Chapter 3

Expansions, FOIL

From *Algebra: For the Enthusiastic Beginner* (Draft version, July 2024)

The first two chapters of this book covered all of the necessary background material, so we're now ready to dive into some real algebra. Algebra, loosely defined, involves working with letters (and some numbers too, of course) instead of only numbers. An important aspect of algebra involves solving for variables (such as x); we'll discuss this in Chapter 5. But there are plenty of other things to do in algebra that don't involve solving for variables, as we'll see throughout the book. This chapter and the following one cover the basics of how to work with letters – writing expressions in different forms, simplifying them, deriving general formulas, and so on. And then in Chapters 5 and onward we'll learn all sorts of applications of our newfound letter skills.

We'll start off in Section 3.1 with the topic of *collecting like terms*, which is a key component in simplifying expressions. Section 3.2 presents an extension of the distributive law called the "FOIL" method, which will come up again and again. Section 3.3 discusses three special cases of FOIL, and then Sections 3.4 and 3.5 cover various generalizations. Section 3.6 takes things a step further with the *binomial expansion* and *Pascal's triangle*. These in turn lead to a nice method of making approximations in Section 3.7.

Once you complete this chapter and the next, your algebra skills will be on firm footing. In a sense, the rest of the book is "just" applications. However, there is still a lot to be learned (as evidenced by the length of the book), due to the fact that the applications are both numerous and very important (for example, solving for a variable like x).

3.1 Collecting like terms

When doing algebra, one of the most common things you will need to do is "collect *like* terms." By "collect" we mean add (or subtract), and by "like" terms we mean terms that are *similar*, or equivalently of the *same type*. So "collect like terms" says the same thing as "collect similar terms." But what do we mean by the adjectives "like" or "similar"?

Consider, for example, $3a^2$ and $6a^2$, or $3 \cdot a^2$ and $6 \cdot a^2$ if you want to include the multiplication dots. These are like (similar) terms because they have the *same power* of a. The multiplicative factors of 3 and 6 out front don't matter, as far as determining like-ness (similarity) goes. These multiplicative factors are called *coefficients*. Coefficients are often numbers, but they can also be letters. For example, later on in Section 6.4.2, we'll see that the slope of the line $y = mx + b$ is m. The m here is the coefficient of x.

Since $3a^2$ and $6a^2$ are like terms (they have the same power or a), we can add them together just as you would think:

$$3a^2 + 6a^2 = 9a^2. \tag{3.1}$$

No matter what the value of a might be, it is always true that $3a^2$ plus $6a^2$ equals $9a^2$. A common task in algebra is simplifying expressions, and collecting like terms is a crucial ingredient in this. $9a^2$ is certainly simpler than $3a^2 + 6a^2$.

The terms you're dealing with may very well involve more than one letter, for example, $2a^2b$ and $5a^2b$. These terms are also of the same type, because they have the same powers of the different letters. In both terms, the a is squared and the b is raised to the first power. (Again, the coefficients of 2 and 5 out front don't affect the type.) Since the powers match up, the terms again add just as you would think:

$$2a^2b + 5a^2b = 7a^2b. \tag{3.2}$$

No matter what the values of a and b might be, it is always true that $2a^2b$ plus $5a^2b$ equals $7a^2b$. For example, if $a = 3$ and $b = 2$, then $a^2b = 3^2 \cdot 2 = 18$. So for these specific values of a and b, Eq. (3.2) becomes

$$2 \cdot 18 \ + \ 5 \cdot 18 = 7 \cdot 18 \quad \Longrightarrow \quad 36 + 90 = 126, \tag{3.3}$$

which is indeed true. But the main point here is that Eq. (3.2) is true for *any* values of a and b you choose. Equivalently, it is true for any value of the quantity a^2b. 2 snorgiborgs plus 5 snorgiborgs always equals 7 snorgiborgs, even if you have no idea what a snorgiborg is!

