Chapter 5

Solving for x

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One of the most important applications of algebra is solving for particular quantities, often called *variables* (or unknowns). Variables are usually denoted by letters, and the most common one to use is x, although any other letter can work just as well. If you have more than one variable, the usual convention is to use letters at the end of the alphabet, like x, y, and z. But really, any letters are fine.

To be sure, algebra isn't *only* about solving for variables. There are plenty of other uses, such as simplifying expressions, writing general formulas, proving theorems, and plotting functions (as we'll see in Chapters 6 and 7). But solving for variables is probably the most important aspect of algebra, so we'll devote an entire chapter to an in-depth discussion of the process. You'll be able to get lots of practice with many examples and exercises.

We'll begin this chapter in Section 5.1 by discussing the underlying principle used in solving equations, namely that if you start with two equal things, and if you make *equal changes* to these equal things, the results are again equal. We'll apply this principle to some simple equations, and then in Sections 5.2 and 5.3 we'll tackle more complicated ones. Section 5.4 deals with a slight variation, where we have *inequalities* instead of equalities. But the general strategy is the same. In Section 5.5 we'll discuss *word problems*, where the main goal is to take the information that is presented in words and translate it into an equation, which can then be solved using the techniques we've learned. Section 5.6 covers *systems of equations*, where we have more than one equation and we're trying to solve for more than one variable. Finally, in Section 5.7 we'll solve some simple *quadratic equations*. This is just a teaser for Chapter 8, where we'll discuss quadratic equations in much greater detail.

We've actually already done some solving for variables in our discussion of roots/zeros in Section 4.3. If someone says, "Solve the equation $x^2-5x+6 = 0$ for x," then the $(x-2)(x-3)$ factorization in Eq. (4.22) (with a replaced by x) tells you that the solutions (the roots, the zeros) are $x = 2$ and $x = 3$. However, the main task in cases like this is the factoring, which we discussed at length in Chapter 4. Once the factoring is done, there isn't much left to do. You just need to solve the equations $x-2 = 0$ and $x-3 = 0$, which is quick; you simply write down the $x = 2$ and $x = 3$ answers, since these values make the equations be true. The goal of this chapter is to learn how to deal with equations that are much more complicated than the $x-2 = 0$ type. See, for example, the lengthy expression in Eq. (5.24).

5.1 Equal changes to equal things

The fundamental ingredient in solving for a variable is the following fact:

> If you start with two equal things, and if you make equal changes to these equal things, the results are again equal.

Let's illustrate this with an example. Consider two buckets of water that contain equal amounts of water, say 4 gallons each. If you add the same amount of water, say 2 gallons, to each, then you'll end up with equal amounts, namely 6 gallons each. Likewise, if you subtract the same amount, say 1 gallon, from the original 4's, you'll again end up with equal amounts, namely 3's for each.

Similarly, if you multiply the original amounts (4's) by the same thing (say 3), then you'll end up with equal amounts (12's). And if you divide the original amounts (4's) by the same thing (say 2), you'll again end up with equal amounts (2's). You could even take the square roots of the original amounts (4's), and you'll end up with equal amounts (2's). The bottom line is: If you start with two equal things, and if you do the *same* thing to both of them (add something, multiply by something, take the square root of each, etc.), you'll end up with two new things that are *again equal.*

This principle works, of course, for any type of setup (not just buckets of water) where you start with two things that are equal. For example, you could be dealing with the numbers of marbles in two jars, or the amounts of money in two piggy banks, or the heights of two trees, or (and this is the one that is relevant for the present purposes) you could be dealing with the *two sides of an equation*. If you write down an equation with two things on either side of an equals sign, then you're saying that the two things are equal. So, as above, if you perform the same operation on these two things (add something, multiply by something, take the square root of each, etc.), you'll end up with two new things that are *again equal*. That is, the "=" sign between them will still hold.

For example, if you start with the equation $5 \cdot 3 = 15$, and if you add 7 (or anything else) to both sides, you'll obtain another true statement, $5 \cdot 3 + 7 = 15 + 7$. You can quickly check that this relation is true; both sides are equal to 22. But the point is that you don't *need* to check that it's true. You already know it's true because you started with two equal things ($5 \cdot 3$ and 15), and then you performed the same operation (adding 7) on both things. So the results must again be equal.

The real usefulness of this fact (that a common operation on two equal things produces two things that are again equal) shows up when we want to solve for the value of a letter that makes an equation be true. People usually use the letter x to stand for the unknown quantity they're trying to solve for, although any other letter works just as well. You actually don't even need to use letters. You can use a little picture of a frog or a house or whatever!

Consider the equation,

$$x - 2 = 7. \tag{5.1}$$

What value of x makes this true? Well, after a little thought, you likely figured out that the answer is 9, and your reasoning was probably something like, "If 2 less than a number is 7, then the number must be 2 more than 7, so it must be 9." Or perhaps you guessed and checked a few numbers and narrowed things down until you arrived at the answer of 9.

Either of the above methods is a perfectly reasonable way to solve for x in this simple case (and likewise in the even simpler case of $x - 2 = 0$ mentioned above). However, for more complicated equations, we'll need to be more systematic. We'll ramp up the difficulty level in the next section, but for now let's start with some easier examples in order to illustrate the basic method. The method uses the above fact that *equal changes to equal things (in particular, the two sides of an equation) produce two things that are again equal.*

This fact is fairly clear, as we saw with the above buckets of water. It's hard to argue with, so you might be wondering why we're emphasizing it so much! There are two reasons: (1) It's really, *really* important. It's the fundamental ingredient in solving for any variable, and you won't get anywhere without it, and (2) The two sides of an equation are a bit more abstract than two buckets of water or two piggy banks. So even though the overall concept of "equal changes to equal things" is fairly clear, it takes a little getting used to when applied to equations. In any case, the concept is so important that we'll repeat it one more time:

> Performing the same operation on both sides of an equation produces two new quantities that are again equal. That is, it produces another valid equation with an "=" sign.

> Though it seems like a trivial claim,
> In fact it's the key to the game:
> The same change to equals
> Results in their sequels
> Again being one and the same.

The "equal changes to equal things produce things that are again equal" principle applies to three or more things too, of course. But we're concerned here with just two things, because we're applying the principle to equations, and equations have only two sides. Let's now apply our "equal changes to equal things" principle to a few simple examples.

Example 5.1 Consider the $x - 2 = 7$ equation in Eq. (5.1). We know that we can perform whatever operation we want, and as long as we do the same thing to both sides of the equation, the resulting two sides will again be equal. For example, we can add 100 to both sides to obtain (the parentheses aren't necessary here, but we'll include them anyway)

$$(x-2)+100 = 7+100 \quad \Longrightarrow \quad x+98 = 107. \tag{5.2}$$

This is a true equation, but it isn't terribly useful. We can also multiply both sides of $x - 2 = 7$ by 3 to obtain

$$3(x-2) = 3 \cdot 7 \quad \Longrightarrow \quad 3x - 6 = 21. \tag{5.3}$$

This is also a true equation, but again it isn't very useful. Fortunately, there is one *special operation* we can perform that *is* useful: We can add 2 to both sides. This gives

$$(x - \cancel{2}) + \cancel{2} = 7 + 2 \quad \Longrightarrow \quad x = 9. \tag{5.4}$$

The 2's cancel on the left side, and the x ends up all alone. So we have successfully solved for x. "Solving for x" basically means getting the x all alone on one side of the equation (it can be either side). You're then left with an equation of the form, "$x =$ something." So you're done.

The original $x - 2 = 7$ equation involved *subtracting* 2, so that's why we *added* 2 to both sides. The goal was to undo the subtraction of 2, so we used the fact that addition is the inverse operation of subtraction. Although there are many (an infinite number of) possible operations we can perform on both sides of the $x - 2 = 7$ equation (for example, adding 100, multiplying by 3, adding 2, etc.), *only one of them is useful*, namely adding 2.

Remark: Remember that, as we noted back in the discussion of Eqs. (1.23) and (1.24), each number or letter comes with the sign in front of it (with a plus sign understood at the front, if there is no sign there). So if the above $x - 2 = 7$ equation were instead written as $-2 + x = 7$, we would again want to add 2 to both sides. When the -2 is combined (added) with the 2, they cancel just like they did above, so we're left with only x on the lefthand side, which is the goal.

If we were instead solving the equation $x+2 = 7$, or equivalently $2+x = 7$, then in both cases we would want to *subtract* 2 from both sides. This would make the 2's cancel, and we would be left with the answer of $x = 5$:

$$(x + \cancel{2}) - \cancel{2} = 7 - 2 \quad \Longrightarrow \quad x = 5. \quad \clubsuit \tag{5.5}$$

Example 5.2 As a second example, consider the equation $3x = 12$. Again there are many (an infinite number of) different operations we can perform on both sides of this equation that will produce new equations that are again true. But only one special operation is actually *useful*. We could add 2 to both sides like we did above, but that yields $3x + 2 = 14$, which isn't useful. It doesn't help us in our goal of getting the x all alone on one side of the equation.

To accomplish this goal, we note that since x is multiplied by 3 in $3x = 12$, and since division is the inverse operation of multiplication, the useful operation we want to perform is division by 3. So we'll divide both sides of the equation by 3:

$$\frac{\cancel{3}x}{\cancel{3}} = \frac{12}{3} \implies x = 4. \tag{5.6}$$

The 3's cancel on the left, leaving x all alone. So we have successfully solved for x.

Once you've identified the useful operation to perform (dividing by 3 here), don't forget to apply it to *both* sides of the equation! Applying it to only one side would result in an equation that is *not* true anymore. Dividing only the left side by 3 would produce the incorrect statement $x = 12$. And dividing only the right side by 3 would produce the incorrect statement $3x = 4$. In terms of the buckets of water we discussed above, if you add 5 gallons of water to only one of them, the resulting amounts of course won't be the same!

Example 5.3 As a third example, consider the equation $x^2 = 25$. Without too much effort, you know that a solution for x is 5. But don't forget that another solution is -5, because it's also true that $(-5)^2 = 25$, since $(-1)^2 = 1$. So there are two solutions, and they can be written as ± 5 (read as "plus or minus 5").

Let's now produce these two answers more systematically by performing the same operation on both sides of $x^2 = 25$. The lefthand side involves the squaring operation, and since we know that the square-root operation undoes the squaring (well, sort of; see the remarks below), we can take the square root of both sides to obtain

$$\sqrt{x^2} = \sqrt{25} \implies x = \pm 5. \tag{5.7}$$

The "$\pm$" always needs to be included when taking a square root, because the $\sqrt{\ }$ symbol is defined to be positive (see the first remark on page 92), whereas there are always two square roots, since $(-1)^2 = 1$.

Again, there are many other things we could have done to both sides of the $x^2 = 25$ equation that would have yielded perfectly true equations. We could have added 2, or divided by 3, or taken the cube root of both sides. However, all of these actions would have resulted in true but *useless* equations.

Remarks:

1. To be a little more precise about Eq. (5.7), we technically should have included one more step and written

$$\sqrt{x^2} = \sqrt{25} \implies |x| = 5 \implies x = \pm 5, \tag{5.8}$$

because as we observed in Remark 2 on page 93, the correct expression for $\sqrt{x^2}$ is $|x|$. (This makes the result always be positive, which is how the $\sqrt{\ }$ symbol is defined.) And then since both 5 and -5 have an absolute value of 5, the $|x| = 5$

equation implies $x = \pm 5$. However, in practice you don't need to be this formal. As long as you throw in a "$\pm$" sign whenever you take a square root, everything will be fine.

2. As we have noted, the $\sqrt{\ }$ *symbol* is defined specifically to be the positive square root. So when we take the $\sqrt{\ }$ of both sides of $x^2 = 25$, we obtain $|x| = 5$, as we wrote in Eq. (5.8). But what do the two *words* "square root" mean when we say, "Take the square root of both sides of an equation"? Do they mean $\sqrt{\ }$ (the positive root)? Or do they mean either the positive or negative root?

   There actually doesn't seem to be a clear consensus on what the phrase "take the square root of both sides of $x^2 = 25$" means. Here are three reasonable interpretations of what this phrase is instructing you to do, all of which produce the same (correct) answer of $x = \pm 5$:

   - Take the $\sqrt{\ }$ (the positive square root) of both sides, which gives $|x| = 5$. You then have to remember that there are two different numbers (namely 5 and -5) that have an absolute value of 5, as we saw in Eq. (5.8).
   - Take either of the two square roots (positive or negative) of both sides. This yields the equation $\pm x = \pm 5$, which in turn yields four possible equations, depending on how you pair up the signs:

     $$x = 5, \qquad x = -5, \qquad -x = 5, \qquad -x = -5. \tag{5.9}$$

     However, the 1st and 4th of these say the same thing (namely $x = 5$), as do the 2nd and 3rd (namely $x = -5$). So we end up with two possibilities, $x = \pm 5$.
   - Answer the question, "What two numbers have their squares equal to 25?" You always get two answers (5 and -5 here) to questions like this, since $(-1)^2 = 1$.

   In the end, it doesn't matter which of the above interpretations you use, because you're going to obtain the same two answers of 5 and -5 in any case. But since the third interpretation is the simplest, we'll generally assume that's what the phrase "take the square root of both sides of an equation" means in this book. But again, any of the three interpretations is perfectly fine.

   Note that when *squaring* (instead of taking the square root of) both sides of an equation, none of the above issues are relevant. You simply square both sides and carry on. Although any particular number (except 0) has two square roots, it has only one square. So you don't need to worry about which option to pick when squaring.

3. The underlying fact that led to the above remarks appearing in the present example, but not in the preceding two, is that there are two numbers whose squares equal 25, and likewise two numbers whose absolute values equal 5. (In contrast, there is only one number with the property that when you subtract 2 you obtain 7, or when you multiply by 3 you obtain 12.) So when you try to undo the squaring operation or the absolute value operation, you don't obtain a unique result. There are always two

possible answers (except for $x = 0$). However, as we've noted, in the end this isn't a big deal, because you just need to throw in a "±," and you're all set.

In the terminology of *functions* (the topic of Chapter 7), the issue is that the squaring and absolute-value operations don't have unique inverse operations. We'll talk about inverse functions in detail in Section 7.5.

4. We have observed that when finding square roots, you just need to throw in a "±." However, this works only for square roots, and *not* for cube roots, or fourth roots, etc. Those require a different procedure. There's no need to worry about that now, since we'll be dealing mainly with square roots for a while. But later on in Example 8.13 we'll learn how to find cube roots. Finding *all* of the cube roots of a number is possible only with the help of imaginary numbers, which we'll introduce in Section 8.5. But you can ignore imaginary numbers for now. ♣

Exercise 5.1 Solve for x in the following equations. Do this by performing the same (useful) operation on both sides, and *not* by guessing and checking. A single operation will get the job done for these equations.

(a) $x + 3 = 8$ (b) $-5 = x/2$ (c) $-x = 3$

(d) $4/x = 1$ (e) $\sqrt{x} = 6$ (f) $x^3 = 27$

We mentioned the following critical point a number of times in the above examples, but let's put it in a box here for good measure:

> Of the infinite number of operations you can perform on both sides of an equation, only one (or perhaps a few, in more complicated equations) is actually *useful*.

> There are zillions of things you can do
> That will make your equations be true.
> But alas, just a few
> Will be *useful* to you.
> All the others won't give you a clue!

This observation that only one (or possibly a few) particular operation is useful is closely related to the observation we made in Section 1.13 when we formed common denominators for the purpose of adding fractions. Eq. (1.87) was an example of some operations that were completely valid but also completely useless.

An important piece of advice: After you're done solving for x (or whatever letter you're solving for), you should *always* plug your answer for x back into the given equation to check that it yields a true statement. It's always possible to make a mistake when doing algebra. And the more complicated the equation, the more likely a mistake. So checking your answer is a wise move. In the three examples we discussed above, it takes only a few seconds to show that our x answers of 9, 4, and ± 5 do indeed yield true statements.

Always plug your answer for x back into the given equation to check that it yields a true statement!

After solving, resist the temptation
To go nuts with a huge celebration.
Your x, you can't trust,
So without fail you must
Plug it in to obtain confirmation.

Letters

The examples we've done so far have mainly involved numerical operations (with the exception of a couple operations involving x in Exercise 5.1). We've added 2, divided by 3, etc. But operations can just as well involve letters. For example, if you're given the equation $a \cdot b = c$, and if you want to solve for b in terms of a and c, you can divide both sides of the equation by a to obtain b:

$$\frac{\cancel{a} \cdot b}{\cancel{a}} = \frac{c}{a} \quad \Longrightarrow \quad b = \frac{c}{a}. \tag{5.10}$$

The a's cancel on the left, leaving b all alone. Note that this equation is equivalent to Eq. (5.6), with 3 replaced by a, and x replaced by b, and 12 replaced by c.

As another example involving letters, you may be familiar with the Pythagorean theorem, which is covered in Section A.6 in Appendix A. (There's no need to read that section now. We're just using the theorem to illustrate a point.) The theorem says that the lengths of the sides of a right triangle satisfy $a^2 + b^2 = c^2$. If you want to solve for a^2 in terms of b^2 and c^2, you can subtract b^2 from both sides of $a^2 + b^2 = c^2$ to obtain

$$(a^2 + \cancel{b^2}) - \cancel{b^2} = c^2 - b^2 \quad \Longrightarrow \quad a^2 = c^2 - b^2. \tag{5.11}$$

If you then want to solve for a itself, you can perform the additional operation of taking the square root of both sides to obtain $a = \pm\sqrt{c^2 - b^2}$. The final step is to note that since a length is necessarily positive, you can thrown away the negative option for the square root. It's technically always there, but it's irrelevant in the present case where we're looking for a length. So the answer is $a = \sqrt{c^2 - b^2}$.

Terminology

A few bits of terminology:

1. The act of adding 2 to both sides of $x - 2 = 7$ can be called "getting the 2 on the other side." This language is quite reasonable, because the 2 disappeared from the left side of Eq. (5.4) and appeared on the right. Likewise, dividing both sides of $3x = 12$ by 3 is called "getting the 3 on the other side," which is what happened in Eq. (5.6).

   However, you have to be careful about how you put the number (or letter) on the other side. You can't just put it anywhere. The 2 originally appeared with a minus sign on the left side of Eq. (5.4), so it appears with a plus sign on the right. Similarly, the 3 originally appeared in the numerator on the left side of Eq. (5.6), so it appears in the denominator on the right. Each number appears in a new manner, so the "get this thing over there" language doesn't quite capture the whole story. At any rate, as long as you perform the same (useful) operation on both sides of an equation, everything will automatically work out, and all the numbers (or letters) will end up where they should be.

2. In an equation like $a + c = b + c$, you can simply cancel (that is, erase) the c's on both sides to obtain

   $$a + \cancel{c} = b + \cancel{c} \quad \Longrightarrow \quad a = b. \tag{5.12}$$

   But to be precise, what you're actually doing is subtracting c from both sides to obtain

   $$(a + \cancel{c}) - \cancel{c} = (b + \bcancel{c}) - \bcancel{c} \quad \Longrightarrow \quad a = b. \tag{5.13}$$

   So technically in Eq. (5.12) you're not canceling the two c's on *opposite* sides of the equation with each other. Instead, you're canceling each c with a new c that you've subtracted on the *same* side, as we observed in Eq. (5.13). But people generally use the shorthand terminology of "canceling common terms on opposite sides," like the c's in Eq. (5.12), even though Eq. (5.13) is what they're really doing.

   A similar thing occurs with multiplication. In the equation $a \cdot c = b \cdot c$, you can cancel (erase) the c's on both sides to obtain

   $$a \cdot \cancel{c} = b \cdot \cancel{c} \quad \Longrightarrow \quad a = b. \tag{5.14}$$

   But again, what you're actually doing is dividing both sides by c obtain

   $$\frac{a \cdot \cancel{c}}{\cancel{c}} = \frac{b \cdot \bcancel{c}}{\bcancel{c}} \quad \Longrightarrow \quad a = b. \tag{5.15}$$

   If you're new to solving equations, it's probably safer to think in terms of the steps in Eqs. (5.13) and (5.15). But after you get the hang of it, you can just cancel things as we did in Eqs. (5.12) and (5.14). That's what people do in practice (once they've had a lot of it!).

   The above cases dealt with addition and multiplication. The same canceling steps also apply to subtraction (as in $a - \cancel{c} = b - \cancel{c}$, where you're actually adding c to both sides), and division (as in $a/\bcancel{c} = b/\bcancel{c}$, where you're actually multiplying both sides by c).

3. It gets tiring to keep repeating the words "both sides of." So instead of saying, "Multiply both sides of the equation by 3," we often just say, "Multiply the equation by 3." This action can also be phrased as "Multiply the equation through by 3," or just "Multiply through by 3." In any case, however you want to say it, it's always understood that you're performing the *same* operation on *both sides* of the equation. ♣

Two-step operations

All of the equations we've solved so far have involved a single operation, with the exception of the Pythagorean-theorem result of $a = \sqrt{c^2 - b^2}$ on page 231 (where we subtracted b^2, and then took the square root), and the second solution to Exercise 5.1(c) (where we added x, and then subtracted 3).

