2a+\left(\sqrt{a^2+b}-a\right)}. \tag{8.45}$$

(b) Eq. (8.45) tells you that $\sqrt{a^2+b}-a$ equals the fraction on the righthand side. You can therefore take that fraction and insert it where $\sqrt{a^2+b}-a$ appears in parentheses on the righthand side, which gives

$$\sqrt{a^2+b}-a=\frac{b}{2a+\cfrac{b}{2a+\left(\sqrt{a^2+b}-a\right)}}. \tag{8.46}$$

You can continue this process of replacing $\sqrt{a^2+b}-a$ with the fraction $b/\big[2a+\big(\sqrt{a^2+b}-a\big)\big]$, to obtain an infinite continued fraction. Use this fact to express $\sqrt{14}=\sqrt{3^2+5}$ as a continued fraction. Then verify with your calculator that if you include enough levels of the continued fraction, you obtain a result that is very close to $\sqrt{14}=3.74166$.

Exercise 8.20 Use a calculator to evaluate the following expression, up to the 10th 1:

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}. \tag{8.47}$$

Does your result look familiar? (For that matter, does the square-root expression itself look familiar? Perhaps on a book cover somewhere?) Use a strategy analogous to the one in Exercise 8.18(b) to find the exact value of the expression when an infinite number of 1's are included.

Distance problems

In Section 5.5.2 we solved some distance problems (mostly involving trains), where we ended up needing to solve *linear* equations, as opposed to *quadratic* or more complicated ones. Let's now solve some distance problems that involve quadratic equations. As in Section 5.5.2, the main ingredient in these problems is the "rate times time equals distance" relation $rt=d$ in Eq. (5.41). Quick corollaries of this equation are (depending on which letter you want to solve for) $t=d/r$ and $r=d/t$. We'll generally use v (for velocity) instead of r (for rate). The $rt=d$ relation then becomes $vt=d$.

Example 8.6 You walk for 3 miles at a certain speed v. You then turn around and head home. Realizing that you're going to be late for dinner, you increase your speed by 2 mph (so it's now $v+2$ mph) and jog home on the return trip. If the entire trip takes an hour and 15 minutes, what is v?

Solution Since $vt=d \implies t=d/v$, and since $d=3$, the outward part of the trip takes a time of $t_{\text{out}}=3/v$. Similarly, since the return speed is $v+2$, the time to get back home is $t_{\text{back}}=3/(v+2)$. We're measuring v in units of miles per hour, and d in units of miles. We won't bother writing out the units here (or in future problems) as we did in Eqs. (5.42) and (5.43), because the equations would get unwieldy. But it is understood that the 3 here has units of miles, and the 2 has units of miles per hour. The v is fine as it is, because the units of miles per hour are already embedded in the v. If you wrote "v miles/hour," that would actually be incorrect.

We're told that the total time is an hour and 15 minutes, which is 5/4 hours. Therefore,

$$t_{\text{out}} + t_{\text{back}} = \frac{5}{4} \quad \Longrightarrow \quad \frac{3}{v} + \frac{3}{v+2} = \frac{5}{4}\,. \tag{8.48}$$

Multiplying this equation through by the common denominator $4v(v+2)$ yields

$$\begin{aligned} 3 \cdot 4(v+2) + 3 \cdot 4v = 5 \cdot v(v+2) \quad &\Longrightarrow \quad (12v + 24) + 12v = 5v^2 + 10v \\ &\Longrightarrow \quad 0 = 5v^2 - 14v - 24. \end{aligned} \tag{8.49}$$

The quadratic formula then gives

$$v = \frac{-(-14) \pm \sqrt{(-14)^2 - 4(5)(-24)}}{2 \cdot 5} = \frac{14 \pm \sqrt{676}}{10} = \frac{14 \pm 26}{10}\,. \tag{8.50}$$

We must pick the "+" sign since the speed v is positive. So we obtain $v = (14{+}26)/10 = 4$. The return speed is then $v + 2 = 6$.

The other root in Eq. (8.50), namely $v = (14 - 26)/10 = -6/5$, satisfies the quadratic equation, but it isn't physical because you can't walk with a negative speed; a speed is by definition a positive quantity.

As a check, plugging our answer of $v = 4$ into Eq. (8.48) does indeed produce an equality. The outward time is 3/4 of an hour, and the return time is $3/6 = 1/2$ of an hour, for a correct total of 5/4 hours. Note that the math in this example is very similar to the math in Exercise 8.14.

Remark: What is your average speed for the entire trip? That is, what speed v_{avg} yields a time of one hour and 15 minutes for the 6-mile roundtrip, if you maintain the same speed v_{avg} for the entire time? Equivalently, what value of v_{avg} satisfies

$$v_{\text{avg}} \cdot t_{\text{total}} = d_{\text{total}} \quad \Longrightarrow \quad v_{\text{avg}} = \frac{d_{\text{total}}}{t_{\text{total}}}\,. \tag{8.51}$$

For the present problem, we have $d_{\text{total}} = 6$ and $t_{\text{total}} = 5/4$. So $v_{\text{avg}} = 6/(5/4) = 24/5 = 4.8$ (in miles per hour).

Note that this $v_{\text{avg}} = 4.8$ result is *not* the average of your outward and return speeds (namely 4 and 6), which is $(4 + 6)/2 = 5$. v_{avg} is less than this. It turns out that this is a general result: The average speed is always less than the average of the outward and return speeds (or equal, if the two speeds are the same). You can demonstrate this fact in Exercise 8.21. ♣

Exercise 8.21 You travel outward a distance ℓ at speed a, and then return the same distance ℓ at speed b. (It would be more natural to label the two speeds as v_1 and v_2, but then the algebra in the solution would get a bit cluttered with all the subscripts. So that's why we're using a and b.)

Write down an expression for the average speed v_{avg} for the entire trip, which equals $d_{\text{total}}/t_{\text{total}}$ from Eq. (8.51). Then show that v_{avg} is always less than or equal to the average of the two speeds, which is $(a+b)/2$. *Hint*: In the $v_{\text{avg}} \le (a+b)/2$ relation you're trying to show is true, perform the same helpful operations on both sides, until you end up with a result that you know is true.

Exercise 8.22 You swim in a river where the water speed is $1/2$ mph. Your swimming speed with respect to the water is v. You swim to a beach one mile upstream, and then swim the mile back downstream to where you started. If the total time is 2 hours and 20 minutes, what is v?

Note: Your actual speed (with respect to the ground) upstream or downstream is obtained by subtracting or adding the river's speed to your swimming speed. For example, if you swim at 3 mph with respect to the water, and if the water speed is 1 mph, then your actual speed upstream is $3 - 1 = 2$ mph (since you are fighting the current), and your actual speed downstream is $3 + 1 = 4$ mph (since the current is helping you).

Exercise 8.23 In the preceding exercise, you found that your swimming speed (with respect to the water) was $v = 1$ mph. If you swam with this $v = 1$ speed for one mile in each direction in a river with *no current*, then your time in each direction would simply be $t = d/v = 1/1 = 1$. So the total roundtrip time would be 2 hours. The time of 2 hours and 20 minutes in the preceding exercise is longer than this. This is consistent with the following fact:

Any nonzero current makes your total time be longer than in the zero-current case. Prove this fact by letting the river's speed be a general speed r and finding the total time in terms of r. (Work with a general still-water speed v, and a general distance d.)

Exercise 8.24 A plane flies 2500 miles west, and then turns around and flies 2500 miles east, back to where it started. If there is no wind, and if the plane's "air speed" (the speed with respect to the surrounding air) is 500 mph, then the total roundtrip time is $t = d/v = (5000 \text{ miles})/(500 \text{ miles/hour}) = 10$ hours. If there is a 100 mph eastward wind, what does the plane's air speed need to be increased to, in order to make the total roundtrip time still be 10 hours? (The air in this exercise is just like the river in Exercise 8.22. The plane is effectively swimming in an air river.)

8.4 The discriminant

8.4.1 The three cases

As we mentioned in the 7th remark on page 491, the number and nature of the roots in the quadratic formula in Eq. (8.30) depend on the sign of the $b^2 - 4ac$ quantity under the square root. This quantity is called the *discriminant*, D:

$$\boxed{D \equiv b^2 - 4ac} \qquad \text{(Discriminant)} \tag{8.52}$$

There are three possibilities for the kinds of solutions that Eq. (8.30) yields, depending on whether the discriminant is positive, zero, or negative:

1. Positive (two real roots): If $b^2 - 4ac > 0$, then $\sqrt{b^2 - 4ac}$ is a standard real number (as opposed to an imaginary number, which we'll define below). So the $\pm$ in Eq. (8.30) leads to *two* distinct real solutions. Nearly all of the cases we've considered so far in this chapter have fallen into this category.

2. Zero (one root): If $b^2 - 4ac = 0$, then $\sqrt{b^2 - 4ac} = 0$, so the $\pm$ in Eq. (8.30) is irrelevant, and we have only one solution which equals $-b/2a$. The quadratic polynomial ax^2+bx+c is a perfect square in this case (see Exercise 8.25), so the two binomial factors are identical, and we have only *one* root. This is referred to as a "double root." There is only one root, but it appears twice.

3. Negative (no real roots): If $b^2 - 4ac < 0$, then taking the square root of the negative quantity $b^2 - 4ac$ presents a problem. We need to find a number whose square is the negative quantity $b^2 - 4ac$. But this doesn't seem possible, because the square of any positive or negative number is positive. For example, $3^2 = 9$ and also $(-3)^2 = 9$. It appears to be impossible to find a number y such that $y^2 = -9$. So is there a number y satisfying this equation or not?

   Well, yes and no. There *isn't* a solution involving the type of numbers we've been dealing with so far (called *real* numbers). But there *is* a solution if we consider *imaginary* numbers. We'll look at these in Section 8.5.

   The general solutions in the $b^2 - 4ac < 0$ case are combinations of real and imaginary numbers, which are known as *complex* numbers. We'll discuss these in Section 8.5. But for now, suffice it to say that as long as you don't restrict the solutions to just real ones, there's nothing wrong with the discriminant being negative. It yields two solutions (albeit complex ones), just like the positive case.

There's really no need to malign
A "minus" discriminant sign.
The solutions aren't real,
But that's not a big deal –
Complex numbers will make it all fine!

Examples of the above three cases for the sign of the discriminant D are given below and shown in Fig. 8.6.

1. Positive (two real roots): $x^2-6x+5=0$: The discriminant D is $b^2-4ac=(-6)^2-4\cdot1\cdot5=16$, which is positive. So Eq. (8.30) gives the two real roots as

$$x=\frac{-(-6)\pm\sqrt{16}}{2\cdot1}=\frac{6\pm4}{2}=5\text{ or }1. \tag{8.53}$$

   The plot of x^2-6x+5 is the bottom parabola in Fig. 8.6. This agrees with what we obtain if we complete the square and rewrite x^2-6x+5 as $(x-3)^2-4$. The minimum value is -4, and it is located at $x=3$. And as the function increases from this value, it crosses the x-axis twice: at $x=1$ and $x=5$. So there are two distinct roots.

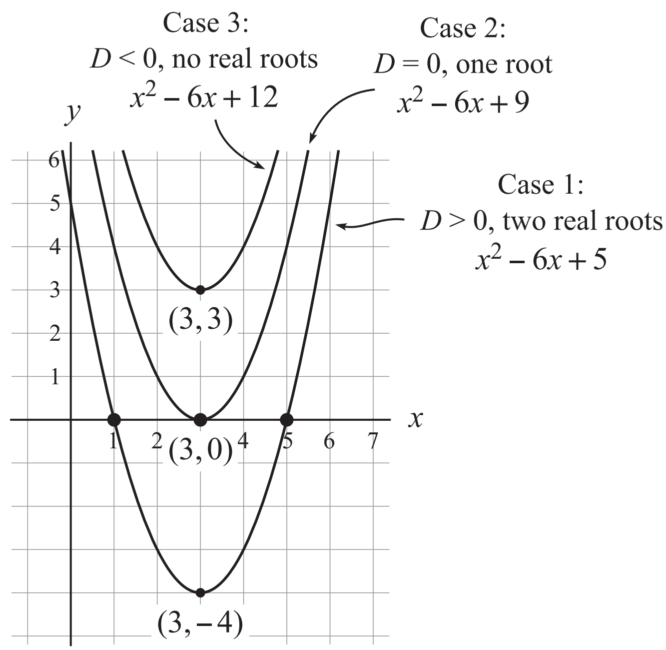

Figure 8.6: The three cases for the discriminant D.

2. Zero (one root): $x^2-6x+9=0$: The discriminant D is $b^2-4ac=(-6)^2-4\cdot1\cdot9=0$. Since this is zero, there is only one (real) root:

$$x=\frac{-(-6)\pm\sqrt{0}}{2\cdot1}=3. \tag{8.54}$$

   The middle parabola in Fig. 8.6 agrees with what we obtain if we write x^2-6x+9 as the perfect square $(x-3)^2$. The value is always at least zero, and the parabola touches the x-axis at a single point at $x=3$.

3. Negative (no real roots): $x^2-6x+12=0$: The discriminant D is $b^2-4ac=(-6)^2-4\cdot1\cdot12=-12$, which is negative. So there are no real roots. There are two

complex roots, which we'll discuss in Section 8.5. The top parabola in Fig. 8.6 agrees with what we obtain if we complete the square and rewrite $x^2 - 6x + 12$ as $(x-3)^2 + 3$. The value is always at least 3, so it can never be zero. There are no real solutions to $x^2 - 6x + 12 = 0$.

If the coefficient a of x^2 is negative, then we have an upside-down parabola. But the same general results hold, as you can check by making some plots in Desmos. The numbers of real solutions in the above three cases for the sign of D are again (in the same order) two, one, and zero.

Example 8.7 Imagine plotting the line $y = 2x + b$ (with a Desmos slider for b), along with the parabola $y = x^2 - 2x + 5$. If $b = 2$, we found in Exercise 8.12 that the line and the parabola have two intersection points at $(1, 4)$ and $(3, 8)$. However, if b is a large negative number then the two plots won't intersect at all, because the line will lie far below the parabola (since b is the y-intercept).

Therefore, there must be a value of b where the transition is made from two intersection points to zero. For this special value of b, there is *one* intersection point, and the line is *tangent* to the parabola, meaning that it just barely touches the parabola as it skims by. Find this special value of b.

Solution Following the procedure in Exercise 8.12, but with the line now having a y-intercept of b instead of 2, we obtain

$$x^2 - 2x + 5 = 2x + b \quad \Longrightarrow \quad x^2 - 4x + (5 - b) = 0. \tag{8.55}$$

The quadratic formula gives

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot (5 - b)}}{2 \cdot 1}. \tag{8.56}$$

The constant "c" in the quadratic formula in Eq. (8.30) is $5 - b$ here. There's nothing wrong with the a, b, or c in the quadratic formula consisting of more than one term.