As we've mentioned, the powers of the letters are what's important, and not the coefficients out front. If we instead have $2a^2b + 5ab$, where the a isn't squared in the second term, then these are *not* like terms, so we can't do much with the sum. Well, we could factor it and write it as $a(2ab + 5b)$ by using the distributive law. (We'll talk about factoring in Section 4.2.) Depending on what we're going to do with it, this factored form might or might not be more simplified and convenient than the original form. In any case, we still have a sum of two terms in the parentheses. There is no way to write the expression as just a single term and have it be true for all values of a and b.

For *specific* values of a and b, a certain relation may very well be true. For example, if $a = 2$, then $2a^2b + 5ab = 9ab$. This is indeed true if $a = 2$, because the lefthand side equals $2 \cdot 2^2b + 5 \cdot 2b = 8b + 10b = 18b$, and the righthand side also equals $9 \cdot 2b = 18b$. So the two sides are equal for any value of b *if* $a = 2$.

However, it doesn't count if a relation is true only for *specific* values of a letter. It needs to be true for *all* values of the letters. If no one has told us yet what the values of a and b are, then our proposed relation needs to be true for *any* numbers they might throw at us. Terms such as $2a^2b$ and $5ab$ can't be combined *in general*, because they aren't like terms; they don't have the same powers of the various letters. Like terms can differ in their coefficients (the numerical factors out front, like the 2 and 5 in Eq. (3.2)), but not in their exponents.

Like terms (equivalently, terms that are *similar*, or of the *same type*) are ones that have the same power of each letter, such as $5a^2b^7c^4$ and $8a^2b^7c^4$. The numerical coefficient out front doesn't affect the type.

Since multiplication is commutative (see Section 2.1.1), the order of the letters within the terms you're comparing doesn't matter, as long as the total power of each letter is the same. So the following five terms are all like terms, that is, they are all of the same type (they all have three a's and two b's multiplied together):

$$2a^3b^2 \qquad -5b^2a^3 \qquad 7a^2bab \qquad abbaa\cdot 8 \qquad -b\cdot 6\cdot a^2\cdot 4\cdot ba \tag{3.4}$$

If we add all five terms together, we obtain $-12a^3b^2$, because the sum of the coefficients is $2-5+7+8-24=-12$.

REMARK: Note how we have been using the word "term" in the above discussion. The *product* of any number of letters (and numbers) is a *single* term. For example, $9a^2b^5$ is just one term. (The same thing is true if we include quotients.) However, the *sum* (or difference) of letters is more than one term. For example, $a+b$ is two terms, $5a^2b-7abc^3$ is also two terms, and $3ab-6cd+a$ is three terms.

Counting the number of terms is made easy due to the convention that writing letters right next to each other means multiplying them. This convention makes it clear that $3ab-6cd+a$ consists of three terms ($3ab$, $6cd$, and a) separated by the $-$ and $+$ signs. These signs take up some physical space, which conveniently separates the different terms from each other, allowing you to easily see that there are in fact three of them. In contrast, if you write $3ab-6cd+a$ like this:

$$3\times a\times b-6\times c\times d+a, \tag{3.5}$$

it isn't as obvious that there are actually three terms. So don't write it that way!

With regard to counting the number of terms, we see that products and sums are *not* treated on equal footing: A product produces a single term, whereas a sum produces two or more. This asymmetry stems from the order-of-operations rule that multiplication (and division) is always performed before addition (and subtraction). After you perform the multiplications/divisions, but before you perform the additions/subtractions, if you count up how many clumps of letters you have (for example, there are three clumps in $3ab-6cd+a$), then that's the number of terms. ♣

Example 3.1 Simplify the following expression by collecting like terms:

$$a^2b+3a-8ba+7ba^2-9b+2ab+2b-4a^2b-5ac. \tag{3.6}$$

Solution The like terms can be grouped as follows (the parentheses are for visualization purposes; removing them wouldn't change the expression):

$$(a^2b+7a^2b-4a^2b)+(-8ab+2ab)+(-9b+2b)-5ac+3a. \tag{3.7}$$

We have used the fact that multiplication is commutative, which means that $7ba^2 = 7a^2b$, and $8ba = 8ab$. We have also used the fact that addition is commutative, which means that we're free to rearrange the order of the terms (provided that each term is still associated with the same sign in front of it; for example, the $9b$ term still has a minus sign in front of it). Note that the $-5ac$ and $3a$ terms are stand-alone terms that aren't like any others. So we can't do anything with them.