Let's now solve some other equations that require two steps. As we'll see, for the first step, there are sometimes two operations (among the infinite number of possible ones) that are useful. In Section 5.2 we'll solve equations that require more than two steps.

Example 5.4 Consider the equation $x/3 - 2 = -7$. There are two ways to solve it. We can add 2 to both sides to obtain $x/3 = -5$, and then multiply both sides by 3 to obtain $x = -15$.

Alternatively, we can start by multiplying both sides of $x/3 - 2 = -7$ by 3, which yields $x - 6 = -21$. Adding 6 to both sides then gives $x = -15$. The two possibilities for a useful first step are therefore adding 2 to both sides, or multiplying both sides by 3.

As a check, plugging $x = -15$ back into the given equation leads to the true statement, $-5 - 2 = -7$. It is understood that the plugging-back-in check should always be done, although we won't explicitly write it out every time we solve an equation. You can do the check in your head in simple cases. But for more complicated equations like the ones we'll solve in the next section, you'll often need to write things out.

Exercise 5.2 Solve for x in the following equations. Each solution involves two steps.

(a) $5x + 2 = 12$ (b) $(x - 6)/3 = 7$ (c) $5/(x - 2) = 1$

(d) $\sqrt{x - 3} = 4$ (e) $x^2/3 = 12$ (f) $1/x = 2$

Exercise 5.3 If $a = b$, we can reverse the sides of this equation to obtain $b = a$. (If two things are equal, it doesn't matter what order we write the equality.) But if you want to be formal about reversing the order, show (by performing the same operations on both sides of the equation) how $a = b$ can be turned into $b = a$. This involves three steps.

Exercise 5.4 We saw in Eq. (1.64) (with different letters) that $a/(b/c)$ equals ac/b. That is, the c "flips up top." Reproduce this result by letting $x = a/(b/c)$, and then multiplying both sides of this equation by b/c, and then solving for x.

REMARK: This is a slightly picky remark, but I'll say it anyway: It's conceivable that someone might object to the above "equal changes to equal things" reasoning by saying, "Consider, for example, the equation $x/3 - 2 = -7$ in Example 5.4. Since we don't yet know what value of x makes this equation be true, how do we know that we're making equal changes to *equal* things when we add 2 to both sides? In fact, since most values of x (more precisely, all of the infinite possible values except one) *don't* make the equation true, we're probably making equal changes to *unequal* things, in which case the results are unequal."

The response to this concern is the following. It is indeed generally true that making equal changes to *unequal* things results in things that are again unequal. (There are a few exceptions, though. For example, multiplying the untrue equation $2 = 3$ by 0 yields the true equation $0 = 0$.) However, this fact is irrelevant, because we're concerned only with the special case where the given equation *is* true, that is, where the two sides *are* equal.

The logical sequence of statements that are relevant when solving the equation are: (a) *If* the equation $x/3 - 2 = -7$ is true, *then* (after adding 2 to both sides) the equation $x/3 = -5$ is also true. And (b) *If* the equation $x/3 = -5$ is true, *then* (after multiplying both both sides by 3) the equation $x = -15$ is also true. Putting these two statements together, we see that *if* the equation $x/3 - 2 = -7$ is true, *then* it must be the case that $x = -15$. So if someone comes along and tells you that $x/3 - 2 = -7$ *is* true, then you know that $x = -15$ is the special value of x that makes it true. You don't care about all the (infinite number of) other x values that don't make it true. ♣

Illegal operations

We have stated a number of times that performing the same operation on both sides of an equation produces an equation that is again true. When making this statement, it is understood that the operation is a *legal* one. The most common *illegal* operation is division by zero, which we discussed in the subsection at the end of Section 1.11.

Consider, for example, the equation $0 \cdot 3 = 0 \cdot 7$. This is a true equation, since both sides equal 0. But if we divide both sides by 0, we obtain $3 = 7$, which is false. It's clear that we performed an (invalid) division by zero here. But in some instances it isn't so obvious, so you need to be careful. Such is the case in the following classic mathematical fallacy, where a sequence of seemingly legitimate steps appears to prove that 2 equals 1.

Starting with the relation $a = b$, perform the following six steps: (1) multiply both sides by a, (2) subtract b^2, (3) factor, (4) divide by $a - b$, (5) replace a with b (since $a = b$), (6) divide by b:

$$\begin{aligned} a = b \quad &\overset{1}{\Longrightarrow} \quad a^2 = ab \quad \overset{2}{\Longrightarrow} \quad a^2 - b^2 = ab - b^2 \quad \overset{3}{\Longrightarrow} \quad (a+b)(a-b) = b(a-b) \\ &\overset{4}{\Longrightarrow} \quad a + b = b \quad \overset{5}{\Longrightarrow} \quad 2b = b \quad \overset{6}{\Longrightarrow} \quad 2 = 1. \end{aligned} \tag{5.16}$$

There clearly must be an invalid step somewhere here, because it certainly isn't true that $2 = 1$! The invalid step was dividing both sides by $a - b$, because this quantity equals zero, due to the initial statement that $a = b$. (All of the other steps are legal.) The zero in the form of $a - b$ here isn't as obvious as the actual 0 that was sitting on the page in $0 \cdot 3 = 0 \cdot 7$, but it's still just as much a zero. So dividing by it is just as illegal.

Summary of important points

Below is a summary of important things to keep in mind when solving equations. All of these items (there are many of them!) were already mentioned at some point above, but it's good to collect them all in one place. You should refer back to this list every now and then as you work through this chapter.

1. The letter x is commonly used to denote a quantity that you're trying to solve for (which is called a *variable* or an *unknown*). But you can use any letter you like; there's nothing special about x. The letters y or z work just as well. Or a or m or whatever. You don't even need to use a letter. You can use symbols like $\clubsuit$ or $\diamond$ or $\odot$, or even a picture of a tree if you feel like it. Really, anything is fine. But x is easy to write.

2. Although there's nothing wrong with using letters at the beginning of the alphabet (a, b, etc.) as unknowns, people generally reserve those letters for things whose value you already know and that you're not trying to solve for. Unknowns are more commonly labeled with letters at the end of the alphabet (x, y, etc.). But this is just convention. There's no right or wrong choice for what letter(s) to use.

3. "Solving for x" means the same thing as "*isolating* x on one side of an equation." This is accomplished by generating successive true equations until you finally arrive at an equation of the form "x = something" (or "something = x"), where the "something" involves numbers and/or other letters.

4. How do you isolate (that is, solve for) x? You do this by *performing the same operation(s) on both sides of an equation*. If you start with two things that are equal (namely, the two sides of an equation) and then do the same thing to each, the results are again equal. When someone is "doing algebra," that could mean various things. But a lot of the time it means doing the same thing to both sides of an equation.

   For example, to solve the equation $x + 3 = 8$, you subtract 3 from both sides to obtain $(x + \cancel{3}) - \cancel{3} = 8 - 3 \implies x = 5$. And to solve the equation $5x = 20$, you divide both sides by 5 to obtain $\cancel{5}x/\cancel{5} = 20/5 \implies x = 4$.

5. When doing the same thing to both sides of an equation, you want to be sure to do something *useful*. If you subtract 7 from both sides of $x + 3 = 8$, you will end up with the perfectly true, but rather *useless*, statement $x - 4 = 1$. Adding/subtracting or multiplying/dividing by random numbers doesn't help in achieving the goal of isolating x on one side of the equation. To be successful at this, you need to make wise choices

about the common operation you apply to both sides. We'll get lots of practice with this in Section 5.2.

As another example of a useless operation, consider the equation $(x+6)/2 = 5$. Subtracting 6 from both sides is *not* useful, because the 6 that appears in the equation is divided by 2, whereas the 6 you're subtracting isn't. So they won't cancel and leave x all alone. What you want to do is first multiply both sides by 2 to obtain $x + 6 = 10$, and *then* subtract 6 from both sides to obtain $x = 4$. Alternatively, you can first perform the division by 2 on the left side to obtain $x/2 + 3 = 5$, and then subtract 3 from both sides to obtain $x/2 = 2$, and then multiply both sides by 2 to obtain $x = 4$.

6. As we just saw, there can be more than one useful first step to take in isolating x. (But there are at most a *few* useful steps, in contrast with the infinite number of useless ones!) As another example of this, consider the equation $x/2 + 3 = 7$. You can first subtract 3 from both sides to obtain $x/2 = 4$, and then multiply both sides by 2 to obtain $x = 8$. Alternatively, you can first multiply both sides by 2 to obtain $2(x/2 + 3) = 2 \cdot 7 \implies x + 6 = 14$, and then subtract 6 from both sides to obtain $x = 8$. Personally, I like the first of these two methods better, because it immediately cuts the number of terms from three down to two. But both methods work fine.

7. Subtracting 3 from both sides of the equation $x + 3 = 8$ is called "getting the 3 on the other side" (where it then gets combined with the 8). Likewise, dividing both sides of the equation $5x = 20$ by 5 is called "getting the 5 on the other side" (where it then partially cancels the 20).

8. In the equation $x + b = a + b$, you can cancel (erase) the b's, to obtain $x = a$. But what's really going on with this cancellation is that you're subtracting b from both sides. Each of the original b's cancels with the b you subtract from that side: $(x+\cancel{b})-\cancel{b} = (a+\cancel{b})-\cancel{b} \implies x = a$. So there are actually two separate cancellations occurring, one on each side. But people rarely write this all out; they simply erase the two original b's.

9. When we say "divide the equation $5x = 20$ by 5," we really mean "divide *both sides of* the equation $5x = 20$ by 5." But we often don't bother saying the "both sides of" part. Likewise, "take the square root of this equation" really means "take the square root of both sides of this equation." The "both sides of" is always understood, because everything always boils down to doing the same thing to both sides of an equation – making equal changes to equal things.

10. After you solve an equation for x (or whatever letter you're solving for), remember to always plug your answer for x back into the equation to make sure it yields a true statement. It's inevitable that you'll make algebra mistakes sooner or later, so you should always check your answer.

Now that you've learned how to solve some simple equations in this section, and also had lots of practice working with letters in the previous two chapters, this is a good time to read Appendix B, which discusses the many benefits of using letters (doing algebra) instead of only numbers (doing arithmetic).

5.2 Undoing the operations

In this section and the rest of the chapter, we'll deal with more complicated problems. We'll ramp things up in four main ways: (1) In this section and the next, the equations will involve many operations, instead of just the one or two operations in the problems we've done so far. (2) In Section 5.5 we'll solve word problems, which require translating words into equations. (3) In Section 5.6 we'll solve for more than one unknown letter. (4) And in Section 5.7 we'll solve a few quadratic equations (ones involving the square of a variable), but just some simple ones. We'll discuss quadratic equations in much greater detail in Chapter 8.

The main thing to keep in mind in this section is that no matter how complicated an equation is, solving it generally boils down to (as we emphasized above) doing the same (useful) things to both sides of the equation, until you end up with x all alone on one side. You have then solved for x. It's just that in more complicated equations, you have to do more steps, and in the proper order. So far, our equations haven't involved too many terms, so the steps were fairly clear. But for complicated equations, how do you determine which operations are the useful ones, and what their order should be? The basic strategy is:

> Choose your operations to be the ones that *undo* the given operations, in the *reverse* order.

This strategy works in cases where the given equation involves a single appearance of x. We'll focus on this type of equation here. Section 5.3 covers equations involving more than one x.

Undoing the operations that appear in the given equation will result in x being all alone on one side of the equation, which is the goal. To see how this works, consider the fairly simple equation,

$$\frac{x-5}{2} = 3. \tag{5.17}$$

If you start with a value of x, the lefthand side of this equation instructs you to:

1. Subtract 5 from x, and then

2. Divide the result by 2.

To solve for x, we need to *undo* the operations, in the *reverse order*. By "undo" we mean perform the inverse operation (on both sides), where addition is the inverse of subtraction (and vice versa), multiplication is the inverse of division (and vice versa), squaring is the inverse of taking a square root (and vice versa, if you throw in a "±"), and so on. So for Eq. (5.17), if we *replace each operation with its inverse* and *reverse the order*, we see that we want to:

1. *Multiply* both sides by 2, and then

2. *Add* 5 to both sides.

The *first* step of *subtracting* 5 becomes the *second* step of *adding* 5. And the *second* step of *dividing* by 2 becomes the *first* step of *multiplying* by 2.

Note the words "both sides" in the reverse/undoing steps. Although it's the lefthand side of Eq. (5.17) that dictates these steps, we must of course apply them to *both sides* of the equation. The equation-solving process always involves the same operations on both sides.

When applied to Eq. (5.17), the above reverse/undoing operations yield:

$$\text{Multiplying by 2:} \quad \cancel{2} \cdot \frac{x-5}{\cancel{2}} = 2 \cdot 3 \quad \Longrightarrow \quad x - 5 = 6, \tag{5.18}$$

and then

$$\text{Adding 5:} \quad (x - \cancel{5}) + \cancel{5} = 6 + 5 \quad \Longrightarrow \quad x = 11. \tag{5.19}$$

We have successfully solved for x by isolating it on one side of the equation.

Note that the other order of operations doesn't work. (More precisely, it leads to a true statement, but one that isn't useful.) If we *first* add 5, we obtain

$$\frac{x-5}{2} + 5 = 3 + 5. \tag{5.20}$$

The addition of 5 on the lefthand side doesn't help us. Splitting up the fraction, we see that the lefthand side equals $x/2 - 5/2 + 5$, which simplifies to $x/2 + 5/2$. So we haven't gained anything in our goal of isolating x on one side.

We stated above that the lefthand side of Eq. (5.17) instructs us to first subtract 5, and then divide by 2. However, if we split up the fraction, we find that there is a second valid way of viewing the instructions, as the following exercise shows.

Exercise 5.5 Rewriting Eq. (5.17) by splitting up the fraction yields $x/2 - 5/2 = 3$. This equation involves a different set of steps. Write out the "forward" and "reverse" steps analogous to the ones we wrote above, and then solve for x.

Example 5.5 Here is a more involved example, where we need to be careful about the order of the operations. We'll solve the following equation by applying the above technique of undoing the operations in the reverse order:

$$\frac{5\sqrt{x+4} - 1}{7} = 2. \tag{5.21}$$

Solution We'll first list the operations specified by the lefthand side of Eq. (5.21). Starting with x, the lefthand side instructs you to perform five operations in the following order:

1. Add 4,
2. Take the square root,
3. Multiply by 5,
4. Subtract 1,
5. Divide by 7.

Undoing these operations in the reverse order means taking these five steps:

1. Multiply by 7,
2. Add 1,
3. Divide by 5,
4. Square the result,
5. Subtract 4.

Applying these five steps to both sides (always both sides!) of the given equation yields (as you can verify)

$$\frac{5\sqrt{x+4}-1}{7} = 2 \implies 5\sqrt{x+4}-1 = 14 \implies 5\sqrt{x+4} = 15$$
$$\implies \sqrt{x+4} = 3 \implies x+4 = 9 \implies x = 5. \tag{5.22}$$

So the solution for x is 5. (You should plug this back in to check that it works.) Again, the order of the "undoing" steps very much matters. If you performed the "Add 1" step first instead of second, you would end up with a true but unhelpful equation.

Although Eq. (5.21) might have seemed a bit complicated at first glance, solving equations of this sort really isn't difficult, provided that you list all the steps and stay organized. If you encounter an equation involving ten steps instead of the five in Eq. (5.21), there's no need to panic, because the only difference is that you'll now end up with lists of operations that are twice as long, and a chain of equations that is twice as long as the chain in Eq. (5.22). The process will therefore take you a little longer, but there won't be anything terribly difficult about it. You just need to systematically march through your list of the ten "undoing" steps.

We mentioned above that inverse operations (ones that undo each other) come in pairs. The pairs we've encountered so far are addition/subtraction, multiplication/division, and squaring/square root (with possibly a "±" when taking the square root). This last pair is a special case of the more general nth-power and $1/n$th-power pair (for example, the $n = 3$ case in Exercise 5.1(f)). Another pair is exponentials and logs, as we'll see in Section 9.2.3.

Yet another inverse-operation pair is reciprocal/reciprocal. That is, the act of taking the reciprocal is its own inverse. To undo a reciprocal, you take another reciprocal. This is true because if you start with a and take the reciprocal, you get $1/a$. And then if you take the reciprocal of this, you get $1/(1/a) = a$. So you're back to where you started. One reciprocal undoes the other. Exercise 5.6 involves a reciprocal.

Exercise 5.6 Solve for x in the following equation by listing the operations involved, and then listing the "undoing" steps in the reverse order.

$$5 \cdot \left(\frac{1}{2x-1} + 2\right) = 11. \tag{5.23}$$

Exercise 5.7 Solve for x in the following equation by listing the operations involved, and then listing the "undoing" steps in the reverse order. There are eight steps here. But as we noted above, the process is just longer, not more difficult!

$$\frac{1}{2} \cdot \left[\left(\frac{\sqrt{5x+1}}{3} + 3\right)^2 - 7\right] = 9. \tag{5.24}$$

You are encouraged to make up your own problems, similar to the above exercises. If you want the numbers to work out nicely, you can create the equation by starting with a particular value of x, and then performing a series of operations (of your choosing) on it. For example, I generated Exercise 5.7 by starting with $x = 7$, and then (arbitrarily) multiplying it by 5, and then adding 1 (so that it would be the perfect square 36), and then taking the square root, etc.

Of course, if you make up problems this way, you'll already know what the solution for x is. But that's fine. You're still going to have to write out the two lists of steps like we did in the above example and exercises, and then do all the work you would need to do if you didn't already know the answer. But if you want some "fresh" problems, you can make up a bunch of them and then set them aside for a week or two, until you forget what the answers are.

As we mentioned in Remark 4 on page 235, "doing algebra" often involves performing the same operations on both sides of an equation. But it isn't helpful to just pick random operations. You need to choose useful ones. And the strategy of writing down the "undoing" list ensures that your operations will in fact be useful.

For the first few complicated equations you solve, like the ones in the above exercises, you should definitely write out both the "forward" and "backward" lists of steps, as we did in the solutions. But once you get comfortable with undoing the operations, you don't need to keep writing out all the steps. With some practice, you'll know what you need to do for each step, and you can proceed directly from Eq. (5.21) to Eq. (5.22), for example. However, if you ever get confused about how to solve a complicated equation involving many operations, you can always fall back on the no-fail method of writing out all of the forward steps, and then all of the backward steps, and then marching through the backward ones.

> For equations that aren't very terse,
> Remember the words of this verse:
> If you need an assist,
> Simply make a nice list,
> And then run all the steps in reverse.

Remember that undoing the operations involves *two* kinds of reversals: You need to reverse (1) the *order* of the operations, and also (2) the *type* of each operation – for example, adding instead of subtracting, multiplying instead of dividing, etc.

Equations are better than words

If you explain the content of an equation with words (instead of numbers and letters), it's invariably much more confusing than the equation itself. For example, in words Eq. (5.21) says that "A seventh of 1 less than 5 times the square root of 4 more than x equals 2." This is a mouthful and very difficult to understand. The equation in Eq. (5.21) is a *much* more compact and clear way of saying the same thing. Equations are far more illuminating than wordy sentences. An equation might not quite be worth 1000 words, but it's still worth a lot!

But what if someone *does* use words to tell you that a seventh of 1 less than 5 times the square root of 4 more than x equals 2, and what if you don't know anything about algebra or the strategy of performing the same operation on both sides of an equation? You can still solve for x by successively deducing the facts in the list below. Starting with the given "a seventh of. . ." sentence, and then working your way along the sentence (which basically means removing the first few words at each step), you will see that:

1. 1 less than 5 times the square root of 4 more than x must equal $7 \cdot 2 = 14$ (so that a seventh of this is 2).

2. 5 times the square root of 4 more than x must equal $14 + 1 = 15$ (so that 1 less than this is 14).

3. The square root of 4 more than x must equal $15/5 = 3$ (so that 5 times this is 15).

4. 4 more than x must equal $3^2 = 9$ (so that the square root is 3).

5. x must equal $9 - 4 = 5$ (so that 4 more than this is 9).

We'll use the term "word logic" to describe the above process. The operations in the list (namely multiplying by 7, adding 1, dividing by 5, squaring, and subtracting 4) are exactly the same as the operations in the "undoing" list in Example 5.5, or equivalently the same as the operations in Eq. (5.22). But I think you'll agree that Eq. (5.22) is *much* easier to follow than the above wordy list! It's hard to avoid getting lost in the successive sentences. So if you find the wordy list confusing, that's a good thing, because it means you'll have a greater appreciation for the algebraic way of solving for x in Eq. (5.22)!

Algebra steps in equations are much easier to follow than "word logic" steps in sentences!

Exercise 5.8 Solve the equation $(3x+2)/5 = 4$ in three different ways:

(a) Guess and check.

(b) Use "word logic" as we did above, to generate successively shorter sentences.

(c) Use algebra, by proceeding through an "undoing" list of steps.

5.3 More than one x

In all of the problems we've done so far, there has been only one x in the given equation. But what if we have more than one? With two or more x's, they might appear on both sides of the equation. For example, the following equation has two terms involving x on the left, and one on the right:

$$5x + 7 - 2x = 3 - x. \tag{5.25}$$

How can we solve this? There isn't just one x here that we perform successive operations on. We're doing different things to the three different x's, so we can't simply undo a single list of operations like we did above. So how do we solve for x?