If the discriminant in Eq. (8.56) is positive, there are two roots. If it is negative, there are no (real) roots. We're concerned with the scenario where it equals zero, in which case there is exactly one root. So we want

$$(-4)^2 - 4 \cdot 1 \cdot (5 - b) = 0 \quad \Longrightarrow \quad 16 - 20 + 4b = 0 \quad \Longrightarrow \quad b = 1. \tag{8.57}$$

Therefore, the line $y = 2x + 1$ is tangent to the parabola $y = x^2 - 2x + 5$. You can verify this in Desmos. Changing the value of the y-intercept b simply shifts the line up and down (while keeping the same slope of 2). The $b = 1$ value makes the line be tangent to (that is, barely touch) the parabola. We'll talk more about tangent lines in Chapter 12.

Plugging $b = 1$ into Eq. (8.56) gives $x = 2$. And then plugging $x = 2$ into the equation for either the parabola or the line (with $b = 1$) gives $y = 5$. So the point of tangency is $(2, 5)$, which you can verify in Desmos.

Exercise 8.25 If the discriminant $b^2 - 4ac$ is zero, show that the polynomial $ax^2 + bx + c$ is a perfect square, by eliminating b in favor of a and c.

Exercise 8.26 In Desmos, plot the parabola $y = x^2$, along with the parabola $y = 2(x-1)^2 + c$ with a slider for c. You will observe that for some values of c there are two intersection points, and for others there are none. And for one special value of c there is one intersection point. Use the discriminant to find this special value (which makes the two parabolas be tangent).

8.4.2 Maxima and minima

The discriminant provides a nice method for finding the maximum or minimum of a quadratic function. In calculus, the standard technique for maximizing or minimizing a function involves the derivative. But if you want to avoid calculus, the discriminant is the way to go (at least for quadratic functions). It takes a little longer than the calculus method, but it gets the job done. And it's nice to know that there's a non-calculus way of maximizing or minimizing quadratic functions.

The main thing you need to know for all of the examples and exercises below is that the parabola in Fig. 8.6 intersects the x-axis twice in the first case (where there are *two* solutions), once in the second case (where there is *one* solution), and zero times in the third case (where there are *no* solutions). Because we'll be dealing only with real numbers in this section, the word "solution" in the preceding sentence is understood to mean "*real* solution." But it's a pain to keep repeating the word "real," so we'll often drop it here. The fact of the matter is that the third case *does* have two solutions; it's just that they involve imaginary numbers (see Section 8.5). For the purposes of this section, imaginary numbers don't exist.

The process of using the discriminant to find the maximum or minimum of a function is illustrated in the following examples. In the first example, we'll use our old method of completing the square, and then the new method involving the discriminant. In the other examples, we'll generally use only the discriminant method. This consists of the following steps:

DISCRIMINANT METHOD for finding the max or min of a quadratic function:

1. Set the given function equal to some value, call it m.
2. Solve for x (in terms of m) with the quadratic formula.
3. Find the max (or min) value of m that generates a real root for x (equivalently, that makes the discriminant be greater than or equal to zero). In other words, find the max (or min) value of m that actually has a (real) x value that produces it.

Example 8.8 Find the maximum value of the function $f(x) = -2x^2 + 8x + 7$ by:

(a) Completing the square.

(b) Using the discriminant method in the above box.

Solution

(a) To complete the square, we'll use the method in Eq. (8.18), where we *factor out* the coefficient of x^2, which is -2 here. Remember, we can't just *divide* the function by -2. We instead need to factor out the -2, because (as we mentioned near the beginning of Section 8.2) we're dealing with a function, not an equation. Dividing by -2 would change the function.

Factoring out the -2 from the first two terms in the function gives $f(x) = -2(x^2 - 4x)+7$. Completing the square involves adding $(-4/2)^2 = 4$ inside the parentheses. But since there is a -2 out front, we've actually subtracted 8. So we need to add 8 in order to make the net change be zero. We therefore obtain

$$f(x) = -2(x^2 - 4x + 4) + 8 + 7 = -2(x - 2)^2 + 15. \tag{8.58}$$

Since $(x - 2)^2$ is always positive (or zero), the $-2(x - 2)^2$ term is always negative (or zero). The maximum value of the function is therefore 15. Additionally, the completed-square form tells us that this maximum occurs at $x = 2$. So the highest point on the (upside-down) parabola is $(2, 15)$. You can verify this with a Desmos plot.

(b) Setting the function equal to m and using the quadratic formula to solve for x gives

$$\begin{aligned} -2x^2 + 8x + 7 = m \implies & \quad 0 = 2x^2 - 8x + (m - 7) \\ \implies & \quad x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4 \cdot 2 \cdot (m - 7)}}{2 \cdot 2}. \end{aligned} \tag{8.59}$$

A real root for x exists only if the discriminant is greater than or equal to zero, which yields

$$64 - 8(m - 7) \geq 0 \quad \Longrightarrow \quad 8 - (m - 7) \geq 0 \quad \Longrightarrow \quad 15 \geq m, \tag{8.60}$$

or equivalently $m \leq 15$. So 15 is the maximum value of m, in agreement with the result in part (a). 15 is the largest value of m that actually has a (real) value of x that produces it; this is clear from your Desmos plot.

REMARKS:

1. Since the maximum is achieved when the discriminant $b^2 - 4ac$ equals zero, the quadratic formula in Eq. (8.30) tells us that the maximum is achieved at

$$\boxed{x = -\frac{b}{2a}} \qquad \text{(Location of maximum or minimum)} \tag{8.61}$$

   This equals $-(-8)/(2 \cdot 2) = 2$ in this example, in agreement with the x value we found in part (a).
2. When writing out the $b^2 - 4ac \geq 0$ condition for the discriminant, as we did in Eq. (8.60), we automatically obtain an inequality with the inequality sign pointing in the correct direction. So we automatically know if the m value is a maximum (as it is in the present case of $15 \geq m$), or a minimum (if we had instead ended up with $15 \leq m$).
3. Although we wrote out the entire quadratic formula in Eq. (8.59), this actually wasn't necessary for the present purposes, so we won't do so for future problems of this type. All we're concerned with is the discriminant $b^2 - 4ac$, and what value of m makes it be zero. If we additionally want to know the value of x where the maximum or minimum occurs, the quadratic formula (with the discriminant equal to zero) quickly tells us that $x = -b/2a$.
4. Note the difference in the strategies in parts (a) and (b). In part (a) we found the maximum value of the *function* $f(x) = -2x^2 + 8x + 7$ (by completing the square), whereas in part (b) we solved the *equation* $-2x^2 + 8x + 7 = m$ (by using the quadratic formula) and found the maximum value of m for which a real solution exists. ♣

Example 8.9 Assuming that x is positive, what is the minimum value of $x + 1/x$?

Solution First note that if x is very small, then the $1/x$ term in $x + 1/x$ is very large. And if x is very large, then the x term in $x + 1/x$ is very large (of course). The sum $x + 1/x$ must therefore achieve a minimum somewhere in between, as the question correctly implies.

Similar to part (b) of the preceding example, our strategy will be to set $x + 1/x$ equal to m and then find the smallest value of m for which there exists a (real) solution for x. Multiplying the equation $x + 1/x = m$ through by x gives

$$x + \frac{1}{x} = m \quad \Longrightarrow \quad x^2 + 1 = mx \quad \Longrightarrow \quad x^2 - mx + 1 = 0. \tag{8.62}$$

The discriminant in the quadratic formula is $(-m)^2 - 4 \cdot 1 \cdot 1 = m^2 - 4$. A (real) solution for x exists only if this is greater than or equal to zero. So we must have $m^2 - 4 \geq 0 \implies m^2 \geq 4 \implies m \geq 2$. (Technically $m \leq -2$ also works, but we're assuming x is positive.) So $m = 2$ is the desired minimum value of $x + 1/x$. This value is achieved when $x = -b/2a = -(-m)/(2 \cdot 1) = m/2 = 2/2 = 1$.

You can check with Desmos that the minimum of $x + 1/x$ is in fact located at the point $(1, 2)$. Any (positive) value of x different from 1 yields a value of $x + 1/x$ that is larger than 2. For example $x = 1.1$ yields a value of about 2.01, as you can verify.

Example 8.10 Two numbers add up to 10. What is the maximum value of their product?

Solution Let the two numbers be x and y. Then we know that $x + y = 10$, and we want to find the maximum value of xy. The given sum tells us that $y = 10 - x$. Plugging this into xy gives the product as $x(10 - x)$. We want to maximize this function of x, so setting it equal to m yields

$$x(10 - x) = m \quad \Longrightarrow \quad 0 = x^2 - 10x + m. \tag{8.63}$$

The discriminant is $(-10)^2 - 4 \cdot 1 \cdot m = 100 - 4m$. A (real) solution for x exists only if this is greater than or equal to zero. So we must have $100 - 4m \geq 0 \implies 100 \geq 4m \implies 25 \geq m$. The maximum value of the product is therefore 25. This value is achieved when $x = -b/2a = -(-10)/(2 \cdot 1) = 5$. So both x and y are equal to 5.

Remark: A quicker solution is to note that since the sum of the two numbers is 10, we can write them as $5 + z$ and $5 - z$, for some value of z. (No matter what the value of z is, the sum of $5 + z$ and $5 - z$ is 10.) The product of the two numbers is then the nice difference of squares, $(5 + z)(5 - z) = 5^2 - z^2 = 25 - z^2$. And since z^2 is always at least zero, we're always subtracting off at least zero from 25. The maximum value of the product is therefore 25.

This quicker method makes it clear that for any given sum S of two numbers, the maximum value of their product is achieved when the two numbers are both equal to $S/2$. And the maximum product is then $S^2/4$. This is true because the two numbers can be written as $S/2 + z$ and $S/2 - z$, so their product is $(S/2)^2 - z^2$, which is always less than or equal to $S^2/4$. This general result can also be demonstrated by replacing the 10 with S in the above discriminant solution. ♣

The discriminant method for finding the maximum or minimum applies only to quadratic functions. So if you want to find the max or min of more complicated functions, you'll need to use calculus. But the discriminant method is plenty sufficient for quadratic functions. It boils down to the observation that real solutions exist only if the discriminant is greater than or equal to zero.

Note that when finding the maximum of a function, you're finding where the function makes the transition from increasing to decreasing. Consider, for example, the parabola on the right in Fig. 8.3. If you start off on the left side and then march rightward (that is, you increase x), then the value of the function increases until you hit the maximum. And then after you pass the maximum the function decreases. Similarly, the minimum of a function is where it makes the transition from decreasing to increasing; see the left parabola in Fig. 8.3.

For an f that till now has increased,
If you want to find where the growth ceased,
A non-calc-y way
Is to cleverly say,
The discriminant's zero (at least)!

Exercise 8.27 Assuming that x is positive, what is the minimum value of $2x + 1/x$?

Exercise 8.28

(a) A rectangle has sides x and y. For a given value of the perimeter $P = 2x + 2y$, what is the maximum value of the area $A = xy$ (in terms of P)?

(b) For a given value of the area $A = xy$, what is the minimum value of the perimeter $P = 2x + 2y$ (in terms of A)?

Exercise 8.29 A cylinder has a given height h. You are free to pick the radius r to be whatever you want. For the purposes here, you can accept the following facts: The (curved) side area of the cylinder equals $A_{\text{side}} = 2\pi r \cdot h$. (This is the base circumference times the height; imagine unwrapping the cylinder into a rectangle.) And the combined area of the two circular ends equals $A_{\text{ends}} = 2 \cdot \pi r^2$.

For a given h, what is the maximum possible value of the difference $A_{\text{side}} - A_{\text{ends}}$? What value of r yields the maximum?

Exercise 8.30 The hypotenuse of a right triangle has a given length c. If one of the legs is x, then the Pythagorean theorem (see Section A.6 in Appendix A) tells us that the other leg is given by $x^2 + b^2 = c^2 \implies b = \sqrt{c^2 - x^2}$. So the product of the legs is $P = x\sqrt{c^2 - x^2}$, and the sum is $S = x + \sqrt{c^2 - x^2}$. Find the maximum values of P and S, in terms of c.

Hint: Square both sides of the above equations for P and S, to get rid of the square root. In the P case, you will end up with a quadratic equation in the variable x^2. In the S case, put the x on the lefthand side first (as we noted in Exercise 8.13).

8.5 The imaginary number i

8.5.1 Definition and properties

Let's now discuss in detail the third of the three cases in Section 8.4.1, namely the one where the discriminant is negative. But first a quick recap of where a negative discriminant comes from:

In our derivation of the quadratic formula, the square on the lefthand side of Eq. (8.26) is equal to the quantity $b^2/4a^2 - c/a$ on the righthand side. If this quantity is negative, then we're setting a square equal to a negative number. And as we saw in the steps leading to Eq. (8.30), a negative value of $b^2/4a^2 - c/a$ is equivalent to a negative value of the discriminant, $b^2 - 4ac$.

As we've noted, the square of any real number is positive (or zero), which means that we can't take the square root of a negative number (at least when restricting ourselves to real numbers). There is no real number whose square is negative. This is consistent with the top parabola in Fig. 8.6, which is the plot of the function $x^2 - 6x + 12$, or equivalently $(x-3)^2 + 3$. All points on this parabola are above the x-axis, so it never crosses the axis. That is, there is no (real) value of x that satisfies the equation $(x-3)^2 + 3 = 0 \implies (x-3)^2 = -3$, since there is no real number whose square is -3.

However, there is another type of number. Since we're at a loss to find a real number whose square is a negative number, let's simply *define* a new kind of number with this property. We hereby define the *imaginary* number i to be a number whose square equals -1. That is, i is a square root of -1:

$$\boxed{i^2 = -1 \quad \implies \quad i = \sqrt{-1}} \tag{8.64}$$

Though negatives seem to defy
Applying square roots, you should try!
It still can be done
With a negative 1,
If you use a new number called i!

You might claim that it's silly to define such a number. You can't have $5i$ apples in a bag, after all. But for that matter, you can't have -5 apples either, and we use negative numbers all the time. To be sure, imaginary numbers take a bit more getting used to than negative numbers. But they are extremely important (absolutely necessary in some cases!) in countless fields of math and science.