Combining (adding/subtracting) all the like terms (the ones within each set of parentheses) in Eq. (3.7) gives

$$(4a^2b) + (-6ab) + (-7b) - 5ac + 3a, \tag{3.8}$$

which can be rewritten as

$$4a^2b - 6ab - 7b - 5ac + 3a. \tag{3.9}$$

This is as far as we can go with the simplification. None of these terms are like any others, so we can't combine them any further.

Example 3.2 In the preceding example, all the individual terms were given to us. But in some cases we first need to expand (multiply out) sub-expressions involving parentheses by using the distributive law. Once we do that, we can then compare the terms and see which ones are of the same type. Use this procedure to simplify the following expression:

$$a^2(b-c) - ab(a+b) - c(b^2 - a^2) \tag{3.10}$$

Solution Using the distributive law to expand the three sets of parentheses gives

$$(\cancel{a^2b} - \cancel{a^2c}) - (\cancel{a^2b} + ab^2) - (cb^2 - \cancel{ca^2}) \;=\; -ab^2 - cb^2. \tag{3.11}$$

We see that there are two terms of the a^2b type, and they have opposite signs, so they cancel. (Canceling is a special case of collecting like terms, when you end up with nothing.) And there are two terms of the a^2c type, and they also have opposite signs (since the two minus signs in front of the second ca^2 make a positive sign), so they also cancel.

Remark: The definition of the word "term" we gave in the remark preceding this set of examples can be slightly generalized. It would be quite reasonable for someone to say that there are three terms in Eq. (3.10), where each term consists of a product of various things, one of which is a sum in parentheses. However, none of these three terms are like each other, so this way of looking at terms isn't helpful. That's why we expanded the products to obtain the six terms in Eq. (3.11). Some of these terms were like each other, allowing the simplification.

On the other hand, if we were given the expression $a^2(b-c)+3a^2(b-c)$, then it *would* be helpful to view this as two terms, and to then combine them to obtain $4a^2(b-c)$. (You can expand this if you wish.) In the present case, the two terms (products of things, one of which is a sum in parentheses) are like each other, because they differ only in their coefficient. ♣

When dealing with fractions, the powers of each letter in like terms must again agree, taking into account whether they're in the numerator or denominator. For example, $3a^2/b^3$ and $5a^2/b^3$ are like terms, but $3a^2/b^3$ and $5a^2/b^4$ are not, because the powers of b aren't the same. $3a^2/b^3$ and $5a^2b^3$ are also not like terms, because although they both involve b^3, one of the b^3's is in the denominator, and the other is in the numerator.

Everything we've said holds for fractional powers too: The powers need to agree, whether they're integers or not. For example, $3a^{7/2}b^{1/2}$ and $5a^{7/2}b^{1/2}$ are like terms, but $3a^{7/2}b^{1/2}$ and $5a^{5/2}b^{1/2}$ are not, because the powers of a aren't the same.

If you try to combine terms whose powers don't match, you'll end up with nonsense. And if you *don't* combine terms whose powers *do* match, your expression won't be as simple as it could be.

> The students all started to gripe,
> "Who *cares* about terms and their type?
> It's all such a chore –
> We'll match powers no more!"
> Unsurprising, their answers were tripe.