Well, when solving for x, the end goal is always the same – to isolate x on one side of the equation (either side is fine). And we can accomplish this by:

1. Performing the same (useful) operations on both sides of the equation, as we've been doing. But now also:

2. Simplifying each side of the equation at various opportune points in the process.

There are many different orders of (useful) actions we can take on Eq. (5.25), but probably the easiest is to start by simplifying the lefthand side by combining the x's. Since $5x-2x = 3x$, the equation becomes

$$3x + 7 = 3 - x. \tag{5.26}$$

Our goal is to isolate x on one side, so we want to get all the x's on one side of the equation (either side), and all the regular numbers on the other side. Let's get the x's on the left, and therefore the regular numbers on the right. Adding x to both sides gives $4x+7 = 3$. And then subtracting 7 from both sides gives $4x = -4$. Finally, dividing both sides by 4 yields $x = -1$. So that's our answer.

If we had chosen to get the x's on the right side and the regular numbers on the left, we would have ended up with $4 = -4x$. Dividing both sides by -4 then gives $-1 = x$, which is the same answer. It doesn't matter which side you choose to get the x's on, but personally I generally try to get them on the side where the final coefficient of x is positive. It's not critical to do this, though. At worst, you'll need to divide by a negative number in the final step, as we did with $4 = -4x$. That's perfectly fine and isn't much of an inconvenience. But my view is that the fewer minus signs you need to deal with, the less likely you are to accidentally drop one.

Example 5.6 Solve this equation for x:

$$7\left(\frac{5x-1}{2} - x\right) = \frac{4-x}{5}\,. \tag{5.27}$$

Solution There are many different routes we can take in isolating x on one side of the equation. Here's one. We'll first combine the x's on the left. Writing x as $2x/2$ to get a common denominator, we obtain

$$7\left(\frac{3x-1}{2}\right) = \frac{4-x}{5}\,. \tag{5.28}$$

Equations without denominators are simpler, so let's multiply both sides by 2 to get the 2 out of the denominator on the left. And likewise multiply both sides by 5. Equivalently we can just multiply by $2 \cdot 5 = 10$ all at once. This gives

$$5 \cdot 7(3x-1) = 2(4-x) \quad \Longrightarrow \quad 105x - 35 = 8 - 2x. \tag{5.29}$$

Adding $2x$, and also 35, to both sides, and then dividing by 107 yields

$$107x = 43 \quad \Longrightarrow \quad x = \frac{43}{107}\,. \tag{5.30}$$

This isn't the prettiest answer, but that's the way it is. You're probably not going to arrive at it by guessing and checking!

There are plenty of other routes that will also lead to x being all alone on one side of the equation. For example, after arriving at Eq. (5.28) (or even before), you can distribute the 7 into the parentheses and separate the fractions on each side of the equation to obtain $21x/2 - 7/2 = 4/5 - x/5$. Adding $x/5$, and also $7/2$, to both sides and simplifying yields $107x/10 = 43/10$, as you can check. Multiplying both sides by $10/107$ then gives $x = 43/107$, as above.

Since I don't like dealing with denominators, I usually try to get rid of them early in the process, by multiplying both sides by them. But that's just a matter of personal preference, and I'm sure there are cases where I apply a strategy more like the one in the preceding paragraph.

Exercise 5.9 Solve these equations for x:

(a) $1 + 2x = 7 - x$ (b) $\frac{1}{2}\left(5x + \frac{2-3x}{4}\right) = \frac{2x-1}{3}$

Cross multiplication

If we take the equation,

$$\frac{a}{b} = \frac{c}{d}\,, \tag{5.31}$$

and multiply both sides by bd and then cancel some factors, we obtain

$$\cancel{b} \cdot d \cdot \frac{a}{\cancel{b}} = b \cdot \cancel{d} \cdot \frac{c}{\cancel{d}} \quad \Longrightarrow \quad ad = bc. \tag{5.32}$$

The process of going from Eq. (5.31) to Eq. (5.32) is called *cross multiplication*. This name makes sense, because the act of multiplying both sides of Eq. (5.31) by bd is equivalent to multiplying the a on the upper-left by the d on the lower-right, and equating the result with the product of the b on the lower-left with the c on the upper-right. This is shown in Fig. 5.1.

$$\frac{a}{b} \qquad \frac{c}{d}$$

Figure 5.1: Cross multiplication.

In your head, you can think of the process as cross-multiplying, but remember that what you're really doing is multiplying both sides by bd. We already encountered cross multiplication (without calling it that) in the multiplication by $2 \cdot 5$ that led to Eq. (5.29), and in the multiplication by $2 \cdot 3$ that led to Eq. (5.85) in the solution to Exercise 5.9(b).

Exercise 5.10 Solve these equations for x:

(a) $\frac{4x-3}{7} = \frac{6-x}{5}$ (b) $\frac{7}{3x-2} = \frac{3}{2x+7}$

5.4 Inequalities

We introduced inequalities back in Section 1.10. To review: The relation $a < b$ means "a is less than b," and the relation $a > b$ means "a is greater than b." So $c > d$ and $d < c$ mean the same thing; the former says that c is greater than d, and the latter says that d is less than c. These are equivalent statements. Said in another way, if you flip the inequality $c > d$ over (imagine literally flipping the paper over and looking through the reverse side), then it becomes $d < c$. Well, the c and d themselves don't look the same when flipped over (the d now looks like a b), but ignore that fact.

Let's say that someone asks you what range of x values satisfy the inequality $3x < 21$. When solving *inequalities* like this, the goal is to find a *range* of x values, as opposed to the *single* value of x we obtain when solving *equalities* (or perhaps a few values, as with quadratic equations, for example). To solve an inequality, we can use the same strategy as with equalities, namely: Isolate x on one side of the inequality by (1) Performing the same operations on both sides of the inequality, and (2) Simplifying each side as needed. For the problem at hand, we just need to divide both sides of the inequality by 3, which yields $x < 7$. So that's our answer; all values of x that are less than 7 satisfy the given inequality $3x < 21$.

The principle behind the above reasoning is nearly the same as our general principle of "equal changes to equal things produce things that are again equal." The slight difference is that with inequalities, we are *not* starting with two things that are equal. So we can't use exactly the same "equal changes to equal things" principle. But a variation gets the job done: In terms of buckets of water, if bucket A has more water than bucket B, and if you divide the amounts of water in the two buckets by the *same* factor of, say, 3, then bucket A will still have more water than bucket B. Likewise, if you add, say, 7 gallons to each, bucket A will still have more water. So doing the same thing to both buckets will keep the amounts in the same order. (However, see the "Multiplying by -1" subsection below for an exception to this rule.)

Example 5.7 Solve this inequality for x:

$$5(x-2)+3 > 3x-1. \tag{5.33}$$

Solution Our goal is to get x all alone on one side of the inequality by performing the same operations on both sides, and simplifying as needed. If we distribute the 5 into the parentheses and simplify, and then subtract $3x$ and add 7 to both sides, and then divide by 2, we obtain

$$(5x-10)+3 > 3x-1 \quad\Longrightarrow\quad 5x-7 > 3x-1 \quad\Longrightarrow\quad 2x > 6 \quad\Longrightarrow\quad x > 3. \tag{5.34}$$

Therefore, all values of x larger than 3 satisfy the given inequality. All of the steps here are exactly the same as if the ">" inequality sign had been an "=" equals sign. But instead of the answer being the single number 3, the answer is "all numbers larger than 3." If the inequality sign had been a "≥" instead of a ">," then the answer would have been "$x \geq 3$," that is, "all numbers greater than *or equal to* 3."

Multiplying by −1

We have seen that solving inequalities involves the same basic steps as solving equalities. However, there is one critical case where the procedure for inequalities differs from equalities. Consider the inequality $-x > 3$. There are two plausible ways to solve it:

1. We can add x to both sides, and subtract 3 from both sides, to obtain $-3 > x$ (or equivalently $x < -3$). Therefore, x is *less* than -3. So -7 satisfies the inequality, for example.

2. We can multiply (or equivalently divide) the inequality by -1 to obtain $x > -3$. We're therefore now saying that x is *greater* than -3. So it seems like -1 should satisfy the inequality, for example.

These two answers don't agree. Which one is correct? Is the given inequality, $-x > 3$, satisfied by values of x that are less than -3, or greater than -3? Well, the second of these answers (namely $x > -3$) says that all numbers greater than -3 satisfy the given inequality, $-x > 3$. In particular, it says that since $x = 100$ is a number greater than -3, it should satisfy the given inequality, $-x > 3$. But this is incorrect, because $-100 > 3$ isn't true. Similarly, the -1 value we mentioned above doesn't satisfy the inequality, because it isn't true that $-(-1) > 3 \implies 1 > 3$. The second of the above answers is therefore wrong.

The first answer is the correct one, because adding x and subtracting 3 from both sides are certainly legal steps. So everything about the first solution is valid. And as a check, $x = -7$ satisfies $-x > 3$, because $-(-7) > 3$ is equivalent to $7 > 3$, which is true.

Since the second of the above answers is wrong, there must have been an invalid step in the solution. Where exactly is the error? The error is the multiplication (or division) by -1. Multiplication by a negative number works perfectly fine with *equalities* (making equal changes to equal things results in things that are again equal). But it does *not* work with inequalities (without a correcting action, as we'll see). The reason, which is summarized in Fig. 5.2, can be explained as follows.

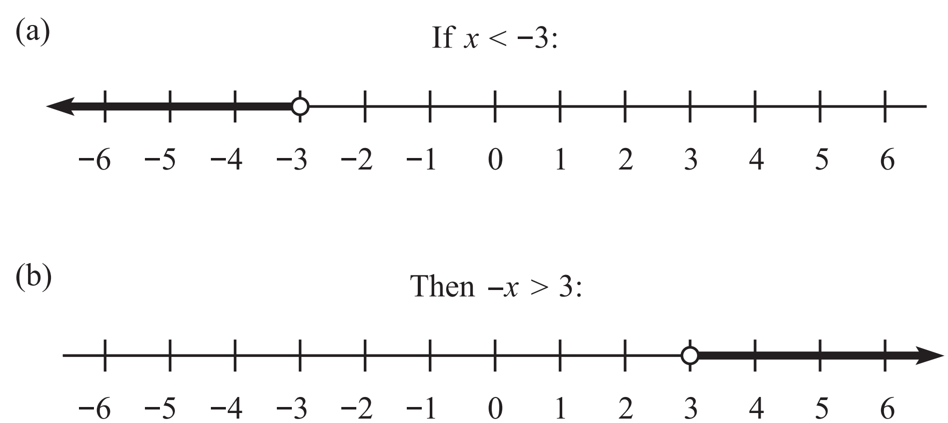

Figure 5.2: If you multiply an inequality through by -1, you must also reverse the sign.

Multiplying by -1 requires reversing the sign: On the number line in Fig. 5.2(a), the set of all numbers satisfying $x < -3$ is indicated by the heavy line. (The arrow signifies that the values extend down to $-\infty$, and the hollow dot indicates that -3 itself isn't included in the set.) This set of numbers includes values such at -3.5, -16, and -207. These all satisfy $x < -3$. Let's now take all of the numbers that satisfy $x < -3$ and *negate* them (that is, throw a minus sign in front of them, so a number x becomes $-x$). We then obtain the range of numbers indicated by the heavy line in Fig. 5.2(b). When negated, the numbers -3.5, -16, and -207 become 3.5, 16, and 207, which are all greater than 3. So we see that if $x < -3$, then $-x > 3$. These are equivalent statements.

The critical thing to note is that if we want to change the $x < -3$ inequality to the equivalent $-x > 3$ inequality, we can't just multiply the inequality through by -1. We must also *reverse the inequality sign*. The "<" becomes a ">." This reversal makes sense, because when you negate a number, you basically flip it over to the other side of the number line (the same distance from 0, but on the other side). Negating the numbers corresponding to the heavy line in Fig. 5.2(a) therefore involves flipping the heavy line over, from the left to the right side of the number line. This yields the heavy line in Fig. 5.2(b). So we see that the range in Fig. 5.2(a), where x is *less* than -3, leads to the range in Fig. 5.2(b), where $-x$ is *greater* than 3. The "less than" becomes "greater than," which means that we have to reverse the inequality sign. So that's the rule:

> If you multiply an inequality by -1, you must reverse the sign.

> Though positive 1's a frivolity,
> A negative 1 has the quality
> That when multiplied through,
> You must flip the sign too,
> If you still want a true inequality.

Of course, there's technically never any need to multiply an inequality by -1. You can always just get x on other side (and also get the regular number, which was 3 here, on the other side), as we did in our first solution above. You can't lose by using that method. But it's often quicker to multiply by -1 and reverse the sign.

This "reversing the sign" rule holds for multiplication (or division) by *any* negative number, not just -1. This is true because if you multiply an inequality by, say -8, this is equivalent to first multiplying by 8 (which doesn't involve reversing the sign), and then multiplying by -1 (which does involve reversing it). The generalization of the above boxed rule for -1 is therefore:

> If you multiply an inequality by any negative number, you must reverse the sign.

In Fig. 5.2(a) we drew a hollow dot at $x = -3$, which signifies that the number -3 is *not* included in the $x < -3$ range. If we were instead dealing with the inequality $x \le -3$ (with $\le$ instead of $<$), then we would draw a solid dot at -3. A solid dot includes the point in the range, whereas a hollow dot doesn't.

Exercise 5.11 Solve these inequalities for x:

(a) $\dfrac{1-2x}{5} < \dfrac{x+4}{2}$ (b) $3x - 5(x-1) > \dfrac{1-5x}{3} + \dfrac{x}{2}$

5.5 Word problems

5.5.1 Translating words into equations

If someone gives you the equation $(x+2)/3 = 4$ and asks you to solve for x, you know what to do: Multiply both sides by 3 to obtain $x + 2 = 12$, and then subtract 2 from both sides to obtain $x = 10$. But what if someone asks a question using words instead of an equation? For example, what if they phrase the above equation as, "If one third of 2 more than a number equals 4, what is the number?" We call such a question, quite naturally, a *word problem*.

REMARK: We should actually say, "If one third of *the quantity* 2 more than a number equals 4," to indicate the presence of the parentheses in $(x + 2)/3 = 4$. Otherwise the question might be interpreted as $x+2/3 = 4$. And technically that's what the original sentence means, because the order of operations is that division is done before addition. However, in the discussion below, we won't bother repeating the "the quantity" part, since the sentence is already long enough. But it's understood to be there. We see once again that equations are much more clear and concise than words! ♣

To solve the above word problem, you could work things out in terms of the "word logic" described at the end of Section 5.2. But as we noted in that discussion, it is invariably *much* easier to first translate the words into an equation, and then solve the equation. The more complicated the question, the more important it is to do this translation. Performing algebraic steps on an equation is far simpler (assuming you've learned the basics of algebra, which you have!) and far more organized than working with long drawn-out sentences. The name of the game when solving word problems is therefore to *translate the words into an equation*. Once you do this, you're back in the realm of algebra, which you know how to navigate.

The first step in translating words into an equation is always the same, and it involves the word "let." In the above example of "one third of 2 more than a number equals 4," the first thing you want to do is say, "Let x be the desired number." (Any other letter would work just as well.) In another problem you might say, "Let x be Serena's age," or "Let x be the area of the rectangle," and so on. This "Let x be..." step is a simple one, but it is critical. If you forget to do it, you're going to have a hard time making any progress. You need to clearly define the thing you're trying to solve for, otherwise you're going to get confused. Bottom line:

> "Let" is a very powerful word. You are free to let x be whatever you want, so you should take advantage of this!

Having defined x to be the desired number in "one third of 2 more than a number equals 4," we can now translate the words into an equation. At the end of Section 5.2, in the "Equations are better than words" subsection, we went in the other direction (the unproductive equation-to-words direction) when we translated Eq. (5.21) into words. We trudged through that tedious task so that you would appreciate the comparative simplicity of working with equations. The task of the present section is to learn how to go in the more productive words-to-equation direction.

With our definition of x, the given question becomes "If one third of 2 more than x equals 4, what is x?" We must now take the operations that are described in words and translate them into algebraic operations in an equation. When doing this, it is generally easiest to start with x and look at the successive operations that are performed.

In the present example, the first thing we do to x is add 2. So we can rewrite the question as "If one third of $x+2$ equals 4, what is the number?" The next thing we need to take care of is the "third" operation. This yields "If $(x+2)/3$ equals 4, what is the number?" (The word "of" in "one third of. . ." means to multiply. That is, "of" means the same thing as "times." So "One fifth of 35 equals 7" says the same thing as "$1/5$ times 35 equals 7.")

Continuing on, we can replace the "equals" word with an "=" sign, which gives "If $(x+2)/3 = 4$, what is the number?" Finally, we can get rid of the "what is the number" language and simply say, "Solve for x in $(x+2)/3 = 4$." We have therefore successfully translated the given word problem into an equation. We can now solve for x as we did in the first paragraph of this section.

To summarize, there are three steps in solving a word problem:

> 1. Use the word "let" to define x as the thing you're trying to find.
> 2. Translate the words into an equation that x needs to satisfy.
> 3. Solve the equation for x via the standard techniques we've learned, namely doing the same (useful) things to both sides of the equation, and simplifying each side as necessary.

The second and third steps take longer than the first one, of course. But there is no hope of doing these other two steps if you haven't already done the first!

Once you've had some practice with algebra and are comfortable solving equations, the third step usually goes smoothly. However, the *translation in the second step* takes some getting used to. It's invariably the part that takes the most thinking. When students get tripped up by word problems, it's usually because of the second step. The translation from words into an equation takes some practice (as does everything!), and during the learning process you might find word problems somewhat daunting. But if you ever get confused about how to translate the word operations into algebraic steps, it's helpful to start with the x you've defined and then think about what operations you perform on it, as the following example illustrates.

Example 5.8 If half of 3 more than the square of 5 less than a number is 42, what is the number?

Solution Let the desired number be x. Then the successive steps we perform on x (which we're told lead to a final result of 42) are:

1. Subtract 5,
2. Square the result,
3. Add 3,
4. Divide by 2.

Note that the "5 less than" operation comes last in the given sentence, but first in the above list. It is the first thing you do to x. Conversely, the "half of" operation comes first in the given sentence, but last in the list. It is the last thing you do.

From the list, we see that the equation we want to solve is

$$\frac{3 + (x - 5)^2}{2} = 42. \tag{5.35}$$

If someone gave you a value of x and instructed you to evaluate the lefthand side, you would perform the above four steps in the stated order. So we have indeed correctly translated the word problem into an equation. To solve this equation, we just need to undo all the steps, as we learned in Section 5.2. The reverse steps are:

1. Multiply by 2,
2. Subtract 3,
3. Take the square root,
4. Add 5.

Performing these operations on both sides of Eq. (5.35) yields

$$3 + (x - 5)^2 = 84 \implies (x - 5)^2 = 81 \implies x - 5 = \pm 9 \implies x = 14,\ -4. \tag{5.36}$$

The square-root operation generates two solutions, due to the "±" that we always include when taking a square root. You can quickly check that if you plug in either 14 or −4 for x in the lefthand side of Eq. (5.35), you do in fact obtain 42.

Remark: If you want (for some strange reason!) to solve the above word problem by using "word logic" instead of algebra, the reasoning goes like this: If half of something (namely 3 more than the square of 5 less than a number) is 42, then the something must be 84. And then if 3 more than a second something (namely the square of 5 less than a number) is 84, then the second something must be 81. And then if the square of a third something (namely 5 less than a number) is 81, then the third something must be 9 or −9. And finally if 5 less than a fourth something (namely the desired number) is 9 or −9, then the fourth something must be 14 or −4, as we found above.

As we noted at the end of Section 5.2, the above word-logic reasoning is extremely confusing. It is *much* harder to follow than the math logic in Eq. (5.36) and quite literally might give you a headache. Although generating successively shorter sentences by doing word logic is basically equivalent to doing algebra, it is *far* less organized and concise. Algebra allows you to avoid suffering through paragraphs like the preceding one. So be thankful for algebra! ♣

Example 5.9 There are 30 people in a room, some of whom are wearing hats. If a third of the people wearing hats leave the room, and three people not wearing hats enter, the fraction of people wearing hats becomes $4/9$. What is the initial number of hat wearers?

Solution Let h be the desired initial number of hat wearers. (x would work fine too, of course, but we've chosen h for "hat.") Then the initial number of hatless people is $30 - h$.

We're told that $1/3$ of the hat wearers leave, which is $h/3$. So $h - h/3 = 2h/3$ hat wearers remain. And we're told that 3 hatless people enter, so there are now $(30 - h) + 3 = 33 - h$ hatless people. The total number of people in the room is therefore now $2h/3 + (33 - h) = 33 - h/3$.