For the present purposes, we'll cover just enough of the basics of imaginary numbers so that we can apply the quadratic formula in the third case where the discriminant is negative. Here are some bits of information on imaginary (and complex) numbers:

1. Powers of *i*

We defined i by saying that $i^2 = -1$. What is i^3? Well, $i^3 = i^2 \cdot i = (-1)i = -i$. The next power is then $i^4 = i^3 \cdot i = (-i)i = -i^2 = -(-1) = 1$. (Or you can just say that $i^4 = (i^2)^2 = (-1)^2 = 1$.) In this manner, if you keep multiplying i by itself, you will obtain:

$$i,\ i^2,\ i^3,\ i^4,\ i^5,\ i^6,\ i^7,\ i^8, \cdots \quad \Longrightarrow \quad \boxed{i,\ -1,\ -i,\ 1}\ ,\ \boxed{i,\ -1,\ -i,\ 1}\ \cdots \tag{8.65}$$

The numbers keep repeating in groups of four, as indicated by the boxes. This is a consequence of the fact that $i^4 = 1$, which means that every additional set of four powers of i you tack on is equivalent to simply multiplying by 1.

2. Negative *i*

There is an ambiguity in our definition of i in Eq. (8.64), because there are *two* square roots of -1, namely i and $-i$, since the minus sign in $-i$ goes away when we square it:

$$(-i)^2 = \big((-1)i\big)^2 = (-1)^2 i^2 = (+1)(-1) = -1. \tag{8.66}$$

So the two square roots of -1 are $\pm i$. This $\pm$ in front of the i is the same $\pm$ we've always put in front of a square root. But now we have the issue of which of the square roots we should be calling i. Is one of the roots somehow more fundamental than the other, and hence more deserving to be called i, while the other one gets called $-i$?

The answer is no. They are equally fundamental, and it doesn't matter which one you label as i, as long as you're consistent and forever stick to your convention. If someone comes along and calls your $-i$ their i, and your i their $-i$, that's perfectly fine. There's no right or wrong choice, as long as the two of you keep your calculations separate, so that you don't use some of one convention and some of the other.

3. Imaginary numbers

The definition of an imaginary number is:

> An *imaginary* number is one whose square is a negative real number (or also zero, technically).

Equivalently, an imaginary number is one that takes the form of a real number (positive, negative, or zero) times i. Both of these definitions tell us that $-4i$ and $7i$ are imaginary numbers. These two numbers satisfy the second definition, and also the first because their squares are $(-4)^2i^2 = 16(-1) = -16$ and $7^2i^2 = 49(-1) = -49$, which are both negative.

Multiplying i by an *imaginary* number does *not* yield an imaginary number. This is true because, for example, $(3i)i = 3i^2 = -3$, and the square of this is 9, which is not a negative real number (or zero), which is the condition for being imaginary.

4. Complex numbers

A more general type of number is a *complex* number, which is defined to be any sum of a real number and an imaginary number. (Or difference, since subtraction is the same as adding the negative of something.) So the general form of a complex number is $a + bi$, where a and b are both real:

$$\boxed{\text{General complex number} = a + bi} \tag{8.67}$$

Appropriately, a is called the "real part" of the complex number, and b is called the "imaginary part." Technically, the full bi should be called the imaginary part, but the convention is to just say the b.

Examples of complex numbers are $-2 + 5i$, $7 - 8i$, $4i$, and 3. If you want, you can write the last two of these as $0 + 4i$ and $3 + 0i$. We see that imaginary numbers and real numbers are special cases of complex numbers. Imaginary numbers have $a = 0$ (so they take the form of bi), and real numbers have $b = 0$ (so they take the form of a). All imaginary numbers are complex, but not all complex numbers are imaginary (for example, $3 + 2i$ isn't imaginary). Likewise, all real numbers are complex, but not all complex numbers are real (again, $3 + 2i$ isn't real). For most complex numbers $a + bi$, neither a nor b is zero. So most complex numbers are neither imaginary nor real.

> If a student goes bonkers and tries
> To add up some reals and some i's,
> This seems really bad –
> Have they gone fully mad?
> No, it's how complex numbers arise!

REMARKS:

1. People sometimes use the words "imaginary" and "complex" as synonyms. But that is incorrect. Imaginary numbers take the form of bi, whereas complex numbers take the more general form of $a + bi$. Sometimes the phrase "pure imaginary" is used to describe a number like $3i$, and to emphasize that it has no real part. But the word "pure" isn't necessary. "Imaginary" by itself is a perfectly sufficient description of $3i$.

2. A real number like 5 is a complex number, because real numbers are a special case of complex numbers (where the imaginary part is zero). But most people wouldn't go out of their way to say that 5 is complex.

   The number 0 is both real and imaginary. It has no imaginary part, so it is certainly real (which is how we've been using it so far in this book). And it also takes the form of $0i$, which is a real number times i, which is what it means to be imaginary.

3. A helpful way of visualizing complex numbers is to plot them in the *complex plane*, which is simply the standard Cartesian x-y plane, with the x-axis representing the real part, and the y-axis representing the imaginary part. So the general complex number $a + bi$ corresponds to the point (a, b). For example, $5 + 3i$ corresponds to $(5, 3)$, and $7 - 4i$ corresponds to $(7, -4)$.

   Real numbers (which take the form of $a + 0i$) lie along the x-axis, because their points are all of the form $(a, 0)$. And imaginary numbers (which take the form of $0 + bi$) lie along the y-axis, because their points are all of the form $(0, b)$. The number $0 = 0 + 0i$ is both real and imaginary, because it corresponds to the origin $(0, 0)$, which lies on both axes. The complex plane makes it clear that most complex numbers are neither real nor imaginary, because most points in the plane aren't on the x or y axes. (For example, even the point (6, 0.000001) isn't on the x-axis.)

   The complex plane is indispensable in more advanced treatments of complex numbers. However, most of its usefulness arises from combining complex numbers with trigonometry, which isn't covered in this book. So we won't be able to appreciate the plane's many benefits here. But see the last remark in Example 8.13 for one interesting application of the complex plane.

4. The word "imaginary" is a somewhat disparaging term, because it suggests that imaginary numbers are fake, and just a fictional amusement. On the contrary, complex numbers $a+bi$ are what naturally appear in many subjects, where the real (a) and imaginary (bi) numbers are simply special cases, with one not being any "better" than the other. It would have been nice if a different term got attached to the bi numbers, but we're stuck with "imaginary."
   ♣

5. Addition and subtraction

When adding or subtracting two complex numbers, you just add or subtract the real and imaginary parts separately. They don't mix. For example,

$$(3 + 7i) + (-4 + 9i) = \big(3 + (-4)\big) + \big(7i + 9i\big) = -1 + 16i. \tag{8.68}$$

The sum of two general complex numbers is

$$\boxed{(a + bi) + (c + di) = (a + c) + (b + d)i} \tag{8.69}$$

Likewise, their difference is $(a - c) + (b - d)i$. The real parts stay separate from the imaginary parts.

6. Multiplication, FOIL

The distributive law, and hence also FOIL (since that's just a few applications of the distributive law), holds for complex numbers. For example, the distributive law tells us that (using the fact that $i^2 = -1$)

$$3i(2 + 5i) = 3i \cdot 2 + 3i \cdot 5i = 6i + 15i^2 = -15 + 6i, \tag{8.70}$$

where we have chosen to write the real part first (the usual convention), although this certainly isn't necessary. In the above steps, we used the fact that the commutative law (for both addition and multiplication) holds for real and imaginary numbers; we can switch the order. We also used the associative law to switch the groupings in the products.

In applying FOIL to the product of two complex numbers, the procedure is exactly the same as with the product of two binomials consisting of real numbers (or real letters). You just need to remember that $i^2 = -1$. For example,

$$\begin{aligned}(2 - 3i)(4 + 7i) &= 2 \cdot 4 + 2 \cdot 7i - 3i \cdot 4 - 3i \cdot 7i = 8 + 14i - 12i - 21i^2 \\ &= 8 + 2i - 21(-1) = 29 + 2i.\end{aligned} \tag{8.71}$$

For the product of two general complex numbers, FOIL gives (using $i^2 = -1$)

$$\begin{aligned}(a + bi)(c + di) &= ac + adi + bci + bdi^2 \\ \Longrightarrow \quad &\boxed{(a + bi)(c + di) = (ac - bd) + (ad + bc)i}\end{aligned} \tag{8.72}$$

For the special case of FOIL that led to the difference of squares in Eq. (3.23), the $i^2 = -1$ relation now leads to a *sum* of squares in the end:

$$\begin{aligned}(a + bi)(a - bi) &= a^2 - (bi)^2 = a^2 - b^2 i^2 = a^2 - b^2(-1) \\ \Longrightarrow \quad &\boxed{(a + bi)(a - bi) = a^2 + b^2}\end{aligned} \tag{8.73}$$

Recall Eq. (4.20), which says that $a^4 - b^4$ can be factored into $(a^2 + b^2)(a + b)(a - b)$. Eq. (8.73) now tells us that this can be further factored if we bring i into the mix:

$$a^4 - b^4 = (a^2 + b^2) \cdot (a + b)(a - b) = (a + bi)(a - bi) \cdot (a + b)(a - b). \tag{8.74}$$

Another special case of the FOIL result in Eq. (8.72) is the square of a complex number $a + bi$. Because $i^2 = -1$, the "diagonal" terms yield the difference $a^2 - b^2$ instead of the sum $a^2 + b^2$ that appeared in Eq. (3.20):

$$(a + bi)^2 = a^2 + 2abi + (bi)^2 = a^2 - b^2 + 2abi. \tag{8.75}$$

We won't need to do any division by complex numbers in this section, so we won't worry about that here.

Note that (as we've seen on a number of occasions above) the product of two imaginary numbers is real, while the product of a real and an imaginary number is imaginary. With R standing for real, and I for imaginary, the outcomes of the various possible products are:

$$\mathrm{R}\cdot\mathrm{R}=\mathrm{R},\qquad \mathrm{R}\cdot\mathrm{I}=\mathrm{I},\qquad \mathrm{I}\cdot\mathrm{R}=\mathrm{I},\qquad \mathrm{I}\cdot\mathrm{I}=\mathrm{R}. \tag{8.76}$$

Exercise 8.31 Evaluate the following four quantities:

$$(1+5i)+(3-4i),\qquad (-6+2i)-(4-3i),\qquad (-3+4i)(5-9i),\qquad (1+2i)^3$$

8.5.2 Examples

The purpose of introducing imaginary numbers (and more generally, complex numbers) in this section is to be able to work with them when they appear in the third case of the quadratic formula in Section 8.4.1, where the discriminant is negative. So let's now solve some problems that involve a negative discriminant.

Example 8.11 Solve $x^2+8x+52=0$ by (a) completing the square, and (b) using the quadratic formula.

Solution

(a) Half of 8 is 4, and the square of this is 16. So we can write the given equation as $(x^2+8x+16)+36=0$, which becomes

$$(x+4)^2=-36 \quad\Longrightarrow\quad x+4=\pm 6i \quad\Longrightarrow\quad x=-4\pm 6i. \tag{8.77}$$

(b) The quadratic formula gives

$$x=\frac{-8\pm\sqrt{8^2-4\cdot 1\cdot 52}}{2\cdot 1}=\frac{-8\pm\sqrt{-144}}{2}=\frac{-8\pm 12i}{2}=-4\pm 6i. \tag{8.78}$$

We can use Eq. (8.14) to check our $-4+6i$ and $-4-6i$ solutions. Their sum is -8, which is correctly the negative of the coefficient of x. And their product is

$$(-4+6i)(-4-6i)=(-4)^2-(6i)^2=16-(-36)=52, \tag{8.79}$$

which is correctly the constant term.

If you want to write out the factorization of the given function $x^2+8x+52$, the above roots tell us that it is

$$\big(x-(-4+6i)\big)\big(x-(-4-6i)\big)=\big(x+4-6i\big)\big(x+4+6i\big). \tag{8.80}$$

Although the given function isn't factorable if you use only real numbers, it factors just fine if you allow imaginary (and hence complex) numbers.

Example 8.12 We saw in Example 8.10 that if two real numbers add up to 10, the maximum value of their product is 25. Find two complex numbers whose sum is 10 and whose product is 30.

Solution We want to find two numbers x and y such that $x + y = 10$ and $xy = 30$. The first equation gives $y = 10 - x$. Plugging this into the second equation yields $x(10 - x) = 30 \implies 0 = x^2 - 10x + 30$. (We could also have just written down this equation without doing any work, by using Eq. (8.14) along with the given sum of 10 and product of 30.) The quadratic formula then gives

$$x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4 \cdot 30}}{2} = \frac{10 \pm \sqrt{-20}}{2} = \frac{10 \pm 2\sqrt{5}i}{2} = 5 \pm \sqrt{5}i. \qquad (8.81)$$

These are the desired two numbers. More precisely, if we pick the $x = 5+\sqrt{5}i$ root, then $y = 10 - x = 5 - \sqrt{5}i$. And if we pick the $x = 5 - \sqrt{5}i$ root, then $y = 10 - x = 5 + \sqrt{5}i$. As a check, Eq. (8.73) correctly gives the product as $5^2 + (\sqrt{5})^2 = 30$.

Example 8.13 What are the cube roots of 1? That is, what values of x satisfy $x^3 = 1$, or equivalently $x^3 - 1 = 0$? An obvious root is $x = 1$. And it's fairly clear that there are no other real roots, because if they're real, they must be positive (so that their cube is positive), but they can't be larger or smaller than 1, if we want their cube to equal 1. So 1 is the only real root. However, there are two other *complex* roots. Find these roots.

Solution Our strategy will be to factor out the $x - 1$ factor (associated with the $x = 1$ root) from $x^3 - 1$, and to then use the quadratic formula on the resulting quadratic polynomial.

The first result in Example 3.7 (with $a \to x$ and $b \to 1$) gives the factoring of $x^3 - 1$:

$$x^3 - 1 = 0 \quad \implies \quad (x - 1)(x^2 + x + 1) = 0. \qquad (8.82)$$

If we set the second factor here equal to zero, the quadratic formula gives the roots of the $x^2 + x + 1 = 0$ equation as

$$x = \frac{-1 \pm \sqrt{1 - 4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i. \qquad (8.83)$$

These are the desired two complex cube roots of 1. As a check, their sum is -1, which is correctly the negative of the coefficient of x in $x^2 + x + 1$. And Eq. (8.73) gives their product as $(-1/2)^2 + (\sqrt{3}/2)^2 = 1$, which is correctly the constant term (the 1).

There are therefore *three* cube roots of 1, namely the real $x = 1$ root, along with the two complex roots in Eq. (8.83).

Remarks:

1. If you want to explicitly show that the cube of the roots in Eq. (8.83) equals 1, you can first square them (we'll do this just for the "+" root here). Using Eq. (8.75), we have

$$\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4}-\frac{3}{4}-2\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2}i = -\frac{1}{2}-\frac{\sqrt{3}}{2}i. \tag{8.84}$$

Multiplying this result by the "+" root again, to obtain the third power, gives (using the sum-of-squares result in Eq. (8.73))

$$\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) = \left(-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2 = 1, \tag{8.85}$$

as desired. The calculation for the "−" root proceeds in a similar manner, as you can check.