REMARK: In my early days of learning algebra, I had the hardest time with collecting like terms. I had no idea what was going on, and there didn't seem to be any rhyme or reason to the various steps that were being made. The phrase "collect like terms" simply made no sense to me. It turns out that I was using the word "like" in the way it is used as a *preposition* in phrases like "collect like mad" or "collect like Michelle collects." So I was reading "collect like terms" as "collect in the same manner that terms do their collecting." You can see why it made no sense! Anyway, once I learned that "like" is a synonym for "similar" (an *adjective*!), it all fell into place and became much more fun. It's hard to do algebra when you don't even have the parts of speech correct! ♣

Exercise 3.1 Simplify the following expressions by collecting like terms:

(a) $6ac+4b-2a^2b+3abc-ca+7ba^2-9bca-3aba-2ac$

(b) $a(a-b)-b(b-a)-a(a+c)$

(c) $\dfrac{a^3 + 3a^2 b}{ab^2} - \dfrac{a}{b}\left(\dfrac{a}{b} + 1\right)$

(d) $\sqrt{a}(\sqrt{a} - \sqrt{b}) - 4a + \sqrt{ab}$

3.2 FOIL

We learned in Section 2.1.3 how to evaluate expressions like $a(b+c)$ by using the distributive law: We just need to "distribute" the a to both the b and c inside the parentheses, which yields $ab+ac$. But what about expressions like $(a+b)(c+d)$, which contain two binomials? (A *binomial* is the sum (or difference) of two terms, such as $a+b$, or $3x+5y$, or $ab^2 - 4a^3c$.) We can evaluate the product $(a+b)(c+d)$ by applying the distributive law twice (or actually three times), as follows.

If we define the sum $c+d$ to be e (just for convenience), we can write

$$(a+b)(c+d) = (a+b)e = ae + be = a(c+d) + b(c+d). \tag{3.12}$$

The first equality here involves writing $c+d$ as e, and the last equality involves the reverse: writing e as $c+d$. The middle equality is the distributive law.

Having used the distributive law once in Eq. (3.12), we'll now use it again (twice) to expand the two $c+d$ terms on the righthand side. This gives

$$\begin{aligned} (a+b)(c+d) &= a(c+d) + b(c+d) \\ &= (ac+ad) + (bc+bd) \\ \Longrightarrow \quad & \boxed{(a+b)(c+d) = ac + ad + bc + bd} \end{aligned} \tag{3.13}$$

We included the parentheses in the second line for clarity. But since they're not necessary, we dropped them in the third line. As a quick check on this result, if all of the letters are equal to 1, then the lefthand side is $(1+1)(1+1)$ which equals 4, and the righthand side is $1+1+1+1$ which also equals 4.

The result on the righthand side of Eq. (3.13) is called the *expansion* of the product $(a+b)(c+d)$. In the original expression $(a+b)(c+d)$, each term gets multiplied by the two terms in the *other* parentheses, but not the other term in its *own* parentheses. That is, the products ab and cd do *not* appear in Eq. (3.13). So there are four terms in the expansion, not six.

Although the previous paragraph is technically all you need to remember when multiplying two binomials together, it is helpful to use the systematic "FOIL" method, which is an acronym for "First, Outside, Inside, Last." As shown in Fig. 3.1, the four terms on the righthand side of Eq. (3.13) can be described as:

ac: First terms in the two binomials
ad: Outside terms
bc: Inside terms
bd: Last terms

$$(a+b)(c+d) \;=\; \overset{\text{F}}{ac} + \overset{\text{O}}{ad} + \overset{\text{I}}{bc} + \overset{\text{L}}{bd}$$

Figure 3.1: FOIL, which is an acronym for "First, Outside, Inside, Last," tells you how to multiply two binomials.

FOIL is the mantra you want to have running through your head whenever you multiply two binomials together. You'll be saying the words "first, outside, inside, last" to yourself many times throughout this book! In the end, FOIL is nothing more than a double application of the distributive law used on the two different "levels" in Eqs. (3.12) and (3.13). But the FOIL acronym makes it easier to be systematic.