Since the final fraction of hat wearers is $4/9$, we have

$$\frac{\text{hats}}{\text{total}} = \frac{4}{9} \quad \Longrightarrow \quad \frac{2h/3}{33 - h/3} = \frac{4}{9}\,. \tag{5.37}$$

We have successfully translated the words into an equation. We must now solve this equation, but at least we're back in the familiar territory of Section 5.3. There are various routes we can take. Here is one:

Multiplying the lefthand side of Eq. (5.37) by $3/3$ to get rid of those denominators, dividing both sides by 2, and then cross multiplying and simplifying, gives

$$\begin{aligned} \frac{2h}{99 - h} = \frac{4}{9} \quad \Longrightarrow \quad \frac{h}{99 - h} = \frac{2}{9} \quad &\Longrightarrow \quad 9h = 2(99 - h) \\ \Longrightarrow \quad 11h = 2 \cdot 99 \quad &\Longrightarrow \quad h = 18. \end{aligned} \tag{5.38}$$

Note that it is advantageous to *not* multiply the 2 and the 99 here, because we can quickly cancel a factor of 11 on each side.

We see that initially there were 18 hat wearers, and hence $30 - 18 = 12$ hatless people. As a check on this result, if a third of the hat wearers (which is 6) leave, then $18 - 6 = 12$ remain. And if 3 hatless people enter, there are now $12 + 3 = 15$ of them. The fraction of people wearing hats is then $12/(12 + 15) = 12/27 = 4/9$, as desired.

REMARK: This example is a good illustration of how to proceed through the three problem-solving steps on page 249. You're not going to make any progress unless you start with the definition, "Let h be the desired initial number of hat wearers." (Well, you could alternatively let p be the initial number of hatless people. After solving for p, your final answer for the initial number of hat wearers would be $30 - p$.)

Once the "let" step is complete, the translating-words-to-equations fun begins. This involves marching slowly through the words and figuring out how to translate each clause into a mathematical expression. In the present problem, the "third of the people wearing hats" means $h/3$. "Leave" means to subtract. "Enter" means to add the 3 people. And "fraction" means to write down the appropriate numerator (the number of hat wearers) and denominator (the total number of people), and set the fraction equal to $4/9$. The result is Eq. (5.37), so the third and final step in the process is to solve this equation by using algebra, as we did in Eq. (5.38).

Assuming that a problem doesn't contain any extraneous information, every clause will tell you something specific that you need to incorporate into your translation of words into an equation. For most people, this translation takes more practice than the other more formulaic aspects of algebra we discussed in Section 5.2. But the more practice you do, the more natural the translation becomes. ♣

Example 5.10 Cindy is 10 years older than Dan. 5 years ago, Dan was half as old as Cindy will be in three years. What are their ages?

Solution Let Dan's present age be x. (You could just as well let Cindy's present age be x; see Exercise 5.13.) Then the first sentence of the given information immediately tells us that Cindy's present age is $x + 10$.

To translate the second sentence into an equation, we note that 5 years ago, Dan's age was $x - 5$. And 3 years from now, Cindy's age will be $(x + 10) + 3 = x + 13$. We're told that the former of these ages is half the latter, so we have

$$x - 5 = \frac{x + 13}{2}. \tag{5.39}$$

Solving this equation by performing useful operations that lead to x being all alone on one side (multiplying by 2, subtracting x, and adding 10) gives

$$2(x - 5) = x + 13 \implies 2x - 10 = x + 13 \implies x - 10 = 13 \implies x = 23. \tag{5.40}$$

So Dan's age is 23, and Cindy's age is $x + 10 = 33$. As a check: 5 years ago, Dan was 18; and 3 years from now, Cindy will be 36. And 18 is correctly half of 36.

We'll do some harder word problems involving ages in Section 5.6.2. The increase in difficulty will be due to having two variables we'll need to solve for.

Exercise 5.12 A friend wants to tell you how many miles she rode her bike yesterday, but in a cryptic manner. She says, "5 more than a third of 2 less than the number of miles equals 9." Solve for the number of miles by using: (a) guess and check, (b) word logic, (c) algebra.

Exercise 5.13 Solve Example 5.10 by letting Cindy's present age be x.

Exercise 5.14 A mug of coffee (plus milk) is 20% milk. (See the Percent entry in the Glossary if needed.) You drink half of the mug and then add one ounce of milk. If the resulting percentage of milk is now 40%, how many total ounces (coffee plus milk) were initially in the mug?

5.5.2 Train problems

If you do enough word problems, sooner or later you'll encounter ones that involve speeds, distances, and times. There is a classic type of word problem that begins with something along the lines of, "If train A leaves a station at 10:00 am. . ." To be able to tackle problems like this, we'll need to give a brief introduction to speeds.

The *speed* of an object is a measure of how fast it is going. We're all familiar with a speedometer in a car. If it reads 40 miles per hour, that means the car would travel a distance of 40 miles in one hour, if it kept the same speed for the whole hour.

We'll be concerned here only with setups where the speed is *constant*, that is, where the speedometer always has the same reading, at least for the period of time we're concerned with. (Obviously the car needs to speed up at some point from being at rest. And eventually it needs to slow down to a stop. It can't keep moving forever.)

If the speed of an object (you, a car, a rabbit, or whatever) is constant, then the main fact we'll be using in this section is that the distance traveled equals the speed times the time of travel. And since "rate" and "speed" mean the same thing, the standard mantra for setups involving constant speed is:

Rate times time equals distance.

Remember that this holds only for *constant speed*. If the speed changes, things get more complicated.

The "rate times time equals distance" statement can be written concisely as

$$\boxed{rt = d} \tag{5.41}$$

where r is the rate (speed), t is the time, and d is the distance. As an example, let's say that you travel at a rate of $r = 20$ miles per hour (mph) for a time of $t = 3$ hours. Then you travel a distance d equal to

$$d = rt = \left(20\,\frac{\text{miles}}{\cancel{\text{hour}}}\right)(3\,\cancel{\text{hours}}) = 60 \text{ miles}. \tag{5.42}$$

(The "per" in "miles per hour" means to form the quotient miles/hour.) It makes sense that the distance is obtained by multiplying 20 by 3. Each hour produces a distance of 20 miles. So 3 hours yield $20 + 20 + 20$ miles, which can be written more concisely as $20 \cdot 3$.

Note that the units of hours canceled in Eq. (5.42), leaving us with units of miles, which are correct since we're finding a distance. (*Units* are things like miles, hours, feet, seconds, kilograms, etc. They tell you what kind of quantity you're dealing with and allow you to describe its size.) If you accidentally *divide* r by t in Eq. (5.42) instead of multiplying, you'll end up with $(20\,\text{miles/hour})/(3\,\text{hours}) = (20/3)\,\text{miles/hour}^2$. These miles/hour^2 units aren't the desired "mile" unit for distance, so you'll know that you must have made a mistake.

If we solve for t in $rt = d$, we obtain $t = d/r$. This tells you that "time equals distance over rate." So, for example, if you travel $d = 60$ miles at a rate of $r = 20$ miles per hour, then the time this takes is

$$t = \frac{d}{r} = \frac{60\,\cancel{\text{miles}}}{20\,\dfrac{\cancel{\text{miles}}}{\text{hour}}} = 3 \text{ hours}. \tag{5.43}$$

The units of miles cancel, leaving us with units of hours, which are correct since we're finding a time. And the numerical result of 3 makes sense, because if you travel for 3 hours, and if you go 20 miles during each hour, then you'll travel a total of $3 \cdot 20 = 60$ miles.

You don't need to remember that "time equals distance over rate." The only thing you need to remember is the "rate times time equals distance" statement in Eq. (5.41). You can then solve for whatever other letter you might be concerned with. For example, solving for r gives $r = d/t$. That is, "rate equals distance over time."

Instead of "r" for rate, people often use "v" for velocity. Or you could use "s" for speed. We'll generally use v here. So the $rt = d$ relation in Eq. (5.41) becomes

$$\boxed{vt = d} \tag{5.44}$$

There is technically a difference between speed and velocity: Speed is the absolute value of the velocity for one-dimensional motion, and it is the magnitude (size) of the velocity vector for two- or three-dimensional motion. But we'll use the words speed and velocity interchangeably, because we'll be working only in one dimension, and we'll generally be assuming that the velocity is positive anyway. So we'll say things like "the speed v."

Here is a classic type of word problem that utilizes $vt = d$:

Example 5.11 Train A leaves a station at noon and travels at 40 mph. Train B leaves the same station at 3:00 pm and travels at 70 mph. When and where does train B pass train A? (Assume they're traveling on parallel tracks.)

Solution Let t be the time that B passes A, and let $t = 0$ correspond to noon. We'll measure t in hours, so $t = 1$ is 1:00 pm, $t = 2$ is 2:00 pm, etc. In problems like this, you usually want to have $t = 0$ correspond to the first time at which something of importance happens, which is noon here. Note that the word "let" appeared twice in the first sentence above. We're free to pick t as the letter representing the time, *and* we're also free to pick noon as the moment corresponding to $t = 0$.

By "B passes A" we equivalently mean that the trains are at the same location. So our goal is to find the time t when the two trains have the same position. We'll do this by finding the positions (in terms of t) of the two trains at an arbitrary time t, and then we'll set these two positions equal to each other, to find the particular value of t that makes them be equal. We'll label the two positions as x_{A} and x_{B}. These x's will take the place of the d in Eq. (5.44). We could just as well use d_{A} and d_{B}, but positions are more commonly denoted by the letter x.

From Eq. (5.44), the position of train A at time t is simply

$$x_{\mathrm{A}} = v_{\mathrm{A}}t \quad \Longrightarrow \quad x_{\mathrm{A}} = 40t. \tag{5.45}$$

We're measuring x_{A} in miles, and v_{A} in miles per hour. The 40 we've written should really be "40 miles/hour," but we won't bother writing out the units here (or in future problems) as we did in Eqs. (5.42) and (5.43), because the equations would get unwieldy. But it's understood that the units are there.

What is B's position at time t? The important thing to realize is that since B starts 3 hours later at 3:00 pm, B has been traveling for 3 hours less than A. So B has been traveling for only $t - 3$ hours. The "$d = vt$" statement for B is therefore

$$x_{\mathrm{B}} = v_{\mathrm{B}}(t-3) \quad \Longrightarrow \quad x_{\mathrm{B}} = 70(t-3). \tag{5.46}$$

As mentioned above, our goal is to have x_{A} be equal to x_{B}. Equating the above two expressions yields

$$40t = 70(t-3) \quad \Longrightarrow \quad 40t = 70t - 210 \quad \Longrightarrow \quad 210 = 30t \quad \Longrightarrow \quad 7 = t. \tag{5.47}$$

Since $t = 0$ corresponds to noon, $t = 7$ corresponds to 7:00 pm. This is when B passes A.

The common position of the trains at this time can be found by plugging $t = 7$ into either of the $x_A = 40t$ or $x_B = 70(t-3)$ expressions above. (They're guaranteed to give the same result, because we explicitly found the value of t that makes them be equal.) The x_A value at $t = 7$ is $40 \cdot 7 = 280$, and the x_B value at $t = 7$ is $70(7-3) = 70 \cdot 4 = 280$, which is correctly the same. The "when" and "where" answers to this problem are therefore 7:00 pm, and 280 miles from the station.

Remark: Another way to solve this problem (which works here, but isn't as clean in more complicated problems) is to note that during the 3 hours that B waits to start, A travels $40 \cdot 3 = 120$ miles. So A has a head start of 120 miles when B leaves the station. Since B is traveling $70 - 40 = 30$ mph faster than A, B gains 30 miles on A during each hour. That is, B closes the initial gap of 120 miles at a rate of 30 mph. Since $120/30 = 4$, it takes B four hours to catch up with A. Finally, because B started traveling at $t = 3$, it therefore catches up to A at $t = 3 + 4 = 7$, as we found above.

Another solution that is basically equivalent to the one in the preceding paragraph is the following. Let $T = 0$ correspond to 3:00. So we're starting our T stopwatch at 3:00. (We're using a new letter T here, since it has a different meaning from our earlier t, which started at noon.) Then B's position at time T is $70T$, and A's position is $120+40T$ because it has already traveled $40 \cdot 3 = 120$ miles by 3:00. (In general, if you start at position d_i (with the i for "initial") and travel at speed v, then your position at time t is $d_\text{i} + vt$ because you've traveled a distance $d = vt$ beyond your initial location of d_i.) Setting these two positions equal to each other gives

$$70T = 120 + 40T \implies (70-40)T = 120 \implies T = \frac{120}{70-40} = \frac{120}{30} = 4. \tag{5.48}$$

When this catching-up time of $T = 4$ is added to 3:00, we obtain 7:00, as above. Note that the $120/(70-40)$ division here is exactly what we did in the preceding paragraph, in a more wordy manner. ♣

As with any other type of word problem, the main task in a train problem (or any $rt = d$ problem in general) is translating words into an equation. As complicated and cryptic as the wordy information may seem, it becomes much simpler once you translate it into an equation. This translation is therefore your main goal. So keep your cool, start with the word "let," and systematically replace words with algebraic operations, until you have created an equation that encapsulates all of the given information.

For problems of wordy persuasion
That seem like they're full of evasion,
Your prime obligation
Is careful translation
Of words to a matching equation.

Exercise 5.15 Train A leaves a station at noon and travels at 50 mph. Train B leaves a different station 220 miles ahead on the tracks at 2:00 pm and travels at 30 mph. When and where does train A catch up to train B?

Exercise 5.16 Two hikers start at two different camps along the same trail. Hiker A starts at noon, and Hiker B starts an hour earlier from a camp that is 6 miles ahead of A's camp. A hikes twice as fast as B. If A catches up to B at 4:00 pm, what are their two speeds?

5.6 Systems of equations

5.6.1 Two equations, two unknowns

All of the equations we've solved so far have contained only one variable, which we've usually called x, or sometimes t, although any letter is fine. But what if we have two variables that we're trying to solve for? For example, what if we want to solve for x and y in this equation:

$$2x + 3y = 7. \tag{5.49}$$

It turns out that we *can't* solve for x and y here. More precisely, there isn't a *unique* (single) solution for each of x and y. Rather, there is an *infinite* number of x,y pairs that solve the equation. For example, one pair is $x = 2$ and $y = 1$, another is $x = 5$ and $y = -1$, and another is $x = 263$ and $y = -173$ (as you can verify). You can easily generate new pairs by solving for x in terms of y in Eq. (5.49) to obtain $x = (7 - 3y)/2$. (Solving for y in terms of x would work just as well.) You can then plug in whatever value of y you want, and the associated x will pop out. That's how the above 263, −173 pair was generated. The pairs don't have to be integers, of course. If you let $y = \pi \approx 3.14$, then $x = (7 - 3\pi)/2 \approx -1.21$.

Given that there isn't a unique solution for x and y in Eq. (5.49), under what circumstances *is* there a unique solution? The basic answer is that if we tack on another equation that x and y must also satisfy (so that we now have two equations and two unknowns), then there is just one x,y pair that simultaneously satisfies both equations. (However, see Section 5.6.3 for a caveat to this.)

Let's therefore introduce a second equation to go along with the one in Eq. (5.49). We'll arbitrarily choose it to be $x - 4y = 9$. So we now want to find the values of x and y that satisfy *both* of these equations:

$$\begin{aligned} 2x + 3y &= 7, \\ x - 4y &= 9. \end{aligned} \tag{5.50}$$

This set of equations is called a *system of equations.* More specifically, the system in Eq. (5.50) is a *system of two equations in two unknowns.* People sometimes also include the word "simultaneous" and use the phrases *simultaneous equations*, or *simultaneous system of equations*.

How do we solve the system of equations in Eq. (5.50) for x and y? There are three standard ways to find the solution:

1. SUBSTITUTION: Use one equation (whichever is more convenient) to solve for one of the variables in terms of the other (it doesn't matter which, but let's say you solve for x in terms of y). Then substitute the result for x into the other equation. This yields an equation involving only y's. So you've reduced the problem to one variable and one equation, which you know how to solve. After solving for y, you can plug the value back into either of the given equations to solve for x. Both options will give the same answer, assuming you haven't made a mistake.

2. EQUATING: Solve for one of the variables in terms of the other (again it doesn't matter which, but let's say you solve for x in terms of y) in each of the two equations. Then equate the two results for x with each other. This yields an equation involving only y's, and you can proceed as above.

3. HELPFUL COMBINATION: Form a helpful combination of the equations (for example, 3 times one equation plus 2 times the other) that makes one of the variables disappear (cancel). This yields an equation with one variable, and you can proceed as above.

All three of these strategies involve *eliminating one of the variables*, thereby reducing the problem to one variable and one equation, which you know how to solve. Once you've solved for that variable, you can plug its value into either of the given equations and solve for the variable you had eliminated. In the end, all three methods are pretty much the same. They're simply organized a little differently.

Example 5.12 Let's apply the first of the above strategies to the system of equations in Eq. (5.50).

Solution The most reasonable thing to do is solve for x in terms of y in the second equation, since the (implicit) coefficient of 1 in front of the x makes this easy. We quickly obtain $x = 4y + 9$. We can then plug this in for x in the first equation, which yields

$$2(4y+9)+3y = 7 \implies (8y+18)+3y = 7 \implies 11y = -11 \implies y = -1. \quad (5.51)$$

Plugging $y = -1$ into either of the given equations in Eq. (5.50) then tells us what x is. We'll pick the first one, which yields $2x + 3(-1) = 7 \implies 2x = 10 \implies x = 5$. (The second equation gives $x - 4(-1) = 9 \implies x + 4 = 9 \implies x = 5$, again.) So the solution to the system of equations in Eq. (5.50) is $x = 5$ and $y = -1$.

As always, you should plug your solution back into the given equations (both of them) to verify that equality holds in both, and that you didn't make an algebra mistake somewhere along the way.

Example 5.13 Let's now solve Eq. (5.50) by using the third of the above methods, where we form a helpful combination of the equations that makes one of the variables disappear. (The second of the above methods is used below in Exercise 5.17.) The easiest combination is the first equation minus twice the second; this makes the x's cancel. You can take that route in Exercise 5.17. But for this example, let's make the y's cancel by adding 4 times the first equation to 3 times the second.

Solution The suggested multiples of the equations are

$$\begin{aligned} 4\big[2x + 3y = 7\big] \quad &\Longrightarrow \quad 8x + 12y = 28, \\ 3\big[x - 4y = 9\big] \quad &\Longrightarrow \quad 3x - 12y = 27. \end{aligned} \tag{5.52}$$

If we now add the right set of equations (that is, set the sum of their lefthand sides equal to the sum of their righthand sides), the y's cancel and we obtain

$$11x + 0 \cdot y = 55 \quad \Longrightarrow \quad x = 5. \tag{5.53}$$

Plugging $x = 5$ into either of the given equations then yields $y = -1$, as we found above.

As we saw in Eqs. (5.52) and (5.53), the phrase "adding 4 times the first equation to 3 times the second" means to multiply both sides of one equation by 4, and both sides of the other by 3, and then add the lefthand sides, and also add the righthand sides, and finally set the two sums equal. The sums are indeed equal, because the equalities in the two original equations tell us that the same thing appears on both sides. So the equations take the general form of $A = A$ and $B = B$. The adding procedure described above then yields an equation of the form $4A + 3B = 4A + 3B$, which is indeed a true statement.

The act of forming a combination of the equations is called forming a *linear combination*. The word "linear" just means that all you're doing to each equation is multiplying it (that is, each side) by a number. You're not squaring it, or cubing it, or taking the square root, or anything like that.

We listed three general methods above for solving a system of two equations in two unknowns. However, each of the three methods involves some choices, namely which variable to solve for first, and also (for the first method) which equation to use. So let's make a list here of all the specific routes you can take. The first four of these routes correspond to the first of the three methods above, the next two to the second method, and the last two to the third. So there are eight specific routes you can take when solving a system of two equations in two unknowns:

1. Solve for x in the first equation, then plug into the second.

2. Solve for x in second, plug into first.

3. Solve for y in first, plug into second.
4. Solve for y in second, plug into first.

---

5. Solve for x in both and equate.
6. Solve for y in both and equate.

---

7. Form a combination where the x's cancel.
8. Form a combination where the y's cancel.

Instead of trying to keep all eight of these routes in your head, it's better to just think about the three general methods we listed earlier, and then remember that each method can be implemented in different (very similar) ways. Depending on the equations you're given, any one of these eight routes might yield the quickest solution. After some practice, a few seconds of considering the options will usually lead you to the most convenient one.

As we mentioned above, all of these routes involve the same basic strategy:

> Eliminate one of the variables (via your choice of the three general methods) so that you then have only one unknown to solve for, which you know how to do.

> If the system you're given contains
> Two letters, here's how to make gains:
> Just do what you may
> To make one go away,
> And then solve for whichever remains.

Exercise 5.17 We have solved the system in Eq. (5.50) twice: In Example 5.12 where we used the 2nd of the above eight routes, and in Example 5.13 where we used the 8th route. Solve Eq. (5.50) six more times by using the remaining six routes (1, 3, 4, 5, 6, 7).

Exercise 5.18 In the solution to the preceding exercise, the fact that the same numbers kept appearing in the different routes implies that they all pretty much say the same thing. Show explicitly why the same numbers kept appearing, by solving this system:

$$\begin{aligned} ax + by &= e, \\ cx + dy &= f, \end{aligned} \tag{5.54}$$

via the three basic methods. For all of these, just solve for x. You don't need to bother plugging your answer back in to find y, which is normally what you would do.