2. There was nothing special about the number 1 in this example. We could just as well have asked for the cube roots of 8 or −13 or 29/11 or anything. The method is nearly the same; Eq. (8.82) now looks like

$$x^3-a^3=0 \quad\Longrightarrow\quad (x-a)(x^2+ax+a^2)=0, \tag{8.86}$$

where a is the real cube root of the number in question. (So for 8 we have $a = 8^{1/3} = 2$, and for −13 we have $a = (-13)^{1/3} \approx -2.35$.) The task is therefore to solve the quadratic equation $x^2 + ax + a^2 = 0$. This follows exactly the same quadratic-formula procedure as in Eq. (8.83), with some additional a's thrown in. The result is simply a times the roots in Eq. (8.83). Finding the two complex roots of a general number therefore isn't any more difficult than finding them for 1. So in the end, all numbers except 0 have three cube roots. This is true for all complex numbers, even though we assumed a was real in the above reasoning.

3. We have found that there are three cube roots of 1 (or any other number). And we know that there are two square roots of 1, namely ±1. How many fourth roots of 1 are there? Well, you can quickly show that the fourth powers of the four numbers 1, −1, i, and $-i$ are all equal to 1. So we see that there are two square roots, three cube roots, and four fourth roots of 1. There definitely seems to be a pattern here! And indeed, it can be shown that the number of nth roots of 1 (or more generally any number except 0) equals n. However, some trigonometry is needed for this, so you'll need to wait until you learn some of that.

   If you want a couple more data points, you can take the square-root technique introduced below in Example 8.14(b) and apply it to each of the cube roots of 1 we found above. This will give you a total of six 6th roots of 1 (including 1 and −1). Similarly, if you repeat Exercise 8.35 for the square roots of $-i$, you will end up with a total of eight 8th roots of 1 (including 1, −1, i, $-i$, along with the two square roots of i).

4. We briefly discussed the complex plane in the third remark on page 513. The points corresponding to the two complex cube roots of 1 in Eq. (8.83) are $(-1/2, \sqrt{3}/2)$ and $(-1/2, -\sqrt{3}/2)$. And the real cube root (which is just 1) is simply the point $(1, 0)$. If you plot these three points in the complex plane, you will discover that they have a very nice relation to one another. (You'll need to use a result from the 30-60-90 triangle in Section A.7 in Appendix A.) You can make a general conjecture about the relation, and then you can plot the nth roots of 1 (there are always n of them) for $n = 2, 4, 6, 8$, to obtain further evidence for your conjecture. Its proof requires trigonometry, which is yet another reason to look forward to learning about that! ♣

Example 8.14 Find the square roots of $16+30i$. That is, find the real numbers x and y that satisfy $(x+yi)^2 = 16+30i$. To do this, equate the real parts of the two sides of this equation, and likewise the imaginary parts. This will produce a system of equations. Solve this system by:

(a) Trial and error (the numbers were chosen nicely so that the solutions are integers).

(b) Formally solving the system by eliminating y and solving for x. You will need to solve a quadratic equation in the variable x^2.

Solution

(a) As in Eq. (8.75), the square of $x+yi$ equals $x^2 - y^2 + 2xyi$. So matching up the real parts, and also the imaginary parts, on the two sides of $(x+yi)^2 = 16+30i$ gives

$$x^2 - y^2 = 16 \qquad \text{and} \qquad 2xy = 30 \implies xy = 15. \tag{8.87}$$

Note that the real and imaginary parts must indeed match up independently. The two parts stay separate and don't mix.

Assuming (correctly) that things have been arranged so that x and y are integers, we quickly observe that $x = 5$ and $y = 3$ satisfy both $x^2 - y^2 = 16$ and $xy = 15$. So we have found our solution. Or at least one of them. The numbers -5 and -3 also work, so the two square roots of $16 + 30i$ are $\pm(5 + 3i)$.

As a check, we can square these. The "±" doesn't matter when squaring, so Eq. (8.75) gives $(5 + 3i)^2 = 5^2 - 3^2 + 2\cdot 5\cdot 3i = 16 + 30i$, as desired.

(b) To formally solve the system of equations in Eq. (8.87), we can solve for y in the second equation to obtain $y = 15/x$, and then we can plug this into the first equation. The result is

$$x^2 - \frac{15^2}{x^2} = 16 \implies x^4 - 16x^2 - 225 = 0. \tag{8.88}$$

Although this might seem like a quartic (4th-power) equation, it's really only a quadratic equation in the variable x^2, since $x^4 = (x^2)^2$. So the quadratic formula gives

$$x^2 = \frac{-(-16) \pm \sqrt{(-16)^2 - 4(1)(-225)}}{2} = \frac{16 \pm \sqrt{1156}}{2} = \frac{16 \pm 34}{2}. \qquad (8.89)$$

Since x^2 is positive (because x is assumed to be real), we must pick the "+" sign. So we obtain $x^2 = (16 + 34)/2 = 25$, and hence $x = \pm 5$. The second equation in Eq. (8.87) then gives $y = \pm 3$ (with the sign being the same as the sign of x, because the product is the positive number $xy = 15$). We therefore obtain the two answers of $\pm(5 + 3i)$, in agreement with the result in part (a).

Alternatively, instead of using the quadratic formula, you can factor the quadratic equation in Eq. (8.88) into $(x^2 - 25)(x^2 + 9) = 0$. The positive root of $x^2 = 25$ is the one we want.

You should verify that if you instead solve for x in terms of y in Eq. (8.87), thereby obtaining a quadratic equation in y^2, you will obtain the same answers.

REMARK: The "−" root in Eq. (8.89) actually also leads to the correct answers. That root is $x^2 = (16-34)/2 = -9$. So $x = \pm 3i$. The second equation in Eq. (8.87) then gives $y = \mp 5i$ (with the sign being opposite to the sign of x, so that the product equals +15). Although these results violate our assumption that x and y are real, they still give the correct answers in the end because

$$x + yi = \pm 3i + (\mp 5i)i = \pm 3i \mp 5(-1) = \pm 5 \pm 3i, \qquad (8.90)$$

as desired. We obtain the same answers; it's just that x now yields the imaginary part, and y yields the real part. ♣

Exercise 8.32 Find two complex numbers whose sum is 2 and whose product is 10.

Exercise 8.33 In the spirit of Example 8.13, find the cube roots of −1.

Exercise 8.34 Find the square roots of $5 - 12i$, via the same two methods as in Example 8.14.

Exercise 8.35 Find the square roots of i.

8.6 Exercise solutions

1. (a) FACTORING: We want to find two numbers that multiply to -112 and add to -6. A little fiddling (by listing out the factors of 112) gives 8 and -14. So the factorization is $(x+8)(x-14)$, which means that the roots are -8 and 14.

   COMPLETING THE SQUARE: Half of -6 is -3, and the square of this is 9. So we'll add 9 to both sides to complete the square. And then we'll add 112 to both sides to remove that from the lefthand side. (Equivalently, we can just add 121 to both sides in one step.) The result is

$$x^2 - 6x + 9 = 121 \implies (x-3)^2 = 11^2 \implies x - 3 = \pm 11. \tag{8.91}$$

   The two roots are therefore $x = 3 \pm 11$, which yields 14 and -8, in agreement with the solutions we found above by factoring.

   (b) We'll first divide by 3 to obtain $x^2 + 8x - 33 = 0$.

   FACTORING: We want to find two numbers that multiply to -33 and add to 8. We quickly obtain 11 and -3. So the factorization is $(x+11)(x-3)$, and the roots are -11 and 3.

   COMPLETING THE SQUARE: Half of 8 is 4, and the square of this is 16. Adding 16 to both sides to complete the square, and then adding 33 to both sides (or just adding 49 all at once), gives

$$x^2 + 8x + 16 = 49 \implies (x+4)^2 = 7^2 \implies x + 4 = \pm 7. \tag{8.92}$$

   So the roots are $x = -4 \pm 7$, which yields 3 and -11, as above.

   (c) We'll first divide by 2 to obtain $x^2 + 10x + 18.75 = 0$.

   FACTORING: Let's rewrite the equation as $x^2 + 10x + 75/4 = 0$, since it's usually easier to work with fractions than with decimals when factoring. We want to find two numbers that multiply to $75/4$ and add to 10. A little guessing and checking gives $5/2$ and $15/2$. So the factorization is $(x + 5/2)(x + 15/2)$, and the roots are $-5/2$ and $-15/2$.

   COMPLETING THE SQUARE: Half of 10 is 5, and the square of this is 25. Adding 25 to both sides to complete the square, and then subtracting $75/4$ from both sides (or just adding $25 - 75/4 = 25/4$ all at once), gives

$$x^2 + 10x + 25 = 25/4 \implies (x+5)^2 = (5/2)^2 \implies x + 5 = \pm 5/2. \tag{8.93}$$

   So the roots are $x = -5 \pm 5/2$, which yields $-5/2$ and $-15/2$, as above.

   (d) The numbers weren't chosen nicely in this equation, so we're not going to guess the factoring. To complete the square, half of -8 is -4, and the square of this is 16. Adding 16 to both sides, and then also adding 5 to both sides (or just adding 21 all at once), gives

$$x^2 - 8x + 16 = 21 \implies (x-4)^2 = 21 \implies x - 4 = \pm\sqrt{21}. \tag{8.94}$$

   So the roots are $x = 4 \pm \sqrt{21}$.

(e) FACTORING: Since there is no constant term in the equation, we quickly see that x is a factor. The factorization is therefore $x(x-2a)$, which means that the roots are 0 and $2a$.

COMPLETING THE SQUARE: Half of $-2a$ is $-a$, and the square of this is a^2. Adding a^2 to both sides to complete the square gives

$$x^2 - (2a)x + a^2 = a^2 \implies (x-a)^2 = a^2 \implies x - a = \pm a. \tag{8.95}$$

So the roots are $x = a \pm a$, which yields $2a$ and 0, as above.

2. Subtracting 4 from both sides of $(x-3)^2 = 4$ and then using Eq. (3.24) to factor the result gives (writing the 4 as 2^2)

$$(x-3)^2 - 2^2 = 0 \implies \big((x-3)+2\big)\big((x-3)-2\big) = 0 \implies (x-1)(x-5) = 0. \tag{8.96}$$

The roots are therefore 1 and 5, as we found in Eq. (8.4).

REMARK: The middle equation in Eq. (8.96) can be rewritten as $\big(x-(3-2)\big)\big(x-(3+2)\big) = 0$. The roots are therefore 3 ± 2, which is exactly what Eq. (8.4) says. So in the end, the method in this exercise (factoring via Eq. (3.24)) is equivalent to the method in Eq. (8.4) (taking the square root of both sides). Either strategy can be used to solve quadratic equations. But the square-root method is a little quicker, so we'll generally use that. In any case, if you're given an equation like Eq. (8.6), both methods require completing the square to obtain the $(x-3)^2 = 4$ form in the first place. That's the important step. ♣

3. If we expand $(x+m)^2 = n$ and put the n on the left, we obtain $x^2+(2m)x+(m^2-n) = 0$. This agrees with the given $x^2 - 6x - 112 = 0$ equation if $2m = -6$ and $m^2 - n = -112$. This is technically a system of equations, but it's an easy one. The $2m = -6$ equation immediately gives $m = -3$. And then $m^2 - n = -112$ gives $n = m^2 + 112 = (-3)^2 + 112 = 121$. So the desired rewritten form is $(x-3)^2 = 121$, in agreement with the result in Exercise 8.1(a).

REMARK: In the end, this formal "system of equations" method is the same as our more informal method in the text (steps 2, 3, and 4 in the list on page 481). In both cases, the m value is half the coefficient of x in the original equation. And as we saw in Exercise 8.1(a), completing the square entails adding $(-3)^2$, and also 112, to both sides, which gives $x^2 - 6x + (-3)^3 = (-3)^2 + 112$. The righthand side here agrees with the n we found above. Bottom line: The formal and informal methods for completing the square accomplish the same thing in the end. But the informal method is quicker, once you use it a few times and get the hang of it. ♣

4. Expanding the given product yields $x^2 - (r+s)x + rs = 0$. To complete the square, we must square half of $r+s$ and add that to both sides, and then subtract rs from both sides. The result is

$$x^2 - (r+s)x + \left(\frac{r+s}{2}\right)^2 = \left(\frac{r+s}{2}\right)^2 - rs. \tag{8.97}$$

We now note that the righthand side can be written as

$$\left(\frac{r+s}{2}\right)^2 - rs = \frac{r^2+2rs+s^2}{4} - \frac{4rs}{4} = \frac{r^2-2rs+s^2}{4} = \left(\frac{r-s}{2}\right)^2. \tag{8.98}$$

Replacing the righthand side of Eq. (8.97) with this expression, and rewriting the lefthand side in its square form, yields

$$\left(x - \frac{r+s}{2}\right)^2 = \left(\frac{r-s}{2}\right)^2 \implies x - \frac{r+s}{2} = \pm\frac{r-s}{2} \implies x = \frac{r+s}{2} \pm \frac{r-s}{2}. \tag{8.99}$$

The "+" root equals r, and the "−" root equals s, as you can quickly verify. So we have successfully reproduced the $x = r$ and $x = s$ solutions. This exercise was a bit silly, of course, because we already knew what the solutions were. But it's good to check that everything does in fact work out, even if you take a roundabout and inefficient route!

5. (a) Following the completing-the-square strategy in Eq. (8.20), we have

$$\begin{aligned} f(x) &= 2(x^2 - 10x + 16) = 2(x^2 - 10x + 25 - 25 + 16) \\ &= 2(x^2 - 10x + 25 - 9) = 2(x-5)^2 - 18. \end{aligned} \tag{8.100}$$

The minimum value of the function is therefore −18, and it occurs at $x = 5$. So the bottom of the parabola is located at the point $(5, -18)$.

Setting the function equal to zero gives

$$2(x-5)^2 - 18 = 0 \implies (x-5)^2 = 9 \implies x - 5 = \pm 3, \tag{8.101}$$

which yields roots of $x = 5 + 3 = 8$ and $x = 5 - 3 = 2$. The parabola therefore passes through the three points $(5, -18)$, $(2, 0)$, and $(8, 0)$. You can check your sketch in Desmos.

(b) In the same manner, we have

$$\begin{aligned} f(x) &= -3(x^2 - 6x + 5) = -3(x^2 - 6x + 9 - 9 + 5) \\ &= -3(x^2 - 6x + 9 - 4) = -3(x-3)^2 + 12. \end{aligned} \tag{8.102}$$

Due to the negative coefficient −3 in front of the square, the *maximum* value of the function is 12, and it occurs at $x = 3$. So the *top* of the parabola is located at the point $(3, 12)$.

Setting the function equal to zero gives

$$-3(x-3)^2 + 12 = 0 \implies (x-3)^2 = 4 \implies x - 3 = \pm 2, \tag{8.103}$$

which yields roots of $x = 3 + 2 = 5$ and $x = 3 - 2 = 1$. The parabola therefore passes through the three points $(3, 12)$, $(1, 0)$, and $(5, 0)$. The parabola opens downward, since the coefficient of x^2 in the function is negative.

6. The solutions below are very similar to the ones in Example 8.2, so we'll be terse here.

(a) FACTORING: Since $x^2 + x - 20$ factors into $(x+5)(x-4)$, the roots are -5 and 4.

COMPLETING THE SQUARE: Half of 1 is $1/2$, and the square of this is $1/4$. So completing the square gives

$$x^2 + x + \frac{1}{4} = 20 + \frac{1}{4} \implies \left(x + \frac{1}{2}\right)^2 = \frac{81}{4} \implies x + \frac{1}{2} = \pm\frac{9}{2}. \tag{8.104}$$

The roots are therefore $x = -1/2 + 9/2 = 4$, and $x = -1/2 - 9/2 = -5$.

QUADRATIC FORMULA: With $a = 1$, $b = 1$, and $c = -20$, Eq. (8.30) gives

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-20)}}{2 \cdot 1} = \frac{-1 \pm \sqrt{81}}{2} = \frac{-1 \pm 9}{2}. \tag{8.105}$$

So the roots are $(-1+9)/2 = 4$ and $(-1-9)/2 = -5$.

PLOTTING: If you plot the function $f(x) = x^2 + x - 20$ in Desmos, you will see that the parabola crosses the x-axis at $x = -5$ and $x = 4$. So those are the roots.

Additionally, the above completion of the square tells us that (if we put everything on the lefthand side of the equation) $f(x)$ can be written as $(x + 1/2)^2 - 81/4$. So the minimum value is $-81/4 = -20.25$, and it occurs at $x = -1/2$. This should agree with your plot (you can zoom in to check).

(b) FACTORING: Since $2x^2 - 4x - 6$ factors into $2(x^2 - 2x - 3) = 2(x+1)(x-3)$, the roots are -1 and 3.

COMPLETING THE SQUARE: We can divide the given equation by 2 to obtain $x^2 - 2x - 3 = 0$. Half of -2 is -1, and the square of this is 1. So completing the square gives

$$x^2 - 2x + 1 = 3 + 1 \implies (x-1)^2 = 4 \implies x - 1 = \pm 2. \tag{8.106}$$

The roots are therefore $x = 1 + 2 = 3$, and $x = 1 - 2 = -1$.

QUADRATIC FORMULA: We could (and probably should) divide by 2 to make the coefficients smaller, but let's just use the original ones: $a = 2$, $b = -4$, and $c = -6$. Eq. (8.30) then gives

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(-6)}}{2 \cdot 2} = \frac{4 \pm \sqrt{64}}{4} = \frac{4 \pm 8}{4}. \tag{8.107}$$

So the roots are $(4+8)/4 = 3$ and $(4-8)/4 = -1$.

PLOTTING: If you plot the function $f(x) = 2x^2 - 4x - 6$ in Desmos, you will see that the parabola crosses the x-axis at $x = -1$ and $x = 3$. So those are the roots.

Additionally, the above completion of the square tells us that (with the overall factor of 2 brought back in) $f(x)$ can be written as $2\big((x-1)^2 - 4\big) = 2(x-1)^2 - 8$. So the minimum value is -8, and it occurs at $x = 1$.

7. Due to the "±" in Eq. (8.30), the square-root term cancels when we add the two roots. The sum is therefore $2(-b/2a) = -b/a$, as desired. In short, because of the "±," the two roots are equally spaced on either side of $-b/2a$. So $-b/2a$ is their average, which means $-b/a$ is their sum.

   Also due to the "±" in Eq. (8.30), the product of the roots yields a nice difference of squares which simplifies greatly:

$$\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right) = \frac{(-b)^2-\left(\sqrt{b^2-4ac}\right)^2}{4a^2} \qquad (8.108)$$
$$= \frac{b^2-(b^2-4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}\,,$$

   as desired.

8. If $b = 0$, Eq. (8.30) reduces to (bringing the $2a$ inside the radical)

$$x = \frac{-0 \pm \sqrt{0^2-4ac}}{2a} = \pm\sqrt{\frac{-4ac}{4a^2}} = \pm\sqrt{\frac{-c}{a}}\,. \qquad (8.109)$$

   This is correct because if $b = 0$, the original quadratic equation $ax^2 + bx + c = 0$ becomes $ax^2 + c = 0 \implies x^2 = -c/a \implies x = \pm\sqrt{-c/a}$. Note that one of a or c needs to be negative if the roots are to be real, as opposed to imaginary. (See Section 8.5 for a discussion of imaginary numbers.)

   If $c = 0$, Eq. (8.30) reduces to

$$x = \frac{-b \pm \sqrt{b^2-0}}{2a} = \frac{-b \pm b}{2a}\,. \qquad (8.110)$$

   The numerator here is either 0 or $-2b$, so the two roots are 0 and $-b/a$. This is correct because if $c = 0$, the original quadratic equation $ax^2 + bx + c = 0$ becomes $ax^2 + bx = 0 \implies x(ax + b) = 0$. This has roots of $x = 0$, and $ax + b = 0 \implies x = -b/a$.

   REMARK: Technically we should have written $\sqrt{b^2} = |b|$ in Eq. (8.110), since the $\sqrt{\ }$ operation is defined to be positive. But this isn't necessary, for the following reason. There are two cases for the absolute value, depending on the sign of b. If $b > 0$ then $|b| = b$, and the above results hold. On the other hand, if $b < 0$ then $|b| = -b$. But the "±" in the quadratic formula makes the minus sign here irrelevant, because the numerator in Eq. (8.110) becomes $-b \mp b$. The two signs, "∓" and "±," give the same two roots in the end. ♣

9. Plugging the new coefficients into Eq. (8.30) gives

$$\frac{-mb \pm \sqrt{(mb)^2-4(ma)(mc)}}{2(ma)} = \frac{-mb \pm \sqrt{m^2b^2-4m^2ac}}{2ma}$$
$$= \frac{-mb \pm m\sqrt{b^2-4ac}}{2ma} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}\,, \qquad (8.111)$$

as desired. The $\sqrt{m^2}$ here should technically be written as $|m|$. But as we noted in the remark in the solution to the preceding exercise, this doesn't affect the final result.

10. The first equation gives $y = -b/a - x$. Plugging this into the second equation yields

$$x\left(-\frac{b}{a} - x\right) = \frac{c}{a} \quad \Longrightarrow \quad -\frac{b}{a}x - x^2 = \frac{c}{a}. \tag{8.112}$$

Multiplying through by a and putting everything on the righthand side gives $0 = ax^2+bx+c$, as desired.

11. Our goal is to find all numbers x with the property that $1/x$ equals $x + 1$. So we want

$$\frac{1}{x} = x + 1 \quad \Longrightarrow \quad 1 = x^2 + x \quad \Longrightarrow \quad 0 = x^2 + x - 1. \tag{8.113}$$

The quadratic formula gives

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2 \cdot 1} = \frac{-1 \pm \sqrt{5}}{2}. \tag{8.114}$$

The two roots are therefore $(-1 + \sqrt{5})/2 \approx 0.618$ and $(-1 - \sqrt{5})/2 \approx -1.618$. Both of these numbers are solutions to the given problem, as you should verify.

REMARKS:

1. The above roots are the negatives of the ones in Example 8.5. So the present roots have the same product as the ones in Example 8.5 (since $(-1)^2 = 1$), but the opposite sum (the negative of it). This is consistent with Eq. (8.15), because the only difference between the quadratic equations in the two problems is the sign of the middle term (the b term) in the equations in Eqs. (8.40) and (8.113).
2. Another way of comparing the roots in the two problems is to note that since the "+1" in the numerator of Eq. (8.41) becomes a "−1" in Eq. (8.114), we have subtracted 2 from the numerator. And then because of the 2 in the denominator, we see that the roots in the present problem are 1 less than the roots in Example 8.5. The full collection of roots in the two problems is: $(\pm 1 \pm \sqrt{5})/2$.
3. From our discussion in the second remark in the solution to Example 8.5, we actually already knew that 0.618 is a solution to the present problem, because we noted that the reciprocal of 0.618 is 1.618, which is 1 more than 0.618. ♣

12. For general x values, the y values (namely $2x + 2$ and $x^2 - 2x + 5$) of the associated points on the line and parabola aren't equal. But for some special x values, they are. These are the intersection points. We therefore want to find the x values for which

$$x^2 - 2x + 5 = 2x + 2 \quad \Longrightarrow \quad x^2 - 4x + 3 = 0. \tag{8.115}$$

The quadratic formula gives

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 3}}{2 \cdot 1} = \frac{4 \pm \sqrt{4}}{2} = 3 \text{ or } 1. \tag{8.116}$$

Alternatively, you can factor the quadratic equation into $(x-1)(x-3) = 0$, which yields roots of 1 and 3. Plugging these x values back into either (or both, as a double check) of the equations for the line or parabola tells you that the intersections points are $(1, 4)$ and $(3, 8)$. You can verify these results by plotting the line and parabola in Desmos.

13. (a) Putting the x on the righthand side and then squaring gives

$$\begin{aligned} \sqrt{2x-1} = 8 - x \quad &\Longrightarrow \quad 2x - 1 = (8-x)^2 \\ \Longrightarrow \quad 2x - 1 = 64 - 16x + x^2 \quad &\Longrightarrow \quad 0 = x^2 - 18x + 65. \end{aligned} \tag{8.117}$$

A little fiddling will tell you that this equation factors into $(x-5)(x-13) = 0$. So the roots are 5 and 13. For practice you should verify that the quadratic formula also produces these roots.

However, only one of the $x = 5$ and 13 roots is actually a solution to the given $x + \sqrt{2x-1} = 8$ equation. $x = 5$ yields $5 + \sqrt{9} = 8$, which is true. But 13 yields $13 + \sqrt{25} = 8$, which is *not* true. So $x = 5$ is the only solution.

The invalid $x = 13$ root was introduced in the squaring operation in Eq. (8.117). If $x = 13$, the first equation in Eq. (8.117) says that $\sqrt{25} = 8 - 13$, or equivalently $\sqrt{25} = -5$. This isn't true, because the $\sqrt{\ }$ operation is defined to be the positive square root. But when we square both sides of the first equation in Eq. (8.117) (when $x = 13$) to obtain the second, we obtain $25 = (-5)^2$, which *is* true. This is why 13 shows up as a solution to the eventual quadratic equation, even though it isn't a solution to the original equation.

If someone long ago had defined the $\sqrt{\ }$ operation to instead be the *negative* option of the two square roots, then 13 would be a solution to the given equation, while 5 would not.

REMARK: For equations like the one we just solved, isolating the square root on one side of the equation, and then squaring both sides, is the standard technique for getting rid of the square root, which is necessary if you want to solve for x (which means isolating x on one side). However, the squaring operation sometimes introduces roots that aren't actually solutions to the given equation. So you always need to plug your answers for x back in, to see if they are in fact solutions.

As we mentioned back on page 231, it's always a good idea to plug your solution for x back into the given equation, as a check against algebra mistakes. But for cases that involve squaring operations like the one above, plugging your solutions back in is more than just a guard against algebra mistakes. It's a necessary step. Even if you did everything correctly, it still might be the case that one of your roots isn't actually a solution to the given equation. ♣

(b) Putting the $\sqrt{x-2}$ on the righthand side (the $\sqrt{2x+3}$ would work fine too) and then squaring gives

$$\sqrt{2x+3} = 2 + \sqrt{x-2} \implies 2x+3 = \left(2+\sqrt{x-2}\right)^2$$
$$\implies 2x+3 = 4 + 4\sqrt{x-2} + x - 2 \implies x+1 = 4\sqrt{x-2}. \qquad (8.118)$$

We now need to square both sides again, to get rid of the remaining square root. This yields

$$x^2 + 2x + 1 = 16(x-2) \implies x^2 - 14x + 33 = 0. \qquad (8.119)$$

A little fiddling tells you that this equation factors into $(x-3)(x-11) = 0$. So the roots are 3 and 11. You should again verify that you obtain the same roots via the quadratic formula. Note that the present equation required two squaring operations, whereas the equation in part (a) required only one.

When you plug $x = 3$ and 11 back into the given equation, you'll find that they both satisfy the equation. So they're both solutions. The squaring operation didn't produce any invalid roots in this case.

You can use Desmos to plot the two functions on the lefthand sides of the given equations in parts (a) and (b) of this exercise. (In Desmos, the $\sqrt{\ }$ symbol is obtained by typing "sqrt." Alternatively, you can raise a quantity to the $1/2$ power.) If you look at where the functions are equal to 8 or 2, it will be clear why the first equation has only one solution, while the second has two.

14. Let Bob's glass hold x cups. Then Alice's glass holds $x + 1/2$ cups. The number of Bob's glassfuls it takes to make 80 cups is the number of times that x fits into 80, which is simply the fraction $80/x$ (from the first interpretation of fractions in Section 1.11). Similarly, the number of Alice's glassfuls it takes to make 80 cups is $80/(x + 1/2)$. We are told that the sum of these two numbers is 72, so

$$\frac{80}{x} + \frac{80}{x + 1/2} = 72. \qquad (8.120)$$

Multiplying through by the common denominator $x(x + 1/2)$ yields

$$\begin{aligned} 80\cdot(x+1/2) + 80\cdot x &= 72\cdot x(x+1/2) \\ \implies (80x+40) + 80x &= 72x^2 + 36x \\ \implies 0 &= 72x^2 - 124x - 40 \\ \implies 0 &= 18x^2 - 31x - 10. \end{aligned} \qquad (8.121)$$

The quadratic formula then gives

$$x = \frac{-(-31) \pm \sqrt{(-31)^2 - 4(18)(-10)}}{2\cdot 18} = \frac{31 \pm \sqrt{1681}}{36}. \qquad (8.122)$$

Since $\sqrt{1681} = 41$, and since we must pick the "+" sign because x is positive, we obtain $x = (31 + 41)/36 = 2$. So Bob's glass holds 2 cups, which means Alice's glass holds 2.5 cups. As a quick check, the two numbers of glassfuls are $80/2 = 40$ and $80/2.5 = 32$, and these correctly add up to 72.