(a) Solve for y in terms of x in the first equation, and plug that into the second, and then solve for x.

(b) Solve for y in terms of x in both equations and set the results equal, and then solve for x.

(c) Form a helpful linear combination of the two equations that eliminates y, and then solve for x.

5.6.2 Age problems

Example 5.10 was a problem dealing with ages, and it involved solving Eq. (5.39), which was one equation in one unknown. However, some age problems lead more naturally to two equations in two unknowns, as the following example illustrates (but see the second remark in the solution). Along with train problems, age problems are another classic type of word problem.

Example 5.14 When Alice was half as old as Bob is now, Bob was 8 years younger than Alice is now. If the sum of their present ages is 88, what are these ages?

Solution Let the present ages of Alice and Bob be a and b, respectively (x and y, or A and B, would work just as well). Then we are told that

$$a + b = 88. \tag{5.55}$$

To translate the rest of the given information into an equation, let T be (for convenience) the time in the past when Alice was half as old as Bob is now. Then Alice's age at T is $b/2$ (by the definition of T). And we are told that Bob's age at T is $a - 8$. So we have the table shown in Fig. 5.3.

We can now use the fact that the difference between Alice's and Bob's ages is always the same, no matter what year we choose to compare them. Presently, the difference is $a - b$. (This will turn out to be negative; that's fine.) And at time T the difference was $b/2 - (a - 8)$. Equating these two differences yields

$$a - b = \frac{b}{2} - (a - 8) \quad \Longrightarrow \quad 2a = \frac{3b}{2} + 8. \tag{5.56}$$

Alternatively, we can use the fact that Alice and Bob both age by the same amount between T and the present time. Alice ages by $a - b/2$, and Bob ages by $b - (a - 8)$. Equating these two amounts of aging yields

$$a - \frac{b}{2} = b - (a - 8) \quad \Longrightarrow \quad 2a = \frac{3b}{2} + 8, \tag{5.57}$$

	Alice	Bob
now	a	b
T	$b/2$	$a-8$

Each person must yield the same difference in ages (how much they age)

Each time must yield the same difference in ages (how much older Bob is than Alice)

Figure 5.3: A table showing Alice's and Bob's ages now, and at time T in the past.

in agreement with Eq. (5.56). Putting this result together with Eq. (5.55) gives us the following system of two equations in two unknowns:

$$\begin{aligned} a + b &= 88, \\ 2a &= \frac{3b}{2} + 8. \end{aligned} \tag{5.58}$$

To solve this system, we can solve for a in the first equation and plug the $a = 88 - b$ result into the second. This yields

$$2(88 - b) = \frac{3b}{2} + 8 \implies 168 = \frac{7b}{2} \implies 48 = b. \tag{5.59}$$

Eq. (5.55) then tells us that $a = 40$. So Alice's present age is 40 and Bob's is 48. We can (and should) double check that these ages do in fact satisfy the first sentence of the given information: When Alice was half as old as Bob is now (which means $48/2 = 24$), that was $40 - 24 = 16$ years ago. So Bob's age then was $48 - 16 = 32$, which is indeed 8 years younger than Alice's present age of 40.

Remarks:

1. There are probably very few people who don't panic at least a little when reading the statement of a problem like this one. (I wrote the problem, and even I get a little anxious reading it!) But just remember to take it slowly and translate the words into equations. Your first step will invariably be to say something like, "Let the present ages be a and b." After that, you'll often be concerned with what is going on at a different time (which was in the past in this exercise, but could be in the future).

2. We technically didn't need to use two variables in this problem. Given the "sum of their present ages is 88" fact, we could have immediately said that Bob's present age is $88-a$, without ever introducing b. However, things are often more organized if you introduce two variables and then solve a system of two equations in two unknowns, even though you can quickly reduce it to one equation in one unknown by using the fact that $a = 88 - b$, as we did in Eq. (5.59).

Conversely, although we solved Example 5.10 by solving one equation in one unknown, you can also treat it as two equations in two unknowns. If Dan's present age is x, and Cindy's present age is y, then the given information tells us that

$$\begin{aligned} y &= x + 10, \\ x - 5 &= \frac{y+3}{2}. \end{aligned} \tag{5.60}$$

Substituting the y from the first equation into the second gives Eq. (5.39).

Example 5.10 was easier than the present example because the other times under consideration were 5 years in the past and 3 years in the future. These 5 and 3 values are definite numbers. In the present example, we don't immediately know how long ago the time T was. So things are more involved, and it's safer to use two unknowns. ♣

Exercise 5.19 When Carol was $4/3$ as old as Alice is now, Alice was 16 years younger than Carol is now. And when Alice eventually reaches Carol's present age, Carol will be twice as old as Alice is now. What are their present ages? (You will need to make a table similar to the one in Fig. 5.3, but now with *three* different rows for three different times.)

5.6.3 Inconsistent, degenerate equations

We mentioned near the beginning of Section 5.6.1 that if we have two equations in two unknowns, then there is a unique (single) x,y pair that simultaneously satisfies both equations. However, we also stated that there was a caveat to this claim. There are actually two caveats, as we'll see below. In short, the two ways of having a non-unique solution are to have zero solutions, or to have infinitely many solutions.

Inconsistent equations (no solutions)

Consider the following system of equations:

$$\begin{aligned} 2x + 3y &= 7, \\ 8x + 12y &= 29. \end{aligned} \tag{5.61}$$

Let's solve this system by solving for x in the first equation and plugging the result into the second. The first equation gives $x = (7 - 3y)/2$. Plugging this into the second equation yields

$$8 \cdot \frac{7-3y}{2} + 12y = 29 \implies 4(7-3y) + 12y = 29 \implies (28 - 12y) + 12y = 29. \tag{5.62}$$

Subtracting 28 from both sides turns this into $0 = 1$, which is a false statement. We conclude that even if x equals $(7 - 3y)/2$ (which must be the case, if we want the first equation to be true), we end up with a false statement, which means that there is no possible way to also make the second equation be true. It is therefore impossible to find an x,y pair that simultaneously satisfies both equations.

Degenerate equations (infinitely many solutions)

There is another closely related case where instead of zero solutions, we have an *infinite* number of solutions. Consider the same system of equations as in Eq. (5.61), except with the 29 replaced by 28:

$$\begin{aligned} 2x + 3y &= 7, \\ 8x + 12y &= 28. \end{aligned} \tag{5.63}$$

If we perform the same steps as above, the last equation in Eq. (5.62) becomes $(28 - 12y) + 12y = 28$, which reduces to $0 = 0$, which isn't very helpful. Our goal was to solve for y by getting it alone on one side of the equation. But all of the y's canceled, so we weren't able to solve for it. What's going on? Why did we fail?

The issue is that the second equation in Eq. (5.63) is an *exact multiple* of the first. It is 4 times the first equation. (That is, each side of the second equation is 4 times the corresponding side of the first.) The second equation therefore contains precisely the same information as the first; it isn't really a second equation. So we actually have only *one* equation. And we know that a single (linear) equation in two unknowns doesn't have a unique solution. As we saw at the beginning of Section 5.6.1, for any arbitrary value of y, an x value of $(7 - 3y)/2$ satisfies the equation. So there are infinitely many solutions in the case where one equation is a multiple of the other.

REMARK: We arrived at the $0 = 0$ result above by using the first of the eight routes on page 259. We did that for instructive purposes, but a quicker method is to use the 7th or 8th route. If we subtract 4 times the first equation in Eq. (5.63) from the second, we obtain $0 = 0$. And then we're done, because this tells us that if we can find a solution to one equation (which we can), then the solution also works in the other.

This is true because the equation you produce by forming a linear combination (such as $0 = 0$ here, or Eq. (5.53) in Example 5.13) tells you what additional fact needs to be satisfied if you want the other equation to be true. But $0 = 0$ is always true, which means that no additional fact is needed. The other equation is automatically true if the first one is.

The goal of the 7th and 8th routes is to find a helpful linear combination of the equations that makes one of the variables disappear. So in a sense we were actually *too* successful here, in that we eliminated *both* of the variables. But when this happens and you end up with $0 = 0$, it means the equations contain the same information.

Note that if we subtract 4 times the first equation in Eq. (5.61) from the second, we obtain $0 = 1$, which tells us that there are no solutions. The lefthand sides of the equations are a multiple (namely 4) of each other, but the righthand sides involve a different multiple (namely

29/7). This would also be the case if we used any other number in place of 29, with the one exception of 28. ♣

Putting everything together, there are three possibilities for the number of solutions to a system of two equations in two unknowns. The following terminology is used for the three cases:

1. CONSISTENT or NON-DEGENERATE equations (one solution): This is the standard case we covered in Section 5.6.1. There is a single value of x and a single value of y that satisfy both equations. The solution is unique.

2. INCONSISTENT equations (no solutions): In this case the parts of the equations involving x and y are multiples of each other, but the constant terms aren't related by the same multiple. The two equations are inconsistent with each other; it is impossible for both of them to be satisfied by a given x,y pair.

3. DEGENERATE equations (infinitely many solutions): In this case the two equations are multiples of each other, so we really have only one equation. For any value of y, as long as x takes on a particular value (or vice versa), both equations are satisfied.

There is a nice geometric interpretation of the above three cases, which should make everything clear. To discuss it, we'll need to get a little ahead of ourselves. A *linear equation* is one where the variable(s) appear only to the first power, like each of the equations in Eq. (5.61). These equations don't contain any squares or cubes, etc., of the variables. We'll see in Chapter 6 that a linear equation represents a *straight line* in the x-y plane. That is, all of the points with coordinates (x, y), whose x and y values satisfy a given linear equation, lie on a straight line. (Let's just accept this fact here; we'll talk about it plenty in Chapter 6.) Therefore, having an x,y pair that is a simultaneous *solution* to two equations is equivalent to having a particular *point* (x, y) in the x-y plane that lies on both of the associated lines.

Now, two lines generally intersect at exactly one point, as shown in Fig. 5.4(a). This is why there is generally a *single* solution to a system of two equations in two unknowns. This is the first of the above three cases – the "consistent" one. However, there are two ways in which we *don't* have a single solution (that is, a single intersection point).

One way not to have a single solution is for the two lines to be parallel, as shown in Fig. 5.4(b). In this case we have *zero* solutions, because the lines never intersect. This is the second case above – the "inconsistent" one.

The other way not to have just a single solution is for the two lines to actually be the same line (that is, one is right on top of the other), as shown in Fig. 5.4(c). In this case there is an *infinite* number of solutions, because every point on the line represents a solution. This is the third case above – the "degenerate" one.

We'll see in Chapter 6 that the relative size of the coefficients of x and y in an equation determines the slope of the line (how tilted it is). In the "consistent" case, the slopes of the lines are different (for example, the coefficient ratios of $3/2$ and $(-4)/1$ in Eq. (5.50) aren't the same). And (infinite) lines with different slopes always intersect at a single point, as in Fig. 5.4(a). In the "inconsistent" and "degenerate" cases, the slopes are the same (for

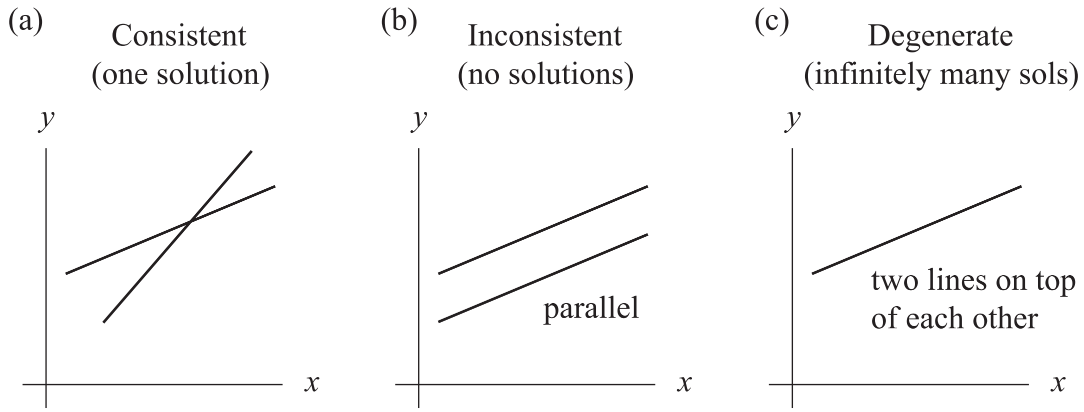

Figure 5.4: The three possibilities for how two lines may or may not intersect.

example, the coefficient ratios of $3/2$ and $12/8$ in Eq. (5.61) are equal). This means that the lines are parallel, which in turn means there are zero solutions as in Fig. 5.4(b), unless the lines are right on top of each other as in Fig. 5.4(c), in which case there are infinitely many solutions.

Each style of system thou hast
Hath solutions whose numbers contrast.
Consistent has one,
Inconsistent has none,
And degenerate? That number's vast!

Exercise 5.20 One of the equations in a system of equations is $3x - 5y = 4$. The other is either (a) $9x - 15y = 12$, or (b) $9x - 15y = 13$, or (c) $9x - 14y = 12$. For each of these three possibilities, state the kind of system (consistent, inconsistent, degenerate), and describe the solution(s).

5.6.4 Three equations, three unknowns

What if we have a system of *three* equations in *three* unknowns? For example:

$$\begin{aligned} x - 2y + 6z &= 17, \\ 2x + 3y - 2z &= -1, \\ -5x + y + 3z &= 20. \end{aligned} \tag{5.64}$$

To solve for the x,y,z triplet that satisfies these three equations, the same general strategy applies as with two equations in two unknowns: The goal is to eliminate variables until there is only one left, and then you can solve for that variable. (You can then use that result to solve for the remaining variables.) It's just that in the present case of three variables, you need to eliminate *two* of them in order to solve for the third variable, instead of eliminating just *one* of them in the earlier case of two variables.

With three equations, a standard technique for eliminating variables is to pick two of the equations (your choice of which two) and eliminate one of the variables (let's say z) via one of the three general methods on page 258. This leaves you with an equation involving just x and y. You can then pick a different pair of equations and again eliminate z, yielding another equation with just x and y. You now have a system of two equations in the two unknowns x and y, which you know how to solve. Once you solve for x and y, you can plug those values into any of the three given equations to solve for z. They will all yield the same result, assuming you haven't made an algebra error.

This is a classic example of the time-honored strategy of reducing a problem to a previously solved one. The goal is to reduce a system of three equations in three unknowns to a system of two equations in two unknowns, which we previously learned how to solve.

In choosing pairs of equations to eliminate one of the variables, you can pick the 1st and 2nd, or the 1st and 3rd, or the 2nd and 3rd. You will need to pick two of these pairs. So there is a choice to be made. You will also need to choose which variable you want to eliminate first. Additionally, you can choose any of the three methods for eliminating variables discussed on page 258. So there are many ways to solve a system of three equations in three unknowns. They all work, but you should try to pick one where the algebra is as clean as possible.

Example 5.15 Solve the system of equations in Eq. (5.64) by first eliminating z.

Solution Let's eliminate z by forming useful combinations of the equations. The combinations we'll use are: (1) the 1st equation plus 3 times the 2nd, and (2) the 1st equation minus 2 times the 3rd. These combinations make the z's cancel, and you can verify that they yield the following two equations:

$$\begin{aligned} 7x + 7y &= 14, \\ 11x - 4y &= -23. \end{aligned} \tag{5.65}$$

We must now solve this system of two equations in two unknowns. Dividing the first equation by 7 (to make it as simple as possible) turns it into $x + y = 2$. So $y = 2 - x$. Plugging this into the second equation gives

$$11x - 4(2 - x) = -23 \quad \Longrightarrow \quad 15x = -15 \quad \Longrightarrow \quad x = -1. \tag{5.66}$$

The $y = 2 - x$ relation then gives $y = 3$. And then plugging $x = -1$ and $y = 3$ into any of the three given equations yields $z = 4$. You can (and should) check that the $x = -1$, $y = 3$, $z = 4$ values do indeed satisfy all three equations.

You could just as well have solved the system in Eq. (5.64) by first eliminating y (see Exercise 5.21), or by first eliminating x (which you should do for practice).

Exercise 5.21 Solve the system of equations in Eq. (5.64) by first eliminating y. The numbers are a bit messier here than in Example 5.15.

In the above example and exercise, we eliminated one of the three variables by forming useful combinations of the equations. But as we've mentioned, any of the three methods on page 258 will get the job done. For example, it will always work to solve for x in terms of y and z in the first equation, and then plug the result into the second and third equations. This will produce two equations in the two unknowns y and z.

You are encouraged to make up some solving-for-ages problems involving *three* ages, following the strategy in the third remark in the solution to Exercise 5.19. You will need to include *three* facts about the ages in the statement of your problem. You will then have three equations in three unknowns. For example, if you arbitrarily pick the ages to be $a = 10$, $b = 12$, and $c = 17$, then one possible fact is: In 6 years, the sum of Alice's and Bob's ages will be twice Carol's present age; this translates to $(a + 6) + (b + 6) = 2c$.

Four equations, four unknowns, etc.

We have solved systems of two equations in two unknowns, and also three equations in three unknowns. What about four equations in four unknowns (call them w, x, y, z)? The strategy is similar: The goal is to eliminate one of the variables and reduce the problem to a system of three equations in three unknowns, which you know how to solve by reducing it to two equations in two unknowns, which you in turn know how to solve by reducing it to one equation in one unknown. Once you solve for that unknown, you can plug it back into the various equations you generated, to gradually find the other unknowns.

To start this reducing process, you can solve for one of the variables (say w) in the first equation, and then plug the result into the other three equations. This will produce three equations in the three unknowns x, y, z. Or you can form linear combinations of the equations. You can pick three different pairs, and use each pair to eliminate w. There are six different possible pairs. If the equations are labeled A, B, C, D, then the pairs are AB, AC, AD, BC, BD, CD. You should try to pick the three pairs that make the algebra the cleanest, although you need to make sure you use all four of the equations at least once, sooner or later.

The same strategy applies to five equations in five unknowns. The goal is to reduce the problem to a system of four equations in four unknowns, which we just learned how to solve. And so on, for any number n of equations and unknowns. The general strategy is always the same:

> Given a system of n equations in n unknowns, the goal is to eliminate one of the variables, to produce a system of $n - 1$ equations in $n - 1$ unknowns. And then eliminate another variable to produce a system of $n - 2$ equations in $n - 2$ unknowns. And so on, until you're left with one equation in one unknown. However, this strategy is extremely time consuming if n isn't small.

In general (leaving aside the inconsistent and degenerate types of equations we discussed in Section 5.6.3), there are three possibilities for the relative number of equations and unknowns:

1. If there are n equations and the *same* number n of unknowns, then in general there is a *single* solution for each of the n variables. The systems in Eqs. (5.50) and (5.64) are examples of this.

2. If there are n equations and *fewer* than n unknowns, then in general there are *zero* solutions. An example of this is two equations and one unknown: $3x = 7$, and $5x = 2$. There is no value of x that satisfies both equations.

3. If there are n equations and *more* than n unknowns, then in general there are *infinitely* many solutions. An example of this is one equation and two unknowns: $2x - 5y = 3$. This has infinitely many solutions.

Again, we have ignored the inconsistent and degenerate types of equations when stating the above three possibilities. If we include those, then the above statements are not necessarily true. Consider, for example, the second statement applied to the two equations $3x = 7$ and $6x = 14$. The second statement is false, because this "system" has one solution, namely $x = 7/3$, instead of the zero solutions stated above. The two equations are degenerate (the second is twice the first), so we really have only one equation in one unknown.

5.7 Quadratic equations

5.7.1 Solving by factoring

We'll discuss quadratic equations in much more detail in Chapter 8. But let's look at a few simple cases here, where we can make use of the factoring we learned in Sections 4.2.2 and 4.3. If you're eager to get to the classic *quadratic formula*, then after reading this section, by all means feel free to jump to Section 8.1, and then on to the derivation of the quadratic formula in Section 8.3. Having worked through the first five chapters, you now know more than enough algebra to prove the formula. The reason why we're waiting until Chapter 8 for the full discussion of quadratic equations is that it's very helpful to be able to plot functions, and we'll learn how to do that in Chapters 6 and 7.

A *quadratic equation* (or just a "quadratic" for short) is one that involves the square of a variable. An example of a quadratic equation is

$$x^2 - 2x - 15 = 0. \tag{5.67}$$

It is understood that x^2 is the highest power of x that appears. If the highest power is instead x^3, then the equation is called a *cubic*. And if x^4 is the highest power, it is called a *quartic*, etc. We'll deal only with quadratics here. Up to this point in this chapter, we've been dealing with *linear* equations, where all of the variables appeared to the first power. There were no squares. And also none of the variables in systems of equations were multiplied together.

As a quick recap of Sections 4.2.2 and 4.3, recall that the method of solving Eq. (5.67) is to factor the polynomial on the lefthand side. And the way we factor it is by finding two numbers that multiply up to -15 and add up to -2. We quickly see that the factorization is $(x-5)(x+3)$. The roots (that is, the solutions) of Eq. (5.67) are therefore 5 and -3, because these numbers make the factors of $x-5$ and $x+3$ be zero. To be complete, we technically need to solve each of the separate linear equations, $x-5=0$ and $x+3=0$. But it's easy to transform these into $x=5$ and $x=-3$, by adding 5 and subtracting 3 from both sides.

If the coefficient of x^2 in an equation isn't 1 (as it was in Eq. (5.67)), then the factoring takes a little more thought, as we saw in Example 4.1. But in any case, the goal is to somehow factor the given quadratic polynomial into two binomials. These then tell you what the solutions are. In this section we'll deal only with polynomials that can be easily factored. We'll discuss more complicated cases in Chapter 8.

Quadratic equations are often easier to solve than linear ones, because linear equations can involve many steps (as we saw in, say, Exercise 5.9(b)), whereas all you need to do with quadratic equations is factor them. You can then simply read off the solutions, as we did with the 5 and -3 above. However, sometimes it isn't obvious how to do the factoring. In those cases you need to use the quadratic formula (see Section 8.3). But even then, the quadratic formula is a very mechanical procedure. You just plug in numbers (or letters), and the answer pops out. There's nothing particularly difficult about it, although at times it can get messy.

When people talk about quadratic equations, they often mean ones that contain all three of the possible terms (quadratic x^2, linear x, and constant), as in the case of Eq. (5.67). However, the term "quadratic" technically still applies even if the middle (linear x) term and/or the last (constant) term is missing. But the first (quadratic x^2) term needs to be there, otherwise the equation is simply a linear one. So the term "quadratic equation" applies to any equation with an x^2 term (as the highest power), regardless of whether the linear or constant terms appear. This means that technically $x^2=0$ is a quadratic equation, although a very simple one.

Example 5.16 If a quadratic equation is missing the *linear* x term, then solving it is quick. For example, the equation $x^2-16=0$ can be solved by factoring the lefthand side. The difference-of-squares result in Eq. (3.24) yields $(x-4)(x+4)=0$, so the roots are 4 and -4. Alternatively, $x^2-16=0$ can be rewritten as $x^2=16$. Taking the square root of both sides gives $x=\pm 4$, where we have remembered to include the usual "±" when taking a square root.

If a quadratic equation is missing the *constant* term, then solving it is again quick. For example, the equation $x^2-7x=0$ easily factors into $x(x-7)=0$, which you can write as $(x-0)(x-7)$ if you want, although this isn't necessary. The roots are $x=0$ and $x=7$.

Recall that in the 12th item in the list of do's and don'ts in Section 4.1, we claimed that the values of a for which a^2 equals $2a$ are $a=0$ and $a=2$. We now see that this conclusion follows from

$$a^2=2a \implies a^2-2a=0 \implies a(a-2)=0 \implies a=0 \text{ or } a=2. \quad (5.68)$$

A common *incorrect* method of solving an equation like $a(a-2) = 0$ is to divide both sides by a to obtain $a - 2 = 0$. This yields only one solution, $a = 2$. So you've missed the $a = 0$ root. The invalid step was the division by a, because if $a = 0$ then you've divided by 0, which isn't allowed.

A linear equation has only one solution. In contrast, since a quadratic equation can always be factored into two binomials (one of which might simply be $(x - 0) = x$, as in the above example), and since each binomial yields a linear equation (like $x - 7 = 0$), a quadratic equation always has *two* solutions, subject to two caveats: (1) The two roots might be the same, as in the case of $x^2 - 6x + 9 = 0 \implies (x-3)^2 = 0$, which has only $x = 3$ as a root, and (2) The two roots might be *complex*; we'll learn about this case in Section 8.5.

When quadratic equations involve all three terms (quadratic, linear, and constant), as in Eq. (5.67), the factoring takes a little more time than in the above example. We already solved some quadratic equations of this type in Section 4.3 by factoring them. Let's now solve a few problems where we need to do some work in order to produce the quadratic equation in the first place. From that point on, it's just the same factoring procedure as in Section 4.3.

Example 5.17 A number is $1/6$ more than twice its reciprocal. What is the number?

Solution Let x be the desired number. Then the mathematical translation of the given statement, "A number is $1/6$ more than twice its reciprocal," is

$$x = \frac{1}{6} + 2 \cdot \frac{1}{x}. \tag{5.69}$$

To solve this equation, we'll multiply through by 6 and x (equivalently, by $6x$) to get rid of the denominators. This gives

$$6x^2 = x + 12 \quad \Longrightarrow \quad 6x^2 - x - 12 = 0. \tag{5.70}$$

We must now factor the lefthand side. The 6 coefficient complicates things a little, but (assuming the numbers were chosen to be nice) the factorization must take the form of either $(6x + ?)(x + ?)$ or $(3x + ?)(2x + ?)$, where the question marks represent numbers that multiply up to -12.

A little FOIL fiddling with the first of these two options will convince you that there's no way to produce the middle $-x$ term in Eq. (5.70). And then a little FOIL fiddling with the second option will tell you that $(3x + 4)(2x - 3)$ works; the coefficient of x in the product is correctly $(3)(-3) + (4)(2) = -1$.

The two solutions for x are obtained by setting each of $3x + 4$ and $2x - 3$ equal to zero. So the solutions are $x = -4/3$ and $x = 3/2$. You can verify that both of these answers satisfy the given statement. There are two solutions, which is always the case for a quadratic equation, as we noted above. The initial equation in Eq. (5.69) didn't look much like a quadratic, but after we multiplied through by $6x$ to get rid of the denominators, it turned into a standard quadratic.

Exercise 5.22 Solve these equations for x:

$$\text{(a)}\quad x + \frac{1}{x} = \frac{10}{3} \qquad\qquad \text{(b)}\quad \frac{1}{x} + \frac{1}{x+2} = \frac{12}{5}$$

Exercise 5.23 Take the reciprocal of a number and then add 2. If the reciprocal of the result is $3/8$ times the reciprocal of 1 more than the number, what is the number?

Exercise 5.24 Consider the quadratic equation $x^2 = 5x$. A student solves this by dividing both sides by x to obtain an answer of $x = 5$. Is this a correct solution? If not, where is the error?

5.7.2 Systems of equations

In Example 5.17 and Exercises 5.22 and 5.23, a single equation led to a quadratic. It's also possible for a system of equations (two equations in two unknowns) to lead to a quadratic if one of the equations isn't linear. When one of the variables is eliminated (which is always the strategy when solving a system of any kind), the resulting equation for the remaining variable is a quadratic. The following example shows how this works.

Example 5.18 If two numbers have a sum of 7 and a product of 12, what are they? Of course, by guessing and checking, you can quickly figure out that the two numbers are 3 and 4. But let's arrive at these values via a more formal method, to see how it works.

Solution The given information tells us that we want to find two numbers x and y that satisfy

$$\begin{aligned} x + y &= 7, \\ xy &= 12. \end{aligned} \tag{5.71}$$

The product xy here means that we don't have a linear equation. (In a linear equation, all the variables must appear to the first power, and they can't be multiplied together.) However, with any system of equations (not just linear), the strategy is the same: Eliminate one of the variables and then solve for the other. For the problem at hand, there are two basic ways to do this. We can solve for y (x would work just as well) in the first equation (which gives $y = 7 - x$) and plug the result into the second. This yields

$$x(7 - x) = 12 \quad\Longrightarrow\quad 0 = x^2 - 7x + 12. \tag{5.72}$$

Alternatively, we can solve for y (x would work just as well) in the second equation (which gives $y = 12/x$) and plug the result into the first. This yields

$$x + \frac{12}{x} = 7 \quad\Longrightarrow\quad x^2 + 12 = 7x \quad\Longrightarrow\quad x^2 - 7x + 12 = 0. \tag{5.73}$$

So either way we end up with $x^2 - 7x + 12 = 0$. To factor this equation, we need to find two numbers that have a sum of -7 and a product of 12. We quickly see that the numbers are -3 and -4, so the factorization is $(x-3)(x-4)$. The solutions for x are therefore 3 and 4. Plugging these back into either of the given equations yields y values of 4 and 3, respectively. The two x,y pairs are therefore 3 and 4; and 4 and 3. So the two numbers are 3 and 4 (in either order). This agrees with what we found above by guessing and checking.

"But wait!" you might say, "The above process was silly, because we just went around in circles!" We started with the problem of finding two numbers whose sum is 7 and whose product is 12, and after a bunch of work we "reduced" the problem to factoring an equation by finding two numbers whose sum is -7 and whose product is 12. So we didn't gain anything; we had to solve the same problem in the end! (The only difference is the -7 instead of the 7, because if a is a root, then $x - a$ is a factor.)

Indeed, for the above example, the method we used *was* in fact silly. We definitely didn't gain anything. We went around in circles and ended up where we started. However, in more complicated problems, the above method falls somewhere between a helpful strategy and an absolute necessity. It is necessary if you end up with a quadratic equation that you need to solve with the quadratic formula (see Chapter 8). And it is helpful if the answers aren't as obvious as the 3 and 4 above. The somewhat systematic guessing and checking you do when factoring a quadratic gives you a better chance of finding the answers than by just guessing solutions to the equations you start with, if they're more involved. The following example illustrates this.

Example 5.19 Solve for x and y in this system of equations:

$$2x + 3y = 16 \qquad \text{and} \qquad x(y+2) = 20. \tag{5.74}$$

Solution Solving for x in the first equation (y would work just as well) gives $x = (16-3y)/2$. Plugging this into the second equation, and using FOIL and multiplying through by 2, yields

$$\frac{16-3y}{2}(y+2) = 20 \quad \Longrightarrow \quad 16y + 32 - 3y^2 - 6y = 40 \quad \Longrightarrow \quad 0 = 3y^2 - 10y + 8 = 0. \tag{5.75}$$

Assuming the numbers were chosen to be nice, the factorization must take the form of $(3y + ?)(y + ?)$, where the question marks represent numbers that multiply up to 8. A little fiddling gives $(3y-4)(y-2)$, so the solutions for y are $4/3$ and 2. Plugging these back into either of the given equations yields x values of 6 and 5, respectively. So there are two solutions for the pair of x,y values: One is $x = 6$ and $y = 4/3$, and the other is $x = 5$ and $y = 2$. You can verify that both of these pairs also satisfy whichever equation you didn't use to generate the x values.

It is highly unlikely that you would arrive at the first pair by guessing and checking in the original equations. The word "solve" in the statement of the problem means to find *all* solutions, not just *a* solution. So if you found the $x = 5$ and $y = 2$ solution by guessing and checking, it wouldn't be correct to say that you had solved the problem (which means to *completely* solve it).

Exercise 5.25 The area of a rectangle is 96, and the perimeter (the sum of all four side lengths) is 40. What are the lengths of the sides?

Exercise 5.26 Solve for x and y in this system of equations:

$$x + \frac{1}{y} = 5 \qquad \text{and} \qquad y + \frac{1}{x} = \frac{5}{6}. \tag{5.76}$$

5.7.3 Quadratic inequalities

In Section 5.4 we solved inequalities involving linear expressions, such as the ones in Example 5.7. Let's now solve some inequalities involving quadratic expressions. For example, let's try to determine the range of x values that satisfy the inequality $x^2 - x - 6 > 0$. To do this, we'll factor the polynomial on the lefthand side. A little trial and error gives the factorization as $(x + 2)(x - 3)$, so the inequality becomes

$$(x + 2)(x - 3) > 0. \tag{5.77}$$

We see that we want the product of two things, namely $x + 2$ and $x - 3$, to be greater than zero. Now, the product of two numbers is greater than zero if the numbers are both positive or both negative. In contrast, the product is negative if one number is positive and the other is negative. The inequality in Eq. (5.77) is therefore satisfied if the factors $x + 2$ and $x - 3$ are both positive, or if they are both negative.

The numbers that make these factors be *equal* to zero are -2 and 3. So the number line in Fig. 5.5 shows the various possibilities. If $x > 3$, then both factors are positive, so the inequality is satisfied. And if $x < -2$, then both factors are negative, so the inequality is again satisfied. However, if x is between -2 and 3, that is, if $-2 < x < 3$, then the $x + 2$ factor is positive, but the $x - 3$ factor is negative. So in that region the product is negative, and the inequality is *not* satisfied.

The answer to the problem is therefore: The inequality $x^2 - x - 6 > 0$ is satisfied if either $x < -2$ or $x > 3$. The signs of the $x + 2$ and $x - 3$ factors are indicated with the $(--)$, $(+-)$, and $(++)$ signs shown below the number line.

Note that if $x = -2$ or $x = 3$ (with equals signs here), the inequality is *not* satisfied, because in both cases the product in Eq. (5.77) is zero, but we're dealing with a strict inequality ">"

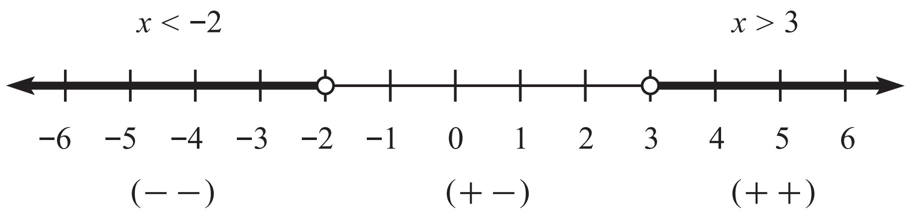

Figure 5.5: The three regions and the associated signs of the factors $x+2$ and $x-3$. The product is positive in the two outer regions.

sign and not a "$\geq$" sign. Hence the hollow dots in the figure at the $= -2$ and $x = 3$ points, consistent with our dot notation from Section 5.4. If the inequality sign in Eq. (5.77) had been a "$\geq$" (greater than or equal to), then we would have drawn solid dots at $= -2$ and $x = 3$, to indicate that the endpoints of the intervals *are* included in the desired ranges.

What if we instead had the inequality $(x+2)(x-3) < 0$, with the opposite sign? We would then want one factor to be positive and the other factor to be negative, so that the product would be negative. The solution to the inequality would then be $-2 < x < 3$ (the $(+-)$ region in the figure), which is the "opposite" of the original answer above.

In general, a quadratic inequality (after dividing through by the coefficient of x^2 and factoring) takes the form of $(x-a)(x-b) > 0$, where a and b are particular numbers. (The inequality sign can more generally be any of $>, <, \geq, \leq$.) The above example had $a = -2$ and $b = 3$, because these numbers make $(x-a)(x-b)$ be equal to $\big(x-(-2)\big)(x-3) = (x+2)(x-3)$, as desired. If we assume for concreteness that $a < b$, then the numbers a and b divide the number line into three regions, namely $x < a$, and $a < x < b$, and $b < x$, as shown in Fig. 5.6. That is, either x is less than a, or it is between a and b, or it is greater than b. We've chosen a to be negative and b to be positive for the sake of drawing the figure, but they could just as well be both positive or both negative (as long as $a < b$ in any case).

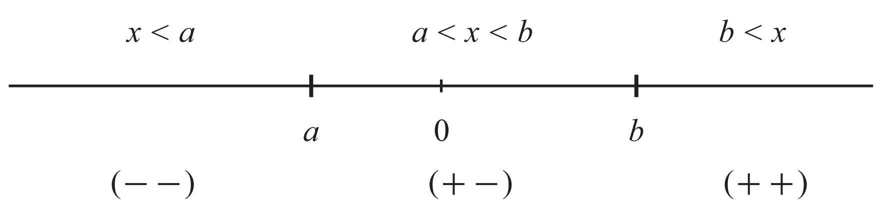

Figure 5.6: The three regions and the associated signs of the factors $x-a$ and $x-b$. This is the general version of Fig. 5.5.

We then simply observe that:

> The product $(x-a)(x-b)$ is positive in the outer two regions in Fig. 5.6, because the two factors are either both positive in the right region, or both negative in the left region. But the product is negative in the inner/middle region, because one factor, the $x-a$, is positive and the other, the $x-b$, is negative.

Therefore, the $(x-a)(x-b) > 0$ inequality is satisfied in the two outer regions, and the $(x-a)(x-b) < 0$ inequality is satisfied in the inner region.

Exercise 5.27 What values of x satisfy the inequality $(2x-1)(x+2) \le x(x+1)+6$?

Exercise 5.28 Explain how the solution to $(x-3)^2 > 0$ is consistent with the reasoning in the above discussion of Fig. 5.5.

Exercise 5.29 On a number line, indicate the values of x that satisfy this inequality: $(x+3)(x-2)(x-4) > 0$.

Exercise 5.30 Consider the inequality $x > 6/(x-1)$. If we multiply both sides by $x-1$ and simplify, we obtain

$$x(x-1) > 6 \quad\Longrightarrow\quad x^2 - x - 6 > 0. \tag{5.78}$$

This is the inequality we solved above in the text, so it seems like the solution to the original $x > 6/(x-1)$ inequality should be given by the heavy-line regions (namely $x < -2$ and $x > 3$) in Fig. 5.5. However, if you plug in, say, $x = -100$, you will find that $x > 6/(x-1)$ is *not* satisfied. Where is the error in the solution we just gave? (*Hint*: Recall an important fact from Section 5.4.)

5.8 Exercise solutions

1. (a) Subtracting 3 from both sides of $x + 3 = 8$ cancels the 3 on the left and leaves you with $x = 5$. Formally, $(x + \cancel{3}) - \cancel{3} = 8 - 3 \implies x = 5$.

   (b) Multiplying both sides of $-5 = x/2$ by 2 cancels the 2 on the right and leaves you with $-10 = x$. Formally, $2 \cdot (-5) = \cancel{2} \cdot x/\cancel{2} \implies -10 = x$.

   If you want, you can reverse the sides and write this as $x = -10$. When solving for x, people usually like to write their answer with the x on the left, although this certainly isn't necessary.

   (c) Multiplying both sides of $-x = 3$ by -1 gives $(-1)(-x) = (-1)(3) \implies x = -3$. Dividing both sides by -1 gives the same result.

   Alternatively, you can arrive at the answer in two steps: You can add x to both sides to obtain $0 = 3 + x$, and then subtract 3 from both sides to obtain $-3 = x$. (Or you can subtract 3 first, and then add x.) The one-step method of multiplying by -1 is quicker.

   (d) Multiplying both sides of $4/x = 1$ by x gives $4 = x$. Yes, it's perfectly fine for the operation to involve the variable you're solving for. We actually already did this when we added x to both sides in the two-step solution in part (c).

   (e) Squaring both sides of $\sqrt{x} = 6$ gives $x = 36$.

   (f) Taking the cube root of both sides (equivalently, raising both sides to the $1/3$ power) gives $(x^3)^{1/3} = 27^{1/3} \implies x = 3$. (This is the only real root.) We used the rule in Eq. (2.35) for compound exponents.

2. (a) Subtracting 2 from both sides of $5x + 2 = 12$ yields $5x = 10$. Dividing both sides by 5 then gives $x = 2$.

   Alternatively, you can first divide both sides by 5 to obtain $x+2/5 = 12/5$. Subtracting $2/5$ from both sides then gives $x = 12/5 - 2/5 = 10/5 = 2$.

   (b) Multiplying both sides of $(x-6)/3 = 7$ by 3 yields $x - 6 = 21$. Adding 6 to both sides then gives $x = 27$.

   Alternatively, you can first do the division by 3 on the left, to obtain $x/3 - 2 = 7$. Adding 2 to both sides yields $x/3 = 9$. And then multiplying both sides by 3 gives $x = 27$. This solution took three steps, although the first one was just rewriting the lefthand side. The other two steps were equal operations applied to both sides.

   (c) Multiplying both sides of $5/(x - 2) = 1$ by $x - 2$ yields $5 = x - 2$. Adding 2 to both sides then gives $7 = x$.

   (d) Squaring both sides of $\sqrt{x - 3} = 4$ yields $x - 3 = 16$. Adding 3 to both sides then gives $x = 19$.

   (e) Multiplying both sides of $x^2/3 = 12$ by 3 yields $x^2 = 36$. Taking the square root of both sides then gives $x = \pm 6$.

   Alternatively, you can first take the square root, to obtain $x/\sqrt{3} = \pm\sqrt{12}$. Multiplying both sides by $\sqrt{3}$ then gives $x = \pm\sqrt{3}\sqrt{12} = \pm\sqrt{36} = \pm 6$.

(f) Multiplying both sides of $1/x = 2$ by x yields $1 = 2x$. Dividing both sides by 2 then gives $1/2 = x$.

Alternatively, you can solve the $1/x = 2$ equation with one operation by taking the reciprocal of both sides. This gives $1/(1/x) = 1/2$, which simplifies to $x = 1/2$.

3. We can first subtract a from both sides of $a = b$ to obtain $0 = b - a$. We can then subtract b from both sides to obtain $-b = -a$. Finally, we can multiply (or divide) both sides by -1 to obtain $b = a$, as desired.

   Alternatively, we can first multiply both sides of $a = b$ by -1 to obtain $-a = -b$, and then add a to both sides, and then add b to both sides. This yields $b = a$.