REMARK: Under the (correct) assumption that the numbers in this problem were chosen nicely, you can obtain the answers of 2 and 2.5 by trial and error. If the amounts were hypothetically both 2 cups, then $80/2 + 80/2 = 40 + 40 = 80$ glassfuls would be needed, which is in the ballpark of 72. So the answers must be somewhere around 2. Therefore, because Alice's glass is half a cup larger, some (nice) possibilities for the two numbers are 1.5 and 2, or 2 and 2.5, or 2.5 and 3. Since you want to decrease the total number of glassfuls slightly from 80 to 72, the 2-and-2.5 option is a good one to try first. And it works. However, if the numbers weren't chosen nicely, there would be no way around solving a quadratic equation, as we did above. ♣

15. Starting with the quadratic formula, the steps that bring you back to the original quadratic equation are: multiplying by $2a$, adding b, squaring, expanding, simplifying, and dividing by $4a$:

$$
\begin{aligned}
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad &\Longrightarrow \quad 2ax = -b \pm \sqrt{b^2 - 4ac} \\
\Longrightarrow \quad 2ax + b = \pm\sqrt{b^2 - 4ac} \quad &\Longrightarrow \quad (2ax + b)^2 = b^2 - 4ac \\
\Longrightarrow \quad 4a^2x^2 + 4axb + \cancel{b^2} = \cancel{b^2} - 4ac \quad &\Longrightarrow \quad 4a^2x^2 + 4abx + 4ac = 0 \\
&\Longrightarrow \quad ax^2 + bx + c = 0.
\end{aligned} \tag{8.123}
$$

REMARK: If you proceed through this sequence of equations in reverse (starting with $ax^2 + bx + c = 0$ and ending with $x = (-b \pm \sqrt{b^2 - 4ac}\,)/2a$), the steps constitute a new derivation of the quadratic formula. This new derivation is a bit cleaner than the one in Eqs. (8.24) through (8.30) in the text, which involved lots of fractions. However, the first step of Eq. (8.123) in reverse is multiplication by $4a$. This generates the terms $4a^2x^2 + 4abx$, which are nicely the first two terms in the perfect square $(2ax + b)^2$, which you can then complete by adding b^2. The downside of this method is that the first step of multiplying by $4a$ might seem a little out of the blue. None of the steps in the standard (but messier) derivation in the text were "tricky" like this.

Note that the reverse of the steps that led to Eq. (8.30) in the text is another way (different from the steps in this exercise) to get back to the original $ax^2 + bx + c = 0$ equation. Starting with Eq. (8.30), the steps are (as you can check): split up the fraction, absorb the $2a$ into the square root, split up the fraction inside the square root, add $b/2a$, square, expand, simplify, and then multiply by a. ♣

16. Factoring an a out of the first two terms in $ax^2 + bx + c = 0$ gives $a\big(x^2 + (b/a)x\big) + c = 0$. To generate a square inside the parentheses, we take half of the coefficient of x (so $b/2a$) and square it to obtain $b^2/4a^2$. We put this inside the parentheses. But due to the factor of a out front, we have actually added $a \cdot b^2/4a^2 = b^2/4a$ to the lefthand side. So we must also add this to the righthand side. If we also subtract c from both sides, we obtain

$$
a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) = \frac{b^2}{4a} - c \quad \Longrightarrow \quad a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - c. \tag{8.124}
$$

Dividing by a then gives

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}, \tag{8.125}$$

in agreement with Eq. (8.26). The derivation proceeds as in the text following Eq. (8.26).

17. Since the two roots take the form of $-b/2a \pm z$, their product (which we know is c/a) yields a difference of squares. So

$$\left(-\frac{b}{2a} + z\right)\left(-\frac{b}{2a} - z\right) = \frac{c}{a} \implies \left(-\frac{b}{2a}\right)^2 - z^2 = \frac{c}{a}$$
$$\implies \frac{b^2}{4a^2} - \frac{c}{a} = z^2 \implies z = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}. \tag{8.126}$$

The $-b/2a \pm z$ form of the roots therefore reproduces Eq. (8.29) and hence Eq. (8.30).

18. (a) The first fraction equals $1/2$. Plugging this result into the second fraction gives a value of $1/(1+1/2) = 1/(3/2) = 2/3$. Plugging this result into the third fraction then gives $1/(1+2/3) = 1/(5/3) = 3/5$. Plugging this result into the fourth fraction then gives $1/(1+3/5) = 1/(8/5) = 5/8$.

Continuing in this manner, if we add on another level of fractions, we obtain $1/(1 + 5/8) = 1/(13/8) = 8/13$. The next level yields $1/(1+8/13) = 1/(21/13) = 13/21$, which equals 0.619. And then the next one yields $1/(1 + 13/21) = 1/(34/21) = 21/34 = 0.6176$. If you want to keep going, the next level gives $1/(1+21/34) = 1/(55/34) = 34/55 = 0.6182$. These results appear to approach 0.618 (or something close). This number should ring a bell, because it is 1 less than the golden ratio $\varphi = 1.618$ we encountered in Example 8.5! Equivalently, as we noted in the second remark in the example's solution, 0.618 equals $1/\varphi$.

(b) Let's now show why the above results for the fractions do in fact approach $\varphi - 1 = 1/\varphi = 0.618$. Our goal is to show that if the 1's extend infinitely far down to the right in Eq. (8.44), the value of the whole fraction is $\varphi - 1 = 1/\varphi = 0.618$.

As suggested in the hint, if we let the entire infinite fraction be x, then the infinite fraction in the box in Eq. (8.44) is also x. So Eq. (8.44) can be rewritten as

$$x = \frac{1}{1+x}. \tag{8.127}$$

Multiplying this equation by $1 + x$ yields $x(1 + x) = 1 \implies x^2 + x - 1 = 0$. The quadratic formula then gives

$$x^2 + x - 1 = 0 \implies x = \frac{-1 \pm \sqrt{1 - 4(-1)}}{2} = \frac{-1 \pm \sqrt{5}}{2}. \tag{8.128}$$

We want the positive root since the fraction in Eq. (8.44) is positive; the negative root is meaningless here. So our answer is $x = (-1 + \sqrt{5})/2 = 0.618$, as desired.

The infinite fraction (including the dots) in Eq. (8.44) is called an infinite *continued fraction*. If we truncate the fraction after a finite number of levels, then it is called a finite continued fraction.

REMARK: The various numbers that appear in the numerators and denominators of the successive fractions in part (a) are:

$$1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ldots \tag{8.129}$$

This sequence of numbers is known as the *Fibonacci sequence*. (The actual sequence has a second 1 at the beginning, so it starts with 1, 1, 2, 3, ...) Each number is the sum of the two preceding ones. So the next number is $34 + 55 = 89$, and then $55 + 89 = 144$, and so on. If you look back at the calculations of the various fractions in part (a), you will see why each of the numbers appearing there is the sum of the two preceding ones. For example, $1 + 8/13 = 13/13 + 8/13 = (13 + 8)/13 = 21/13$. The 21 here is $13 + 8$.

From our calculations of the successive fractions in part (a), it is apparently the case that the ratio of two consecutive Fibonacci numbers approaches $1/\varphi = 0.618$ (or $\varphi = 1.618$ if you take the ratio of the larger one to the smaller). We'll see later on in Exercise 10.31 why this is the case. ♣

19. (a) Combining the a's in the denominator, and then multiplying both sides of Eq. (8.45) by the denominator, and using the difference-of-squares result in Eq. (3.23), yields

$$\begin{aligned}\sqrt{a^2+b}-a=\frac{b}{\sqrt{a^2+b}+a} \quad &\Longrightarrow \quad \big(\sqrt{a^2+b}-a\big)\big(\sqrt{a^2+b}+a\big)=b\\ \Longrightarrow \quad (a^2+b)-a^2=b \quad &\Longrightarrow \quad b=b,\end{aligned} \tag{8.130}$$

which is indeed true. Equivalently, you can start with the true equation $b = b$ and then reverse all of the above steps to produce Eq. (8.45).

(b) If we make an infinite number of the suggested replacements (and then also add a to both sides of the equation), Eq. (8.46) turns into

$$\sqrt{a^2+b}=a+\cfrac{b}{2a+\cfrac{b}{2a+\cfrac{b}{2a+\cfrac{b}{2a+\ddots}}}}\,. \tag{8.131}$$

This is the desired continued-fraction representation of $\sqrt{a^2+b}$. In the case where $a = 3$ and $b = 5$, we obtain

$$\sqrt{14}=\sqrt{3^2+5}=3+\cfrac{5}{6+\cfrac{5}{6+\cfrac{5}{6+\cfrac{5}{6+\ddots}}}}\,. \tag{8.132}$$

With a calculator, you can show that this "level-4" continued fraction equals 3.74154, which is very close to the true value, $\sqrt{14} = 3.74166$. Including more levels will bring you closer to the true value, as you can check.

REMARK: You can also write 14 as $2^2 + 10$. So letting $a = 2$ and $b = 10$ in Eq. (8.131) will also yield $\sqrt{14}$, as you can verify with your calculator. Writing 14 as $1^2 + 13$ also works, as you can check (although you'll need to include more levels in order to get a good result). If you enjoy this kind of activity, there are plenty more (an infinite number of!) continued-fraction representations that you can verify. For example, $\sqrt{19} = \sqrt{4^2 + 3}$, so $a = 4$ and $b = 3$. And $\sqrt{29} = \sqrt{6^2 - 7}$, so $a = 6$ and $b = -7$. Yes, it's perfectly fine if b is negative. We never assumed in part (a) that b was positive. We also never assumed that a and b were integers. So non-integer values work fine too, although the continued fraction doesn't look as nice. Note that in the special case where $b = 0$, Eq. (8.131) is certainly true, since it reduces to $a = a$. ♣

20. If you evaluate the expression up to the 10th 1, you will obtain 1.6180165. This number looks suspiciously similar to the golden ratio $\varphi = 1.618$ (or 1.6180340 with a few more digits), which we encountered in Example 8.5 and Exercise 8.18. We can demonstrate that this is in fact the correct value, as follows. We'll label the given infinite expression in Eq. (8.47) as x, by putting an "$x =$" on its left. Squaring both sides of the equation then gives

$$x^2 = 1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}} \quad \Longrightarrow \quad x^2 = 1 + x, \tag{8.133}$$

where we have used the fact that the square-root expression here also equals x, because it extends infinitely far, just like the one in Eq. (8.47), which we defined as x. The quadratic formula applied to the $x^2 = 1 + x$ equation then yields

$$x^2 - x - 1 = 0 \quad \Longrightarrow \quad x = \frac{1 + \sqrt{5}}{2} = 1.618034 = \varphi, \tag{8.134}$$

where we have chosen the positive root, since the square root in Eq. (8.47) is positive.

REMARK: If all of the 1's in Eq. (8.47) are replaced with another number n, then the quadratic equation in Eq. (8.133) becomes $x^2 = n + x \implies x^2 - x - n = 0$. The quadratic formula gives the (positive) solution for x as

$$x = \frac{1 + \sqrt{1 + 4n}}{2}. \tag{8.135}$$

For any n, this expression is the x value of the square-root expression. For example, if $n = 2$, it reduces nicely to $x = 2$. And if $n = 6$, it reduces to $x = 3$.

Going the other way, the question "What n value produces a desired x value?" is answered by rewriting the $x^2 - x - n = 0$ quadratic equation as $x^2 - x = n$. This tells us that if we pick n to be $x^2 - x$, then the value of the square-root expression is x. Equivalently, solving for n in terms of x in Eq. (8.135) yields $n = x^2 - x$ (since that's the equation that generated the solution in Eq. (8.135) in the first place), as you can verify.

For example, if we want $x = 2$, we need to pick $n = 2^2 - 2 = 2$. And if we want $x = 3$, we need to pick $n = 3^2 - 3 = 6$. (These results agree with the first paragraph in this remark.) If we want $x = 9$, we need to pick $n = 9^2 - 9 = 72$. As a check on this, the solution for x in Eq. (8.135) becomes

$$x = \frac{1 + \sqrt{1 + 4 \cdot 72}}{2} = \frac{1 + \sqrt{289}}{2} = \frac{1 + 17}{2} = 9. \quad \clubsuit \tag{8.136}$$

21. The times for the outward and return parts of the trip are ℓ/a and ℓ/b. The total time is therefore $t_{\text{total}} = \ell/a + \ell/b$. The total distance is simply $d_{\text{total}} = 2\ell$, so Eq. (8.51) gives the average speed for the entire trip as

$$v_{\text{avg}} = \frac{d_{\text{total}}}{t_{\text{total}}} = \frac{2\ell}{\frac{\ell}{a} + \frac{\ell}{b}} = \frac{2}{\frac{1}{a} + \frac{1}{b}} = \frac{2}{\frac{b+a}{ab}} = \frac{2ab}{a+b}. \tag{8.137}$$

Having found v_{avg}, our goal is to show that the following inequality is always true, for any (positive) values of a and b:

$$v_{\text{avg}} \le \frac{a+b}{2} \quad \Longrightarrow \quad \frac{2ab}{a+b} \le \frac{a+b}{2}. \tag{8.138}$$

Multiplying both sides by $2(a + b)$ and simplifying yields

$$\begin{aligned} 4ab \le (a+b)^2 \quad &\Longrightarrow \quad 4ab \le a^2 + 2ab + b^2 \\ \Longrightarrow \quad 0 \le a^2 - 2ab + b^2 \quad &\Longrightarrow \quad 0 \le (a-b)^2. \end{aligned} \tag{8.139}$$

This inequality is always true, because a square is always positive (or zero). The original inequality in Eq. (8.138) is therefore also always true. More precisely, all of the above steps are reversible, so you can start with the true inequality $0 \le (a - b)^2$ and then work your way back to Eq. (8.138). Since a and b are positive, you don't need to worry about reversing the inequality sign during the division by $a + b$.

Equality holds in Eq. (8.139) when the two speeds are equal, so the same is true for Eq. (8.138). This makes sense because if you have only one speed, your average speed equals that speed.

REMARK: Although it might not be intuitively obvious that $v_{\text{avg}} \le (a + b)/2$ is true for general values of a and b, it *is* clear for some special cases. (See the subsection at the end of Section B.2 in Appendix B for a discussion of checking special cases.) Two helpful cases to consider are when one of the speeds is very large or very small:

- One speed is very *large*: Let's say that $\ell = 10$, and you head outward with a speed of $a = 5$, which means that your outward time is $\ell/a = 10/5 = 2$. And then let's say that you head back with a huge speed, which we'll hypothetically declare to be $b = \infty$. It then takes you essentially zero time to get back. So you travel a total distance of $2\ell = 20$ in a time of $2 + 0$, which means that your average speed is

$v_{\text{avg}} = 20/(2+0) = 10$. (This is twice your outward speed of 5, because in the end you travel twice the distance in the same time.) This $v_{\text{avg}} = 10$ result is less than the average of the two speeds, which is $(5+\infty)/2 = \infty$. So in the extreme case where one of the speeds is very large, Eq. (8.138) is certainly true.

- One speed is very *small*: Again let $\ell = 10$ and $a = 5$, but now let your return speed be very small, let's say $b \approx 0$. It then takes you (roughly) an infinite time to get back. Your overall average speed is therefore $v_{\text{avg}} = 20/(2+\infty) = 0$. This is less than the average of the two speeds, which is $(5+0)/2 = 2.5$. So in the extreme case where one of the speeds is very small, Eq. (8.138) is again true.

If you forget which way the inequality sign in Eq. (8.138) points, and if you don't want to work through Eq. (8.139) again, you can just use one of the above extreme cases to determine the direction of the sign. ♣

22. As mentioned in the statement of the problem, since you are fighting the current when swimming upstream, the river's speed of 1/2 mph gets *subtracted* from your still-water speed v. You are moving slower than v with respect to the banks of the river. Your actual upstream speed (with respect to the banks) is therefore $r = v - 1/2$. And since the distance is $d = 1$ (in miles), the $t = d/r$ relation gives your upstream time as $t_{\text{up}} = 1/(v - 1/2)$.

In a similar manner, the river's speed gets *added* to your v when going downstream, yielding an actual speed of $v + 1/2$. You are moving faster than v with respect to the banks of the river. So your downstream time is $t_{\text{down}} = 1/(v + 1/2)$. This is correctly smaller than t_{up}, because you're now moving faster.

Since the total time is 2 and 1/3 hours (which is 8/3 hours), we have

$$t_{\text{up}} + t_{\text{down}} = \frac{8}{3} \quad \Longrightarrow \quad \frac{1}{v - 1/2} + \frac{1}{v + 1/2} = \frac{8}{3}. \tag{8.140}$$

Multiplying through by the common denominator $3(v - 1/2)(v + 1/2)$ yields

$$\begin{aligned} 3(v + 1/2) + 3(v - 1/2) &= 8(v - 1/2)(v + 1/2) \\ \Longrightarrow \quad 6v &= 8(v^2 - 1/4) \\ \Longrightarrow \quad 0 &= 4v^2 - 3v - 1. \end{aligned} \tag{8.141}$$

The quadratic formula then gives

$$v = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(4)(-1)}}{2 \cdot 4} = \frac{3 \pm \sqrt{25}}{8} = \frac{3 \pm 5}{8}. \tag{8.142}$$

We must pick the "+" sign since the speed v is positive. So we obtain $v = (3+5)/8 = 1$. The upstream and downstream speeds of $v - 1/2$ and $v + 1/2$ are therefore 1/2 and 3/2. So the upstream and downstream times in Eq. (8.140) are 2 and 2/3, which correctly add up to 8/3.

Alternatively, instead of using the quadratic formula, you can factor the $4v^2 - 3v - 1 = 0$ quadratic equation into $(4v + 1)(v - 1) = 0$, which yields a positive root of 1.

Consistent with the result in Exercise 8.21, the average speed for the entire trip is $v_{\text{avg}} = d_{\text{tot}}/t_{\text{tot}} = 2/(8/3) = 3/4$, which is correctly less than the average of the two speeds, which is $(1/2 + 3/2)/2 = 1$.

23. If your speed with respect to the water is v, and if the river's speed is r, then your upstream and downstream speeds are $v - r$ and $v + r$, respectively (slower upstream, faster downstream). Let the distance each way be d. Then the total time is (adding the two fractions by getting a common denominator $(v - r)(v + r)$)

$$t_{\text{up}} + t_{\text{down}} = \frac{d}{v-r} + \frac{d}{v+r} = \frac{d}{v-r}\cdot\frac{v+r}{v+r} + \frac{d}{v+r}\cdot\frac{v-r}{v-r} \tag{8.143}$$

$$= \frac{2vd}{v^2 - r^2} = \frac{2d/v}{1 - r^2/v^2}\,. \tag{8.144}$$

The last expression here is obtained by dividing both the numerator and denominator by v^2 (or equivalently by multiplying by 1 in the form of $(1/v^2)/(1/v^2)$). We have done this so that we can easily compare the above result for the total time with the total time in the zero-current case, which is simply $t_{\text{zero}} = (2d)/v$, since the total roundtrip distance is $2d$ and the speed is always v in the zero-current case (that is, $r = 0$).

Due to the $1 - r^2/v^2$ denominator in Eq. (8.144) (which is always smaller than 1 for any nonzero r), the time in Eq. (8.144) is always larger than the zero-current time of $t_{\text{zero}} = 2d/v$, as we wanted to show. The basic reason for this is that (although this might not be obvious) the slower speed upstream always hurts you more than the faster speed downstream helps.

You should convince yourself why the result in this exercise says essentially the same thing as the result in Exercise 8.21.

REMARK: We can check an extreme case here, similar to the very-small-speed case in the solution to Exercise 8.21. If r is only slightly less than v, it's fairly clear that the time in Eq. (8.144) is larger than the zero-current time, even without doing the algebra that produces the final expression. For example, let's say $d = 1$, $v = 1$, and $r = 0.99$. Then your upstream speed is only $v - r = 1 - 0.99 = 0.01$, which means that $t_{\text{up}} = d/(v - r) = 1/0.01 = 100$. So it takes you 100 hours to swim upstream!

Does your short downstream time make up for this long upstream time? No chance, because the upstream 100-hour time is already longer than the *total* time with zero current, which is $t_{\text{zero}} = 2d/v = 2 \cdot 1/1 = 2$ hours. Even if you travel infinitely fast so that the downstream time is zero, the total time will still be 100 hours, which is longer than 2.

The actual downstream time is $t_{\text{down}} = d/(v + r) = 1/(1 + 0.99) \approx 1/2$. So it takes about half an hour to swim downstream. But as we just noted, the long 100-hour upstream time *far* outweighs the short 1/2-hour downstream time (even if it were hypothetically equal to zero). So the total time of (approximately) 100.5 hours is far longer than the $t_{\text{zero}} = 2d/v = 2$-hour total time with zero current. ♣

24. Let v be the desired air speed. Then the westward (upstream) and eastward (downstream) speeds are $v - 100$ and $v + 100$. So in the same manner as in the solution to Exercise 8.22,

setting the total roundtrip time equal to 10 hours gives

$$t_{\text{west}} + t_{\text{east}} = \frac{2500}{v - 100} + \frac{2500}{v + 100} = 10. \tag{8.145}$$

Multiplying through by the common denominator $(v - 100)(v + 100)$ yields

$$\begin{aligned} 2500(v + 100) + 2500(v - 100) &= 10(v - 100)(v + 100) \\ \implies \quad 5000v &= 10(v^2 - 10{,}000) \\ \implies \quad 0 &= v^2 - 500v - 10{,}000. \end{aligned} \tag{8.146}$$

The quadratic formula then gives

$$v = \frac{-(-500) \pm \sqrt{(-500)^2 - 4(1)(-10{,}000)}}{2} = \frac{500 \pm 538.5}{2}. \tag{8.147}$$

We must pick the "+" sign since the speed v is positive. This yields $v \approx 519$ mph. The westward and eastward speeds of $v - 100$ and $v + 100$ are therefore 419 and 619. So the westward and eastward times in Eq. (8.145) are 5.97 hours and 4.04 hours, which correctly add up to 10 hours (rounding errors aside).

We see that the plane needs to increase its air speed from 500 to 519, in order to keep the total time at 10 hours. This is consistent with the result in Exercise 8.23, which states that if the air speed stayed at 500, the time would be longer than 10 hours. (It would be $2500/400 + 2500/600 = 6.25 + 4.17 = 10.42$ hours.) The air speed must therefore increase, to keep the total time at 10 hours.

REMARK: The *jet streams* that circle the globe are "rivers" of eastward-moving air that can reach speeds of 200 mph or more. There are four main jet streams – two in the northern hemisphere and two in the southern. Planes traveling eastward can hitch a ride for a quick trip, while planes traveling westward regularly deviate substantially from the shortest (straight-line) path, if that path happens to lie within the jet stream. The extra time due to the longer path is often less than the extra time due to the reduced upstream speed. So avoiding the jet stream is usually the best option. ♣

25. If $b^2 - 4ac = 0$, then $b^2 = 4ac \implies b = \pm 2\sqrt{ac}$. Therefore,

$$ax^2 + bx + c = ax^2 \pm 2\sqrt{ac}\,x + c = \left(\sqrt{a}\,x \pm \sqrt{c}\right)^2, \tag{8.148}$$

which is the desired perfect square.

For the $b = +2\sqrt{ac}$ case, the quadratic equation $ax^2 + bx + c$ takes the form of $\left(\sqrt{a}\,x + \sqrt{c}\right)^2 = 0$. And this has only one solution, namely $x = -\sqrt{c}/\sqrt{a}$. This is consistent with the fact that the quadratic formula in Eq. (8.30) gives only one solution when the discriminant is zero. Furthermore, when $b = +2\sqrt{ac}$ the one solution in Eq. (8.30) is $(-b \pm 0)/2a = -2\sqrt{ac}/2a = -\sqrt{c}/\sqrt{a}$, in agreement with what we just obtained.

The other option for b, namely $b = -2\sqrt{ac}$, yields the one solution $x = +\sqrt{c}/\sqrt{a}$. In either case, for each particular value of b there is only one solution.

Alternatively, if we instead eliminate c in favor of a and b, then $b^2 = 4ac$ gives $c = b^2/4a$, so

$$\begin{aligned} ax^2 + bx + c = ax^2 + bx + \frac{b^2}{4a} &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) \\ &= a\left(x + \frac{b}{2a}\right)^2 = \left(\sqrt{a}\,x + \frac{b}{2\sqrt{a}}\right)^2, \end{aligned} \tag{8.149}$$

where we have absorbed the a coefficient into the square (where it becomes a $\sqrt{a}$). Similarly, you can also produce a perfect square by eliminating a in favor of b and c.

26. Our goal is to have the two curves intersect at exactly one point (as opposed to two or zero points). Setting the two functions equal to each other to find the intersection point gives

$$2(x-1)^2 + c = x^2 \implies 2(x^2 - 2x + 1) + c = x^2 \implies x^2 - 4x + (2 + c) = 0. \tag{8.150}$$

There will be exactly one solution if the discriminant is zero. So we want

$$(-4)^2 - 4\cdot 1\cdot(2+c) = 0 \implies 16 - 4(2+c) = 0 \implies 4 - (2+c) = 0 \implies c = 2. \tag{8.151}$$

The quadratic equation in Eq. (8.150) is then $x^2 - 4x + 4 = 0$, which factors into $(x-2)^2 = 0$. So the intersection occurs at $x = 2$. (Or you can use the quadratic formula, where the discriminant is zero.) We then quickly see that the associated y value (for both functions) is $y = 4$. So the intersection point (the point of tangency) is $(2, 4)$.

Changing the value of c raises or lowers the parabola. If $c < 2$, there are two intersection points; the discriminant in Eq. (8.151) is positive. And if $c > 2$, there are no intersection points; the discriminant is negative. You can verify these statements with Desmos.

27. This exercise is the same as Example 8.9, except for the 2 in front of the x. Setting $2x + 1/x$ equal to m and then multiplying through by x gives

$$2x + \frac{1}{x} = m \implies 2x^2 + 1 = mx \implies 2x^2 - mx + 1 = 0. \tag{8.152}$$

The discriminant is $(-m)^2 - 4\cdot 2\cdot 1 = m^2 - 8$. A (real) solution for x exists only if this is greater than or equal to zero. So we must have $m^2 \geq 8 \implies m \geq 2\sqrt{2}$. So $m = 2\sqrt{2} \approx 2.83$ is the desired minimum value of $2x + 1/x$. This value is achieved when $x = -b/2a = -(-m)/4 = 2\sqrt{2}/4 = 1/\sqrt{2} \approx 0.71$. You can check with Desmos that the minimum of the $2x + 1/x$ function is in fact located at the point $(0.71, 2.83)$.

It makes sense that the minimum value (namely $2\sqrt{2} \approx 2.83$) for the present $2x + 1/x$ function is *larger* than the minimum value (namely 2) for the $x + 1/x$ function in Example 8.9, due to the factor of 2 in the $2x$ here. It also makes sense that the $x = 1/\sqrt{2} \approx 0.71$ value where the minimum is achieved is *smaller* than the $x = 1$ value in Example 8.9, because (although this might not be obvious) decreasing x from 1 decreases the value of $2x$ more (at least initially) than it increases the value of $1/x$. And we're trying to find the minimum value of the sum $2x + 1/x$.

28. (a) If the perimeter $P = 2x + 2y$ is given, then solving for y in terms of x yields $y = (P - 2x)/2$. Plugging this into the area gives $A = xy = x \cdot (P - 2x)/2$. We therefore want to maximize the function $A = (1/2)(-2x^2 + Px)$. Setting this equal to m yields

$$(1/2)(-2x^2 + Px) = m \quad \Longrightarrow \quad 0 = 2x^2 - Px + 2m. \tag{8.153}$$

The discriminant is $(-P)^2 - 4(2)(2m) = P^2 - 16m$. A (real) solution for x exists only if this is greater than or equal to zero. So we must have $P^2 \geq 16m \implies P^2/16 \geq m$. The maximum value of the area is therefore $P^2/16$.

The associated x value is $x = -b/2a = -(-P)/(2 \cdot 2) = P/4$. y is then also $P/4$, as you can verify. So the shape is a square (as you might guess). Each side of a square with perimeter P is $P/4$, which yields an area of $(P/4)(P/4) = P^2/16$.

We're finding the maximum area here, as opposed to the minimum, because for any given perimeter, we can make the area be as small as we want by making the rectangle be very thin (with the other length being essentially $P/2$). So the minimum area is simply zero.

This exercise is basically the same as Example 8.10, since we're told that the sum $x + y$ equals $P/2$, and we're trying to maximize the product xy.

(b) If the area $A = xy$ is given, then solving for y in terms of x yields $y = A/x$. Plugging this into the perimeter gives $P = 2x+2y = 2x+2\cdot A/x$. We therefore want to maximize the function $P = 2x + 2A/x$. Setting this equal to m and multiplying through by x yields

$$2x + \frac{2A}{x} = m \quad \Longrightarrow \quad 2x^2 + 2A = mx \quad \Longrightarrow \quad 2x^2 - mx + 2A = 0. \tag{8.154}$$

The discriminant is $(-m)^2 - 4(2)(2A) = m^2 - 16A$. A (real) solution for x exists only if this is greater than or equal to zero. So we must have $m^2 \geq 16A \implies m \geq 4\sqrt{A}$. The minimum value of the perimeter is therefore $4\sqrt{A}$.

The associated x value is $x = -b/2a = -(-m)/(2 \cdot 2) = m/4 = (4\sqrt{A})/4 = \sqrt{A}$. y is then also $\sqrt{A}$, as you can verify. So the shape is a square (again, as you might guess). Each side of a square with area A is $\sqrt{A}$, which gives a perimeter of $4\sqrt{A}$.