4. Multiplying both sides of $x = a/(b/c)$ by b/c gives

$$x = \frac{a}{b/c} \quad \Longrightarrow \quad x \cdot b/c = \frac{a}{\cancel{b/c}} \cdot \cancel{b/c} \quad \Longrightarrow \quad x \cdot \frac{b}{c} = a. \tag{5.79}$$

   Multiplying both sides of this equation by c/b then gives

$$x \cdot \frac{\cancel{b}}{\cancel{c}} \cdot \frac{\cancel{c}}{\cancel{b}} = a \cdot \frac{c}{b} \quad \Longrightarrow \quad x = \frac{ac}{b}, \tag{5.80}$$

   as desired. Our first step of multiplying both sides by b/c could be called "un-solving for x," because we already had x isolated on one side of the equation. Normally you don't want to un-isolate x, because the general goal is to get x all alone on one side. However, in this problem we were pretending that we didn't know how to deal with fractions in a denominator. So our strategy was to multiply both sides by the fraction b/c, to get it out of the denominator. We *then* proceeded with isolating x on one side.

5. The rewritten equation, namely $x/2 - 5/2 = 3$, involves a new set of steps. Starting with x, the lefthand side instructs you to:

   1. Divide x by 2, and then
   2. Subtract $5/2$ from the result.

   The undoing operations in the reverse order are:

   1. Add $5/2$ to both sides, and then
   2. Multiply both sides by 2.

   The first of these steps turns $x/2 - 5/2 = 3$ into $x/2 = 3 + 5/2$, which can be simplified to $x/2 = 11/2$. The second step then turns this into $x = 11$, in agreement with Eq. (5.19).

   To be sure, you could very well solve the $x/2 - 5/2 = 3$ equation by first multiplying both sides by 2. This would just be the solution we gave in the text. But in more complicated problems, it's safer to work with the order of the operations you would perform if someone gave you a value of x and asked you to evaluate the lefthand side.

6. Starting with x, the lefthand side of Eq. (5.23) instructs you to perform five operations in the following order:

    1. Multiply by 2,
    2. Subtract 1,
    3. Take the reciprocal,
    4. Add 2,
    5. Multiply by 5.

    Undoing these operations in the reverse order means taking these five steps:

    1. Divide by 5,
    2. Subtract 2,
    3. Take the reciprocal,
    4. Add 1,
    5. Divide by 2.

    Applying these five steps to both sides of the given equation yields (as you can verify)

$$5 \cdot \left(\frac{1}{2x-1} + 2\right) = 11 \implies \frac{1}{2x-1} + 2 = \frac{11}{5} \implies \frac{1}{2x-1} = \frac{1}{5}$$
$$\implies 2x - 1 = 5 \implies 2x = 6 \implies x = 3. \tag{5.81}$$

    Alternatively, you can deal with the $1/(2x-1) = 1/5$ equation by multiplying both sides by 5, and also by $2x - 1$. This yields $5 = 2x - 1$, which is equivalent to the $2x - 1 = 5$ equation we obtained above by taking the reciprocal. The reciprocal method is a little quicker. And also the reciprocal operation is one of the five steps the lefthand side of the given equation instructs you to do. So it's more natural to take that path.

7. Starting with x, the lefthand side of Eq. (5.24) instructs you to perform eight operations in the following order:

    1. Multiply by 5,
    2. Add 1,
    3. Take the square root,
    4. Divide by 3,
    5. Add 3,
    6. Square,
    7. Subtract 7,
    8. Divide by 2.

    Undoing these operations in the reverse order means taking these eight steps:

    1. Multiply by 2,
    2. Add 7,
    3. Take the square root,
    4. Subtract 3,
    5. Multiply by 3,
    6. Square,
    7. Subtract 1,
    8. Divide by 5.

Applying these eight steps to both sides of the given equation yields (as you can verify)

$$\begin{aligned}
&\frac{1}{2}\cdot\left[\left(\frac{\sqrt{5x+1}}{3}+3\right)^2-7\right]=9 \quad\Longrightarrow\quad \left(\frac{\sqrt{5x+1}}{3}+3\right)^2-7=18 \\
&\Longrightarrow\quad \left(\frac{\sqrt{5x+1}}{3}+3\right)^2=25 \quad\Longrightarrow\quad \frac{\sqrt{5x+1}}{3}+3=5 \quad\Longrightarrow\quad \frac{\sqrt{5x+1}}{3}=2 \\
&\Longrightarrow\quad \sqrt{5x+1}=6 \quad\Longrightarrow\quad 5x+1=36 \quad\Longrightarrow\quad 5x=35 \quad\Longrightarrow\quad x=7.
\end{aligned} \tag{5.82}$$

REMARK: When taking the square root of 25 in the third step in Eq. (5.82), you might think that we should have written ± 5. However, the -5 value isn't relevant here, because the square root $\sqrt{5x+1}$ is defined to be a positive quantity, which means that $\sqrt{5x+1}/3+3$ is also positive. So it can't be equal to -5. There's technically nothing wrong with throwing in the $\pm$ sign by habit every time you take a square root. And to play it safe, that's probably what you should always do. But then be sure to check if there is some reason (like a square root necessarily being positive) that rules out one of the two options.

If Eq. (5.24) instead had a minus sign in front of the $\sqrt{5x+1}$, then the -5 value would be relevant (and the 5 wouldn't be). The remaining steps in Eq. (5.82) would be

$$\begin{aligned}
&\frac{-\sqrt{5x+1}}{3}+3=-5 \quad\Longrightarrow\quad \frac{-\sqrt{5x+1}}{3}=-8 \quad\Longrightarrow\quad -\sqrt{5x+1}=-24 \\
&\Longrightarrow\quad 5x+1=576 \quad\Longrightarrow\quad 5x=575 \quad\Longrightarrow\quad x=115. \quad \clubsuit
\end{aligned} \tag{5.83}$$

8. (a) We'll start guessing with $x = 4$. This makes the lefthand side of the given equation equal $14/5$, which is smaller than the desired 4. With $x = 5$ we obtain $17/5$, which is still too small. But then $x = 6$ gives $20/5$, which correctly equals 4. Of course, this guessing-and-checking method worked here only because the numbers were chosen to be nice. If the answer had been $x = 3.27481$, it would have been much harder to guess.

   (b) The given equation says that "A fifth of 2 more than 3 times x equals 4." Therefore,

      1. 2 more than 3 times x equals $5 \cdot 4 = 20$ (so that a fifth of this is 4).
      2. 3 times x equals $20 - 2 = 18$ (so that 2 more than this is 20).
      3. x equals $18/3 = 6$ (so that 3 times this is 18).

   (c) Starting with x, the given equation instructs us to: (1) multiply by 3, (2) add 2, (3) divide by 5. So the reverse "undoing" steps are: (1) multiply by 5, (2) subtract 2, (3) divide by 3. These steps (which are the same as the ones in part (b)) yield

$$\frac{3x+2}{5}=4 \quad\Longrightarrow\quad 3x+2=20 \quad\Longrightarrow\quad 3x=18 \quad\Longrightarrow\quad x=6. \tag{5.84}$$

9. (a) Subtracting 1 from both sides of $1 + 2x = 7 - x$ gives $2x = 6 - x$. Adding x to both sides then gives $3x = 6$. Finally, dividing both sides by 3 yields $x = 2$.

Alternatively, if you want to get x alone on the right: Subtracting 7 from both sides gives $-6 + 2x = -x$. Subtracting $2x$ from both sides then gives $-6 = -3x$. Finally, dividing both sides by -3 yields $2 = x$.

(b) Multiplying both sides by $2 \cdot 3 = 6$ gets rid of the 2 and 3 denominators and leaves us with

$$3\left(5x + \frac{2-3x}{4}\right) = 2(2x-1) \quad\Longrightarrow\quad 15x + \frac{6-9x}{4} = 4x - 2. \tag{5.85}$$

Multiplying both sides by 4 and simplifying (getting all the x's on the left, and all the regular numbers on the right) yields

$$60x + (6-9x) = 16x - 8 \quad\Longrightarrow\quad 35x = -14 \quad\Longrightarrow\quad x = -\frac{14}{35} = -\frac{2}{5}. \tag{5.86}$$

Alternatively, in Eq. (5.85) we could have subtracted $4x$ from both sides before multiplying by 4. We also could have split up the $(6-9x)/4$ fraction. But I like getting rid of the denominators. To eliminate them all right away, we could have multiplied by $2 \cdot 3 \cdot 4 = 24$ all at once at the start.

10. (a) Multiplying both sides by $7 \cdot 5$ (equivalently, cross multiplying) gives

$$5(4x-3) = 7(6-x) \quad\Longrightarrow\quad 20x - 15 = 42 - 7x. \tag{5.87}$$

Adding $7x$, and also 15, to both sides yields

$$27x = 57 \quad\Longrightarrow\quad x = \frac{57}{27} = \frac{19}{9}. \tag{5.88}$$

(b) Multiplying both sides by $(3x-2)(2x+7)$ (equivalently, cross multiplying) gives

$$7(2x+7) = 3(3x-2) \quad\Longrightarrow\quad 14x + 49 = 9x - 6 \tag{5.89}$$

Subtracting $9x$, and also 49, from both sides yields

$$5x = -55 \quad\Longrightarrow\quad x = -11. \tag{5.90}$$

11. (a) Multiplying both sides of the inequality by 10 (equivalently, cross multiplying by 5 and 2) and simplifying gives

$$2(1-2x) < 5(x+4) \implies 2-4x < 5x+20 \implies -9x < 18 \implies x > -2, \tag{5.91}$$

where we have switched the direction of the inequality sign after dividing by -9. Alternatively, if we put the x's on the righthand side instead of the left, we obtain

$$2 - 4x < 5x + 20 \quad\Longrightarrow\quad -18 < 9x \quad\Longrightarrow\quad -2 < x. \tag{5.92}$$

This "-2 is less than x" statement is equivalent to the above "x is greater than -2" statement.

(b) Combining the x terms on the left, and then multiplying both sides of the inequality by $3 \cdot 2 = 6$ and simplifying, gives

$$\begin{aligned} 6(-2x+5) > 2(1-5x)+3x \quad &\Longrightarrow \quad -12x+30 > 2-7x \\ &\Longrightarrow \quad -5x > -28 \quad \Longrightarrow \quad x < 28/5, \end{aligned} \tag{5.93}$$

where we have switched the direction of the inequality sign after dividing by -5. Alternatively, if we put the x's on the righthand side instead of the left, we obtain

$$-12x+30 > 2-7x \quad \Longrightarrow \quad 28 > 5x \quad \Longrightarrow \quad 28/5 > x, \tag{5.94}$$

which is equivalent to $x < 28/5$.

12. (a) Guessing and checking works only if the numbers are chosen to be nice, which they are in this case. Let's start with a random guess of 8 miles. Then 2 less than 8 is 6, and a third of 6 is 2. So 5 more than a third of 2 less than 8 equals 7. This is smaller than the 9 that was specified, so we need to increase our guess.

    We can gradually increase the guess until we obtain the correct answer. Or we can note that each increase of 3 miles yields an increase of 1 in the final result, due to the act of taking a third. Since we need to increase the final result by $9-7=2$, this means that we need to increase the guess by $2 \cdot 3 = 6$. The answer is therefore $8+6=14$ (which you can quickly check).

    (b) If 5 more than something (namely a third of 2 less than a number) is 9, then the something must be 4. And then if a third of a second something (namely 2 less than a number) is 4, then the second something must be 12. And finally, if 2 less than a third something (namely the desired number) is 12, then the number must be 14.

    (c) The translation of "5 more than a third of 2 less than the number of miles equals 9" into an equation is

$$5 + \frac{x-2}{3} = 9. \tag{5.95}$$

    Solving this equation yields

$$3 \cdot 5 + (x-2) = 3 \cdot 9 \quad \Longrightarrow \quad x+13 = 27 \quad \Longrightarrow \quad x = 14. \tag{5.96}$$

    To summarize the above three methods: Guessing and checking works only if the numbers are nice. The word-logic method is guaranteed to work, but it's a pain. The algebra way is guaranteed to work, and it's quick and easy.

13. If Cindy's present age is x, then the first sentence of the given information tells us that Dan's present age is $x-10$. For the second sentence, we note that 5 years ago, Dan's age was $x-15$. And 3 years from now, Cindy's age will be $x+3$. Since the former of these ages is half the latter, we have

$$x - 15 = \frac{x+3}{2}. \tag{5.97}$$

Solving for x gives

$$2(x-15) = x+3 \implies 2x-30 = x+3 \implies x-30 = 3 \implies x = 33. \tag{5.98}$$

So Cindy's age is 33, and Dan's age is $x - 10 = 23$, in agreement with the ages we found in Example 5.10.

14. Let x be the desired initial total number of ounces in the mug. Since $20\% = 20/100 = 1/5$, we're told that the initial number of milk ounces is $x/5$. This then means that the initial number of (pure) coffee ounces is $x - x/5 = 4x/5$.

    After you drink half the mug, the amounts of milk and coffee are each cut in half to $x/10$ and $4x/10$. And then when you add 1 ounce of milk, the amount of milk goes up to $x/10+1$. The total number of ounces in the cup is now $(x/10+1)+(4x/10) = 5x/10+1$. This correctly equals $x/2+1$, since you drank half of the x and then added 1.

    Since the new percentage of milk is $40\% = 40/100 = 2/5$, we have

$$\frac{\text{new milk}}{\text{new total}} = \frac{2}{5} \implies \frac{x/10+1}{5x/10+1} = \frac{2}{5}. \tag{5.99}$$

    Multiplying the lefthand side by $10/10$ to get rid of the 10's in the denominators, and then cross multiplying and simplifying, gives

$$\frac{x+10}{5x+10} = \frac{2}{5} \implies 5x+50 = 10x+20 \implies 30 = 5x \implies 6 = x. \tag{5.100}$$

    As a check on this result, the initial amount of milk is $(1/5)\cdot 6 = 6/5$ ounces, and the initial amount of (pure) coffee is $(4/5)\cdot 6 = 24/5$ ounces. Cutting these in half yields $3/5$ and $12/5$. Adding 1 to the former gives $3/5+1 = 8/5$. The final fraction of milk is therefore correctly

$$\frac{8/5}{8/5+12/5} = \frac{8}{8+12} = \frac{8}{20} = \frac{2}{5} = 40\%. \tag{5.101}$$

15. Let $t = 0$ correspond to noon. Then A's position is given by $x_A = 50t$. B starts 220 miles ahead, but travels for 2 hours less, so B's position is $x_B = 220 + 30(t-2)$. A catches up to B when the two positions are equal, which happens when

$$50t = 220+30(t-2) \implies 50t = 220+30t-60 \implies 20t = 160 \implies t = 8. \tag{5.102}$$

    A therefore catches up to B at 8:00. Plugging $t = 8$ into either of the above expressions for x_A or x_B then gives the common position as $x = 400$ miles. B travels $30 \cdot 6 = 180$ miles from its starting position of 220.

    For more practice, you can also solve this exercise by using each of the two methods presented in the remark in the solution to Example 5.11.

    Having worked through a few train-and-station problems, you are encouraged to solve the problem presented near the end of the video of the First Drafts of Rock: "Blowin' in the Wind" parody song (www.youtube.com/watch?v=XA4mlUPrdl4) performed on the Jimmy Fallon show. Pause the video and solve the problem before "Peter, Paul, and Mary" give the answer!

16. Let B's speed be v. Then A's speed is $2v$. Our goal is to solve for v. The given information tells us that A travels for 4 hours, and B travels for 5 hours. Let A's camp be located at $x = 0$, which means that B's camp is at $x = 6$. Then at 4:00 their two positions are $x_A = 2v \cdot 4$ and $x_B = 6 + v \cdot 5$ (since B starts at $x = 6$ and then travels an additional distance of $v \cdot 5$). Setting these two positions equal to each other gives

$$2v \cdot 4 = 6 + v \cdot 5 \quad \Longrightarrow \quad 3v = 6 \quad \Longrightarrow \quad v = 2. \tag{5.103}$$

So B's speed is 2, and A's speed is 4. Both of these speeds are measured in miles per hour, because with the units included, the above $3v = 6$ equation is actually $(3\text{ hours})v = 6$ miles, which yields $v = 2$ miles/hour.

Plugging $v = 2$ into either of the above expressions for x_A or x_B gives the common position as $x = 16$ miles. So A travels 16 miles in 4 hours (at 4 mph). And B travels 10 miles in 5 hours (at 2 mph) but had a 6 mile head start.

17. The two given equations are

$$\begin{aligned} 2x + 3y &= 7, \\ x - 4y &= 9. \end{aligned} \tag{5.104}$$

The remaining six routes (1, 3, 4, 5, 6, 7) are:

1. Solve for x in first, plug into second: The first equation gives $x = (7 - 3y)/2$. Plugging this into the second equation yields

$$\frac{7-3y}{2} - 4y = 9 \implies (7-3y) - 8y = 18 \implies -11y = 11 \implies y = -1. \tag{5.105}$$

Plugging $y = -1$ into either of the given equations then yields $x = 5$. In the remaining solutions below, we won't bother writing out this last step where we plug our answer back in to find the other variable.

3. Solve for y in first, plug into second: The first equation gives $y = (7 - 2x)/3$. Plugging this into the second equation yields

$$x - 4 \cdot \frac{7-2x}{3} = 9 \implies 3x - (28 - 8x) = 27 \implies 11x = 55 \implies x = 5. \tag{5.106}$$

4. Solve for y in second, plug into first: The second equation gives $y = (x - 9)/4$. Plugging this into the first equation yields

$$2x + 3 \cdot \frac{x-9}{4} = 7 \implies 8x + (3x - 27) = 28 \implies 11x = 55 \implies x = 5. \tag{5.107}$$

5. Solve for x in both and equate: This gives

$$\frac{7-3y}{2} = 9 + 4y \implies 7 - 3y = 18 + 8y \implies -11 = 11y \implies -1 = y. \tag{5.108}$$

6. Solve for y in both and equate: This gives (cross multiplying)

$$\frac{7-2x}{3} = \frac{x-9}{4} \implies 28 - 8x = 3x - 27 \implies 55 = 11x \implies 5 = x. \tag{5.109}$$

7. Form a combination where the x's cancel: Multiplying the second equation by -2 gives

$$\begin{aligned} 2x + 3y = 7 \quad &\Longrightarrow \quad 2x + 3y = 7 \\ -2\big[x - 4y = 9\big] \quad &\Longrightarrow \quad -2x + 8y = -18 \end{aligned} \tag{5.110}$$

Adding the right set of equations makes the x's cancel, and we obtain

$$0 \cdot x + 11y = -11 \quad \Longrightarrow \quad y = -1. \tag{5.111}$$

As you can see, the above solutions involve many of the same numbers (in particular, they all involve 11). So the methods are all very similar in the end. It's your choice which one you want to use. They all work, although the calculations might be quicker in some than in others.

Technically, each of the eight possible routes can be further subdivided into two sub-routes when we take into consideration which equation you plug your answer back into, to obtain the other variable. But we won't count these as separate solutions.

You are encouraged to make up your own systems of equations and solve them. You can work backwards by picking an arbitrary solution, say $x = 7$ and $y = 3$, and then writing down two true statements involving these x and y values (just to the first power; don't square them or anything like that). For example, you can arbitrarily choose to multiply x by 3 and then subtract 5 times y. This gives $3x - 5y = 6$, as you can quickly check. Another true equation is $-2x + 3y = -5$. You have now generated a system of two equations in two unknowns, which you can solve via the method of your choice.

Of course, if you make up problems this way, you'll already know the answer. But as mentioned on page 240, if you want some "fresh" problems, you can make up a bunch of them and then set them aside for a week or two, until you forget what the answers are. In any case, the point is to work through all the steps, even if you already know the answer.

18. (a) Solving for y in the first equation gives $y = (e - ax)/b$. Plugging this into the second equation (and multiplying through by b) yields

$$cx + d\left(\frac{e - ax}{b}\right) = f \quad \Longrightarrow \quad (bc - ad)x = bf - de \quad \Longrightarrow \quad x = \frac{bf - de}{bc - ad}. \tag{5.112}$$

(b) Solving for y in the two equations gives $y = (e - ax)/b$ and $y = (f - cx)/d$. Setting these two results equal (and cross multiplying) yields

$$\frac{e - ax}{b} = \frac{f - cx}{d} \quad \Longrightarrow \quad d(e - ax) = b(f - cx) \quad \Longrightarrow \quad (bc - ad)x = bf - de, \tag{5.113}$$

which is the same as the middle equation in Eq. (5.112).

(c) If we subtract d times the first equation from b times the second, the y terms cancel, and we obtain

$$b(cx + \cancel{dy}) - d(ax + \cancel{by}) = bf - de \quad \Longrightarrow \quad (bc - ad)x = bf - de, \tag{5.114}$$

as above. In all three methods, the coefficient of x on the left is $bc - ad$, and the expression on the right is $bf - de$. If you compare how these terms arise in the three methods, you'll see that you're doing roughly the same thing in all of them. Having the same *letters* appear in the three methods in this exercise is more convincing than having the same *numbers* appear in Exercise 5.17, which theoretically could have been just a coincidence.

For the specific case of Eq. (5.50), if we compare that system with Eq. (5.54), we can read off the values of the six letters $a, \ldots, f$. Plugging those values into the quantities $bc - ad$ and $bf - de$ gives

$$\begin{aligned} bc - ad &= (3)(1) - (2)(-4) = 11, \\ bf - de &= (3)(9) - (-4)(7) = 55. \end{aligned} \tag{5.115}$$

So the above $(bc - ad)x = bf - de$ equations all become $11x = 55$, in agreement with the analogous equations in the solution to Eq. (5.17).