We're finding the minimum perimeter here, as opposed to the maximum, because for any given area, we can make the perimeter be as large as we want by making the rectangle be very thin and long (with the product xy equalling A). So there is no maximum perimeter; it can approach infinity.

The $P = 2x + 2A/x$ form of the perimeter we found above takes the same general form (up to some constants) as the $x + 1/x$ function in Example 8.9, and also the $2x + 1/x$ function in Exercise 8.27. So the strategy here is basically the same as in those problems.

29. Our goal is to find the maximum value of the difference $A_{\text{side}} - A_{\text{ends}} = 2\pi rh - 2\pi r^2$, as a function of r. Setting this equal to m yields

$$2\pi rh - 2\pi r^2 = m \quad \Longrightarrow \quad 0 = (2\pi)r^2 - (2\pi h)r + m. \tag{8.155}$$

The discriminant is $(-2\pi h)^2-4(2\pi)(m) = 4\pi^2h^2-8\pi m$. A (real) solution for r exists only if this is greater than or equal to zero. So we must have $4\pi^2h^2 \geq 8\pi m \implies \pi h^2/2 \geq m$. So $\pi h^2/2$ is the desired maximum value of the difference.

This value is achieved when $r = -b/2a = -(-2\pi h)/(2\cdot 2\pi) = h/2$. (Note that this means that the diameter $2r$ of the cylinder equals the height h.) As a check, when $r = h/2$ the side area is $A_{\text{side}} = 2\pi rh = 2\pi(h/2)h = \pi h^2$, and the area of the ends is $A_{\text{ends}} = 2\pi r^2 = 2\pi(h/2)^2 = \pi h^2/2$. The difference between these areas is correctly the $\pi h^2/2$ value we found above.

REMARKS:

1. Saying that the maximum of the function $f(r) = 2\pi rh-2\pi r^2$ equals $\pi h^2/2$ is equivalent to saying that (after dividing by 2π and relabeling r as x) the maximum of the function $f(x) = hx - x^2$ equals $h^2/4$. You can verify this by making a Desmos plot with a slider for h.
2. This $f(x) = hx-x^2$ function can be written as $f(x) = x(h-x)$. And since $x+(h-x) = h$, we see that maximizing $f(x)$ is equivalent to maximizing the product of two numbers, x and $h - x$, whose sum is h. This is the same kind of problem as Example 8.10, and the $h^2/4$ result here is correctly the same as the $S^2/4$ result in the remark in the solution to Example 8.10.
3. It makes sense that the difference $A_{\text{diff}} \equiv 2\pi rh - 2\pi r^2$ achieves a maximum for some value of r, due to the following reasoning. If r is tiny, then the value of A_{diff} is tiny, because the cylinder is very thin. Both the side area and the end areas are tiny, so $A_{\text{diff}} \approx 0 - 0 = 0$. If we then increase r, but keep it smaller than h, then the value of A_{diff} is positive, because it can be written as $2\pi r(h - r)$. If we then further increase r so that it is larger than h, then A_{diff} becomes negative. (For very large r, imagine a squat cylinder a mile in diameter and an inch tall. A_{diff} is certainly negative, because A_{ends} is much larger than A_{side}.)

   We see that if we start with r equalling zero and then increase it, A_{diff} starts at zero, then becomes positive, and then becomes negative when $r > h$. It must therefore achieve a maximum value for some r between 0 and h. And we showed above that this special value of r is $h/2$. ♣

30. (In this exercise, the letters P and S take the place of the m we've been using.) For the product P, if we square both sides of $P = x\sqrt{c^2 - x^2}$, we obtain

$$P^2 = x^2c^2 - x^4 \quad\implies\quad (x^2)^2 - c^2x^2 + P^2 = 0. \tag{8.156}$$

This is a quadratic equation in the variable x^2. That's fine; it doesn't matter what the variable is. The quadratic formula is still valid. A (real) solution for x^2 (and hence x) exists only if the discriminant is greater than or equal to zero, which means

$$(-c^2)^2 - 4P^2 \geq 0 \quad\implies\quad \frac{c^4}{4} \geq P^2 \quad\implies\quad \frac{c^2}{2} \geq P. \tag{8.157}$$

So the maximum value of the product of the legs is $P_{max} = c^2/2$. This is achieved when $x^2 = -b/2a = -(-c^2)/(2 \cdot 1) = c^2/2$, which yields $x = c/\sqrt{2}$. The other leg is then also $c/\sqrt{2}$, as you can verify. We therefore have a 45-45-90 right triangle (see Section A.7). This is the shape that maximizes the product of the legs (for a given hypotenuse c).

REMARK: The area of a right triangle with legs a and b is $A = ab/2$ (see Section A.4.2), which can be written as $P/2$. So the maximum product P yields the maximum area A. Therefore, a 45-45-90 right triangle is the shape with the maximum area (for a given hypotenuse c). Since we found above that the maximum P is $c^2/2$, the maximum area is $A_{max} = P_{max}/2 = (c^2/2)/2 = c^2/4$. This scenario corresponds to Fig. 8.7(a). The area of each of the four triangles is $1/4$ of the area of the square, which is c^2. If x (which we're calling a here) isn't equal to the optimizing value of $c/\sqrt{2}$ (that is, if we don't have a 45-45-90 right triangle), then we have the scenario in Fig. 8.7(b). The area of each triangle is less than $c^2/4$, due to the "missing" square in the middle. ♣

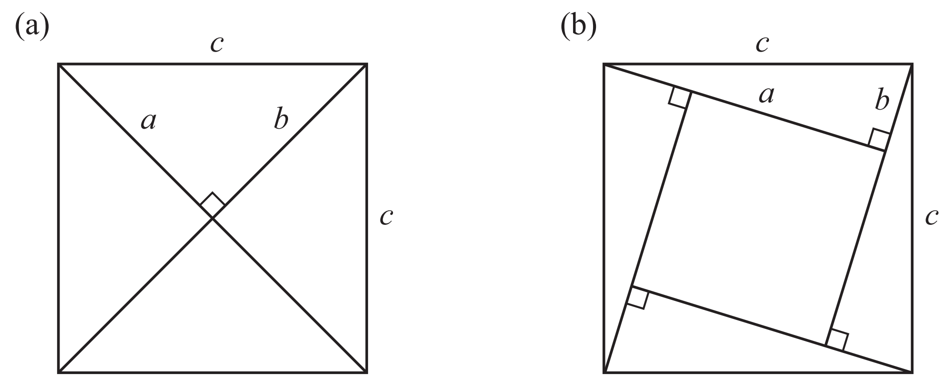

Figure 8.7: For a given hypotenuse c, a 45-45-90 right triangle yields the maximum product ab of the legs, or equivalently the maximum area $ab/2$.

For the sum S, putting the x on the lefthand side of $S = x + \sqrt{c^2 - x^2}$ and then squaring both sides gives

$$(S-x)^2 = c^2 - x^2 \implies S^2 - 2Sx + x^2 = c^2 - x^2 \implies 2x^2 - 2Sx + (S^2 - c^2) = 0. \quad (8.158)$$

A (real) solution for x exists only if the discriminant is greater than or equal to zero, which means

$$(-2S)^2 - 4(2)(S^2 - c^2) \geq 0 \implies 8c^2 \geq 4S^2 \implies \sqrt{2}c \geq S. \quad (8.159)$$

So the maximum value of the sum of the legs is $S_{max} = \sqrt{2}c$. This is achieved when $x = -b/2a = -(-2S)/(2 \cdot 2) = S/2$. Therefore, when $S = \sqrt{2}c$ the corresponding x is $(\sqrt{2}c)/2 = c/\sqrt{2}$. The other leg is then also $c/\sqrt{2}$, as you can verify. So we again have a 45-45-90 right triangle. This is the shape that maximizes the sum of the legs (for a given hypotenuse c). As an exercise, you can think about why (by just looking at the figure and not doing any algebra) the sum of the legs in Fig. 8.7(b) is less than in Fig. 8.7(a).

31. 1. For $(1+5i)+(3-4i)$, we need to add the real parts, and likewise the imaginary parts. So we obtain

$$(1+3)+(5-4)i = 4+i. \tag{8.160}$$

2. For $(-6+2i)-(4-3i)$, we need to subtract the real parts, and likewise the imaginary parts:

$$(-6-4)+\big(2-(-3)\big)i = -10+5i. \tag{8.161}$$

3. For $(-3+4i)(5-9i)$, we can use FOIL:

$$\begin{aligned}(-3+4i)(5-9i) &= -3\cdot 5 - 3\cdot(-9i) + 4i\cdot 5 + 4i\cdot(-9i) = -15+27i+20i-36i^2\\ &= -15+47i-36(-1) = 21+47i.\end{aligned} \tag{8.162}$$

4. For $(1+2i)^3$, we can first calculate $(1+2i)^2$ with FOIL:

$$\begin{aligned}(1+2i)(1+2i) &= 1\cdot 1 + 1\cdot 2i + 2i\cdot 1 + 2i\cdot 2i = 1+4i+4i^2\\ &= 1+4i+4(-1) = -3+4i.\end{aligned} \tag{8.163}$$

We can then multiply this result by the third factor of $(1+2i)$:

$$\begin{aligned}(-3+4i)(1+2i) &= (-3)\cdot 1 + (-3)\cdot 2i + 4i\cdot 1 + 4i\cdot 2i = -3-6i+4i+8i^2\\ &= -3-2i+8(-1) = -11-2i.\end{aligned} \tag{8.164}$$

32. We want to find two numbers x and y such that $x+y=2$ and $xy=10$. The first equation gives $y = 2-x$. Plugging this into the second equation yields $x(2-x) = 10 \implies 0 = x^2-2x+10$. The quadratic formula then gives

$$x = \frac{-(-2)\pm\sqrt{(-2)^2-4\cdot 10}}{2} = \frac{2\pm\sqrt{-36}}{2} = \frac{2\pm 6i}{2} = 1\pm 3i. \tag{8.165}$$

These are the desired two numbers. More precisely, if we pick the $x = 1+3i$ root, then $y = 2-x = 1-3i$. And if we pick the $x = 1-3i$ root, then $y = 2-x = 1+3i$. As a check, Eq. (8.73) correctly gives the product as $1^2+3^2 = 10$.

33. We want to find the values of x that satisfy $x^3 = -1$, or equivalently $x^3+1 = 0$. An obvious root is $x=-1$. To find the other roots, we can factor out the $x-(-1) = x+1$ factor (associated with the $x=-1$ root) from x^3+1, and then use the quadratic formula on the resulting quadratic polynomial, as we did in Example 8.13.

The second result in Example 3.7 (with $a \to x$ and $b \to 1$) gives the factoring of x^3+1:

$$x^3+1 = (x+1)(x^2-x+1). \tag{8.166}$$

The quadratic formula then gives the roots of the $x^2-x+1 = 0$ equation as

$$x = \frac{1\pm\sqrt{1-4}}{2} = \frac{1\pm\sqrt{-3}}{2} = \frac{1}{2}\pm\frac{\sqrt{3}}{2}i. \tag{8.167}$$

These are the desired two complex cube roots of -1. The only difference from the roots in Eq. (8.83) is the sign of the real part.

As a check, the sum of the two roots in Eq. (8.167) is 1, which is correctly the negative of the coefficient of x in x^2-x+1. And Eq. (8.73) gives their product as $(1/2)^2+(\sqrt{3}/2)^2 = 1$, which is correctly the constant term (the 1).

If you want to explicitly show that the cube of each root equals -1, the calculation is very similar to the one we did in the first remark in Example 8.13.

34. (a) The square of $x + yi$ equals $x^2 - y^2 + 2xyi$. Matching up the real and imaginary parts of this with $5 - 12i$ gives

$$x^2 - y^2 = 5 \qquad \text{and} \qquad 2xy = -12 \implies xy = -6. \tag{8.168}$$

We quickly see that $x = 3$ and $y = 2$ satisfy $x^2 - y^2 = 5$. But to get the sign right in the $xy = -6$ equation, one of the variables must be negative. So we have either $x = 3$ and $y = -2$, or $x = -3$ and $y = 2$. The two square roots of $5 - 12i$ are therefore $\pm(3 - 2i)$. You can quickly check that the square of each of these is correctly $5 - 12i$.

(b) Solving for y in the second equation in Eq. (8.168) to obtain $y = -6/x$, and then plugging this into the first equation, gives

$$x^2 - \frac{(-6)^2}{x^2} = 5 \implies x^4 - 5x^2 - 36 = 0. \tag{8.169}$$

This is a quadratic equation in x^2, so the quadratic formula gives

$$x^2 = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-36)}}{2} = \frac{5 \pm \sqrt{169}}{2} = \frac{5 \pm 13}{2}. \tag{8.170}$$

Since x^2 is positive (because x is assumed to be real), we must pick the "+" sign. So we obtain $x^2 = (5+13)/2 = 9$, and hence $x = \pm 3$. The second equation in Eq. (8.168) then gives $y = \mp 2$ (with the sign being the opposite of the sign of x, because the product is the negative number $xy = -6$). We therefore obtain the two answers of $\pm(3 - 2i)$, in agreement with the result in part (a).

The same comments we made in the solution to Example 8.14 are again relevant here: (1) You can alternatively factor the quadratic equation in Eq. (8.169) into $(x^2 - 9)(x^2 + 4) = 0$, and we want the positive $x^2 = 9$ root. (2) You can begin the solution by instead solving for x in terms of y. (3) The $x^2 = -4$ root also leads to the correct answer in the end.

35. As in the preceding exercise, matching up the real and imaginary parts of $x^2 - y^2 + 2xyi$ and $0 + 1 \cdot i$ gives

$$x^2 - y^2 = 0 \qquad \text{and} \qquad 2xy = 1. \tag{8.171}$$

The second equation yields $y = 1/2x$. Plugging this into the first equation gives

$$x^2 - \frac{1}{4x^2} = 0 \implies 4x^4 = 1. \tag{8.172}$$

Taking the square root then gives $2x^2 = \pm 1$. We want the positive sign, since x^2 is positive. So we have $2x^2 = 1 \implies x^2 = 1/2$. Taking the square root yields $x = \pm 1/\sqrt{2}$. And then the second equation in Eq. (8.171) gives $y = \pm 1/\sqrt{2}$ (with the sign being the same as the sign of x, because the product is the positive number $xy = 1/2$). The two square roots of i are therefore $\pm(1/\sqrt{2} + i/\sqrt{2})$. You can quickly check that the square of each of these is correctly $0 + i$.

Interestingly, since taking the square root of -1 requires a new type of number, namely the imaginary number i, you might think that taking the square root of i would require yet another new type of number. However, we just showed that this isn't the case. The $\pm(1/\sqrt{2} + i/\sqrt{2})$ square roots of i are comprised of real and imaginary pieces, with no need for anything new.