19. Let the present ages of Alice and Carol be a and c, respectively. Let T_1 be the past time associated with the first sentence of the given information. And let T_2 be the future time associated with the second sentence. Then we have the table shown in Fig. 5.7 (explanation below).

	Alice	Carol
T_2	c	$2a$
now	a	c
T_1	$c-16$	$4a/3$

Each person must yield the
same difference in ages
(how much they age)

Each time must yield the
same difference in ages
(how much older Carol is than Alice)

Figure 5.7: A table showing Alice's and Carol's ages now, and at two other times.

The first of the given sentences says that at T_1 when Carol was $4/3$ as old as Alice is now (which means $4a/3$), Alice was 16 years younger than Carol is now (which means $c - 16$); this is the T_1 row in the table. The second of the given sentences says that at T_2 when Alice is eventually as old as Carol is now (which means c), Carol will be twice as old as Alice is now (which means $2a$); this is the T_2 row in the table.

As in Example 5.14, we can use the fact that the difference between Alice's and Carol's ages is always the same, no matter what year we choose to compare them. Presently, the difference is $a - c$. (This will turn out to be negative; that's fine.) And at time T_1 the difference was $(c - 16) - 4a/3$. And at time T_2 the difference will be $c - 2a$. These three differences at the three times must all be equal. So we can equate the $a - c$ difference with each of the other two:

$$\begin{aligned} a - c = (c - 16) - 4a/3 \quad &\Longrightarrow \quad 7a/3 = 2c - 16, \\ a - c = c - 2a \quad &\Longrightarrow \quad 3a = 2c. \end{aligned} \tag{5.116}$$

We can also equate the T_1 and T_2 differences, which gives $(c-16)-4a/3 = c-2a$. But this equation doesn't yield any new information, because it can be obtained by subtracting the above two equations, or equivalently by equating their righthand sides (since the lefthand sides are equal).

Replacing $2c$ with $3a$ in the first of the above equations (which the second equation tells us we can do) gives

$$7a/3 = 3a - 16 \quad \Longrightarrow \quad 16 = 2a/3 \quad \Longrightarrow \quad 24 = a. \tag{5.117}$$

And then from $3a = 2c$ we quickly find that $c = 36$. So Alice's present age is 24, and Carol's is 36. You can (and should) verify that these ages do indeed satisfy both sentences of the given information.

REMARKS:

1. As in Example 5.14, we can alternatively use the fact that Alice and Carol both age by the same amount between T_1 and the present time (this yields the equation $a - (c - 16) = c - 4a/3$), and also between the present time and T_2 (this yields the equation $c-a = 2a-c$). These two equations can be rewritten as the two in Eq. (5.116).
2. This exercise was a little more difficult than Example 5.14, because we needed to create a table with three rows here. There were only two rows in Example 5.14, because one of the bits of information was the simple $a+b = 88$ equation. In any case, although age problems like this one might seem difficult at first, they're quite doable if you make a table that incorporates the information at each of the given times (which were T_1, now, and T_2 in the present exercise).

> If you feel that you're simply unable
> To solve things with "Age" in the label,
> Just write down the ages
> At each of the stages,
> And gather them all in a table!

3. In the spirit of the comment in the solution to Exercise 5.17, you are encouraged to make up your own solving-for-ages problems. You can work backwards by picking two arbitrary ages, such as $a = 18$ and $b = 22$, and then writing down true statements about these ages. Two simple statements are: the sum $a + b$ is 40, and the difference $b - a$ is 4. Another one is: 7 years ago, Alice was half as old as Bob is now. (This translates to $a - 7 = b/2$.) Another is: When Alice is 10 years older than Bob is now,

Bob will be twice as old as Alice is now. (This translates to saying that there is a future time T when Alice's and Bob's ages are, respectively, $b + 10$ and $2a$, which you can put in a table and then proceed as above.) When writing your problem, you need to state only *two* of the facts, as we did in this exercise and in Example 5.14. You will then have two equations in two unknowns. ♣

20. (a) This system is degenerate, because the equation $9x - 15y = 12$ is 3 times $3x - 5y = 4$. Solving for x in either equation gives $x = (4 + 5y)/3$. (Solving for y works fine too.) We see that for any arbitrary value of y (of which there is an infinite number of possibilities), if we pick x to be equal to $(4 + 5y)/3$, then the x,y pair is a solution to both equations. So there is an infinite number of x,y pairs that solve the system.

    (b) This system is inconsistent, because $9x - 15y$ is 3 times $3x - 5y$ on the lefthand sides, but 13 isn't 3 times 4 on the righthand sides. The two multiples are different. Equivalently, the $9x - 15y = 13$ equation minus 3 times the $3x - 5y = 4$ equation gives $0 = 1$, which is false. So there are no solutions.

    (c) This system is consistent, because the parts containing the variables aren't multiples of each other: $9x - 14y$ isn't a multiple of $3x - 5y$. To solve the system, let's subtract 3 times $3x - 5y = 4$ from $9x - 14y = 12$. The x's cancel, and we end up with $(-14+15)y = 0 \implies y = 0$. (There's nothing wrong with a solution being 0.) Either equation then gives $x = 4/3$. So the single x,y pair that solves the system is $4/3$, 0.

21. We'll eliminate y by forming useful combinations of the equations. The combinations we'll use are: (1) the 1st equation plus 2 times the 3rd, and (2) the 2nd equation minus 3 times the 3rd. These combinations make the y's cancel, and you can verify that they yield the following two equations:

$$\begin{aligned} -9x + 12z &= 57, \\ 17x - 11z &= -61. \end{aligned} \tag{5.118}$$

We must now solve this system of two equations in two unknowns. Dividing the first equation by 3 turns it into $-3x + 4z = 19$. So $z = (19 + 3x)/4$. Plugging this into the second equation gives

$$17x - 11\left(\frac{19 + 3x}{4}\right) = -61 \quad \Longrightarrow \quad (17\cdot 4 - 11\cdot 3)x = -61\cdot 4 + 11\cdot 19, \tag{5.119}$$

which simplifies to $35x = -35$, so $x = -1$. The $z = (19 + 3x)/4$ relation then gives $z = 4$. And then plugging $x = -1$ and $z = 4$ into any of the three given equations yields $y = 3$, in agreement with the result in Example 5.15.

REMARK: The combinations of equations in this exercise (for example, the 1st equation plus 2 times the 3rd) were simple because the coefficient of y in the third equation was 1. Similarly, the combinations of equations in Example 5.15 were simple because the coefficient of z in the first equation (namely 6) was a multiple of the coefficients of z in the other two equations (namely -2 and 3). This meant that in the linear combination of equations, the first one didn't need to be multiplied by anything. However, in more general cases, both equations need to be multiplied by something, for example by b and d in Exercise 5.18(c). ♣

22. (a) Getting rid of the denominators by multiplying through by $3x$ yields

$$3x^2 + 3 = 10x \quad \Longrightarrow \quad 3x^2 - 10x + 3 = 0. \tag{5.120}$$

A little fiddling tells you that the factorization is $(3x - 1)(x - 3) = 0$. Setting each of these binomials equal to zero gives the solutions as $1/3$ and 3. You can verify that both of these answers satisfy the given equation.

(b) Getting rid of all the denominators by multiplying through by $5x(x + 2)$ yields

$$\begin{aligned} 5(x+2) + 5x = 12x(x+2) \quad &\Longrightarrow \quad 10x + 10 = 12x^2 + 24x \\ \Longrightarrow \quad 0 = 12x^2 + 14x - 10 \quad &\Longrightarrow \quad 0 = 6x^2 + 7x - 5. \end{aligned} \tag{5.121}$$

To factor this equation, we note that (assuming the numbers were chosen to be nice) the factorization must take the form of $(6x + ?)(x + ?)$ or $(3x + ?)(2x + ?)$, where the question marks represent numbers that multiply up to -5. A little fiddling with the first option will convince you that it can't work. And then a little fiddling with the second option will lead to $(3x+5)(2x-1)$. So the solutions are $-5/3$ and $1/2$. Again, you can verify that both of these answers satisfy the given equation.

23. The given information translates to the equation,

$$\frac{1}{1/x + 2} = \frac{3}{8} \cdot \frac{1}{x + 1}\,. \tag{5.122}$$

Multiplying the lefthand side by 1 in the form of x/x turns it into $x/(1+2x)$. Alternatively, you can first add the $1/x$ and 2 to obtain $(1 + 2x)/x$. The denominator x then flips up top in the overall fraction. There are other possible first steps, but it is usually best to get rid of any denominators within denominators right at the start.

We can get rid of all the remaining denominators by multiplying through by $8(2x+1)(x+1)$. This gives

$$\begin{aligned} \frac{x}{2x+1} = \frac{3}{8} \cdot \frac{1}{x+1} \quad &\Longrightarrow \quad 8(x+1)x = 3(2x+1) \\ \Longrightarrow \quad 8x^2 + 8x = 6x + 3 \quad &\Longrightarrow \quad 8x^2 + 2x - 3 = 0. \end{aligned} \tag{5.123}$$

To factor this equation, we note that (assuming the numbers were chosen to be nice) the factorization must take the form of $(8x + ?)(x + ?)$ or $(4x + ?)(2x + ?)$, where the question marks represent numbers that multiply up to -3. A little fiddling with the first option will convince you that it can't work. And then a little fiddling with the second option will lead to $(4x + 3)(2x - 1)$. So the solutions are $-3/4$ and $1/2$. You can verify that both of these answers satisfy the given statement.

24. The solution is incorrect, in that it is incomplete; it misses a root. The correct way to solve the equation is to get all the terms on one side, and then factor the polynomial, as we did in Eq. (5.68):
$$x^2 = 5x \quad \Longrightarrow \quad x^2 - 5x = 0 \quad \Longrightarrow \quad x(x-5) = 0. \tag{5.124}$$
The roots are therefore $x = 0$ and $x = 5$. It's certainly true that $x = 0$ satisfies the $x^2 = 5x$ equation, since both sides are zero.

The error in the student's incorrect solution was the division by x. It isn't legal to divide by zero, so when dividing by x, the student basically *assumed* that x isn't zero. But this is an incorrect assumption since $x = 0$ is in fact a root. When solving equations by performing the same operations on both sides, it is understood that the operations must be *legal* ones. Division by zero is illegal.

This division-by-zero error is the main ingredient in variety of bogus math "proofs," such as the one presented in Eq. (5.16), where we "proved" that $2 = 1$.

25. Let the two side lengths be x and y. Then the area is xy, and the perimeter is $2x + 2y$ because there are two sides of each length. The given information therefore says:
$$xy = 96 \qquad \text{and} \qquad 2x + 2y = 40. \tag{5.125}$$
Solving for y in the second equation gives $y = 20 - x$. Plugging this into the first equation yields
$$x(20 - x) = 96 \quad \Longrightarrow \quad 0 = x^2 - 20x + 96. \tag{5.126}$$
A little trial and error shows that this factors into $(x-8)(x-12) = 0$. So the solutions for x are 8 and 12. Plugging each of these into either of the given equations yields corresponding y values of 12 and 8. The two x,y pairs are therefore 8 and 12; and 12 and 8. So either way, the two side lengths are 8 and 12.

In retrospect, this exercise was roughly as silly as Example 5.18, because (after dividing the perimeter equation by 2), the two given equations are $xy = 96$ and $x + y = 20$. So we're looking for two numbers that multiply to 96 and add to 20. And this is exactly what we needed to determine when we factored the $x^2 - 20x + 96 = 0$ equation. Therefore, not much was gained in this exercise by producing a quadratic equation and then factoring it.

26. First solution: The most straightforward strategy is to solve for x in the first equation (which quickly gives $x = 5 - 1/y$), and then plug the result into the second one. (Solving for y in the second equation and plugging the result into the first one would work just as well.) This turns the second equation into
$$y + \frac{1}{5 - 1/y} = \frac{5}{6} \quad \Longrightarrow \quad y + \frac{1}{(5y-1)/y} = \frac{5}{6} \quad \Longrightarrow \quad y + \frac{y}{5y-1} = \frac{5}{6}. \tag{5.127}$$
Multiplying through by $6(5y-1)$ to get rid of the denominators yields
$$y \cdot 6(5y-1) + y \cdot 6 = 5 \cdot (5y-1) \quad \Longrightarrow \quad (30y^2 - 6y) + 6y = 25y - 5. \tag{5.128}$$

Getting all the terms on the lefthand side and dividing by 5 gives

$$30y^2 - 25y + 5 = 0 \quad \Longrightarrow \quad 6y^2 - 5y + 1 = 0. \tag{5.129}$$

To factor this equation, we note that (assuming the numbers were chosen to be nice) the factorization must take the form of $(6y+?)(y+?)$ or $(3y+?)(2y+?)$, where the question marks represent numbers that multiply up to 1 (which means they must both be 1, or both be -1, assuming they are integers). We quickly see that the first option doesn't work. But the second one yields $(3y-1)(2y-1)$. So the solutions for y are $1/3$ and $1/2$. Plugging these into the first equation in Eq. (5.76) produces the corresponding x values of 2 and 3. So there are two pairs of x,y values that solve the problem: 2 and $1/3$, and 3 and $1/2$. You can verify that each of these pairs also satisfies the second equation in Eq. (5.76).

Second solution: We can take advantage of the similar nature of the lefthand sides of the given equations. If we multiply the equations through by y and x, respectively, to get rid of the denominators, they become

$$xy + 1 = 5y \qquad \text{and} \qquad xy + 1 = \frac{5x}{6}. \tag{5.130}$$

Since the lefthand sides of these equations are equal, the righthand sides must also be equal. Therefore,

$$5y = \frac{5x}{6} \quad \Longrightarrow \quad y = \frac{x}{6} \quad \Longrightarrow \quad 6y = x. \tag{5.131}$$

So $x = 6y$. Plugging this into the first equation in Eq. (5.130) yields

$$(6y)y + 1 = 5y \quad \Longrightarrow \quad 6y^2 - 5y + 1 = 0, \tag{5.132}$$

in agreement with Eq. (5.129). The solution proceeds as above.

27. Expanding the products and getting everything on one side of the inequality yields

$$2x^2 + 3x - 2 \le x^2 + x + 6 \quad \Longrightarrow \quad x^2 + 2x - 8 \le 0. \tag{5.133}$$

With a little fiddling, this factors into $(x+4)(x-2) \le 0$. So the numbers analogous to -2 and 3 in Fig. 5.5 are now -4 and 2; these are the values that make the factors be zero. And since the present inequality is the *less* than (or equal to) type, we want the middle $-4 \le x \le 2$ region in the figure analogous to Fig. 5.5. Values of x in this region make the $x+4$ factor positive and the $x-2$ factor negative, yielding a negative product. The endpoints of the $-4 \le x \le 2$ region are included in the answer, because of the "$\le$" sign. So the dots that were hollow in Fig. 5.5 are now solid.

28. Since the square of any nonzero number is positive, every value of x except $x = 3$ satisfies the inequality $(x-3)^2 > 0$. This is consistent with the discussion of Fig. 5.5, for the following reason. First note that $(x-3)^2$ can be written as $(x-3)(x-3)$. If instead we had $(x-2.99)(x-3) > 0$, this would involve replacing the x value of -2 in Fig. 5.5 with 2.99. So we would need to move the dot at -2 up to 2.99, very close to the dot at 3.

The middle region (the part with no heavy line) in Fig. 5.5 would now be very small. All numbers on the number line except the ones in this tiny region between 2.99 and 3 satisfy the $(x-2.99)(x-3) > 0$ inequality. The heavy line covers all of the number line except the tiny region with length 0.01.

If we then imagine changing the 2.99 value to 2.999, and so on, until we eventually reach 3, we'll end up with the inequality $(x-3)(x-3) > 0$, and the tiny region will have zero length. This means that every x value except $x = 3$ (since the dot is hollow) will satisfy the inequality, as we wanted to show.

29. The product of three factors is positive if all three of them are positive, or if one is positive and two are negative. (The product is negative if all three factors are negative, or if one is negative and two are positive.) The given factors equal zero when x equals -3, 2, or 4. These three points divide the number line into four regions, as shown in Fig. 5.8. If $x > 4$, then all three factors are positive, so the product is positive. And if $-3 < x < 2$, then the $x + 3$ factor is positive, while the other two are negative, so the product is again positive. The regions $x > 4$ and $-3 < x < 2$ therefore satisfy the inequality. The other two regions do not, since either one or three of the factors are negative.

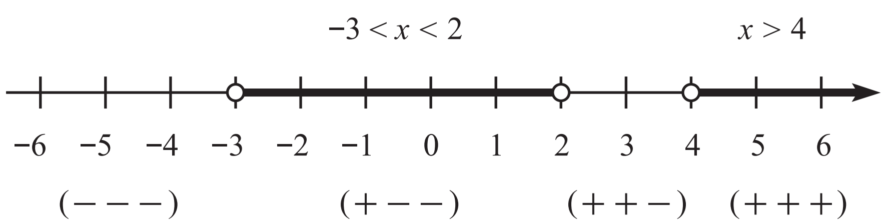

Figure 5.8: The four regions and the associated signs of the factors $x+3$, $x-2$, and $x-4$. The product is positive in the two regions indicated by the heavy lines.

REMARK: More generally, if there are n factors of the form

$$(x-a_1)(x-a_2)(x-a_3)\cdots(1-a_n), \tag{5.134}$$

then they divide the number line into $n+1$ regions. As you move from one region to the next, the product of the n factors alternates in sign, with exactly one of the factors changing its sign when a boundary value is crossed. So the general solution to an inequality involves alternating regions.

There are two possibilities for which set of alternating regions you want, depending on whether it includes the region that extends up to $x = +\infty$. If the inequality sign is "$>$" (or "$\geq$"), then you start with the rightmost region that extends to $x = +\infty$, and then choose alternating regions as you march leftward, as we did in Fig. 5.8. If instead the inequality sign is "$<$" (or "$\leq$"), then you skip the rightmost region that extends to $x = +\infty$, but then likewise choose alternating regions as you march leftward. ♣

30. The issue is that we multiplied the given inequality by $x-1$. And as we saw in Section 5.4, multiplication by a negative number requires reversing the inequality sign. This means

that if x is less than 1 (which makes $x - 1$ negative), we should have reversed the sign. To solve the problem correctly, we need to solve two separate problems, depending on whether x is greater than or less than 1:

1. Assume that $x > 1$. Then the quantity $x - 1$ is positive, so we don't have to reverse the inequality sign. We therefore just need to solve the inequality in Eq. (5.78), which we already did in Fig. 5.5. *However*, only the $x > 1$ part of the number line is valid now, because that is our present assumption. So at this point, all we have is the $x > 3$ heavy line in Fig. 5.5. The $x < -2$ heavy line doesn't satisfy our $x > 1$ assumption, so we must discard it. Presently, we can say nothing about values of x that are less than 1.
2. Assume that $x < 1$. Then the quantity $x - 1$ is negative, so we now *do* have to reverse the inequality sign. We therefore need to solve the $x^2 - x - 6 < 0$ inequality, with a "<." The solution to this is the opposite of the solution in Fig. 5.5. We now want the middle $-2 < x < 3$ region. *However*, only the $x < 1$ part of the number line is valid now, because that is our present assumption. The part of the $-2 < x < 3$ region that also satisfies the $x < 1$ assumption is $-2 < x < 1$. So this is the interval we want. The other $1 < x < 3$ part doesn't satisfy our $x < 1$ assumption, so we must discard it.

Putting the above two results together, the solution to the original $x > 6/(x-1)$ inequality consists of the two regions $-2 < x < 1$ and $x > 3$. These regions are shown in Fig. 5.9.

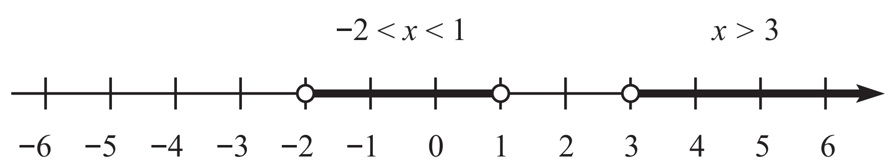

Figure 5.9: The $x > 6/(x - 1)$ inequality is satisfied in the two regions indicated by the heavy lines.

Remark: The reason why Figs. 5.5 and 5.9 look different is that there were only two special points (namely −2 and 3) in Fig. 5.5, whereas there are three special points (namely −2, 1, and 3) in Fig. 5.9. By "special point," we mean a point where a quantity changes sign. In addition to the original two binomials $x + 2$ and $x - 3$, we now also have $x - 1$. The moral of this exercise is: When solving inequalities, be careful when multiplying (or dividing) by expressions involving x, because they may be negative. If there is ever a possibility that you may be multiplying an inequality by a negative quantity, you must solve separate problems, depending on whether that quantity is positive or negative.

When working with *equalities*, the main thing you need to worry about is dividing by zero (as in Eq. (5.16)). But when working with *inequalities*, you additionally need to worry about multiplying or dividing by a negative number, since that reverses the sign. ♣