There's really no need to recoil
From double distributive toil.
Don't rip out your hairs,
Just do products in pairs.
It's as simple as spelling out FOIL!

There is a simple geometric interpretation of the four FOIL terms. Let's say we want to calculate the product $(2+5)(3+8)$, which we know equals $7 \cdot 11 = 77$. The four FOIL terms in the expansion in Eq. (3.13) are the numbers of little boxes in the four sub-rectangles in Fig. 3.2. The upper left rectangle has $ac = 2 \cdot 3 = 6$ boxes, the upper right has $ad = 2 \cdot 8 = 16$ boxes, the lower left has $bc = 5 \cdot 3 = 15$ boxes, and the lower right has $bd = 5 \cdot 8 = 40$ boxes. So the total number of boxes is $6 + 16 + 15 + 40 = 77$, as it should be.

The two sides of Eq. (3.13) therefore represent two different ways of counting the total number of little boxes. The lefthand side counts them all at once (as in $7 \cdot 11 = 77$), while the righthand side counts them via the four sub-rectangles.

The part of Fig. 3.2 that is above the internal bold horizontal line is the visual representation of the distributive statement $a(c + d) = ac + ad$, which in the present case is $2(3 + 8) = 2 \cdot 3 + 2 \cdot 8$. If we replaced the boxes with dots, we would have a figure similar to Fig. 2.7, but with different numbers (2 rows instead of 3, etc.). Likewise, the part of Fig. 3.2 that is *below* the bold horizontal line represents the distributive statement $b(c+d) = bc+bd$.

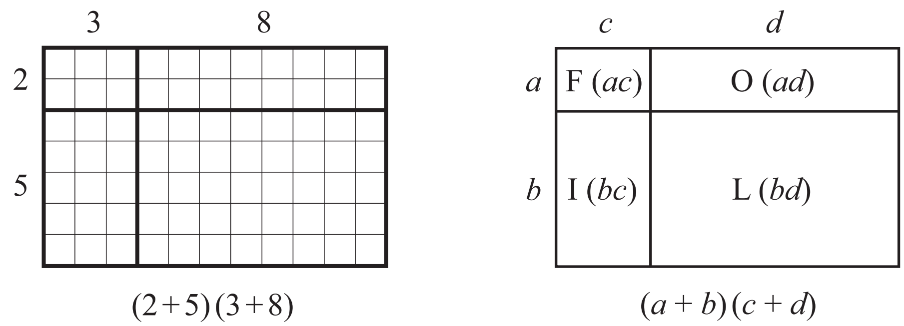

Figure 3.2: Counting the number of little boxes in the four sub-rectangles provides a geometric interpretation of FOIL.

In the same manner, the part to the *left* of the *vertical* bold line represents the distributive statement $(a+b)c = ac + bc$. And the part to the right of the vertical bold line represents the distributive statement $(a+b)d = ad + bd$.

Remarks:

1. In Eq. (3.12) (as in Eq. (2.19) earlier) we defined $c+d$ as e for convenience. But there technically wasn't any need to do this. You can simply think of $c+d$ as a "thing" and use the fact that $(a+b)$ times a thing equals a times the thing plus b times the thing. This brings you directly from the first expression in Eq. (3.12) to the last.

   However, although technically not necessary, defining a group of terms to be a single letter can often save you a great deal of work. The above case involving $c+d$ wasn't too complicated, but sooner or later you'll encounter much messier expressions where it is highly advantageous to temporarily sweep the mess under the rug and replace it with a single letter. The distributive law holds for *any* values of the letters, so we're free to let them be whatever we want. Each one can be a huge expression involving a hundred terms and all sorts of crazy operations – that's completely fine.

2. A common error when using FOIL is not distributing minus signs properly. Consider these three equal expressions:

$$-(a+b)(c+d) = (-a-b)(c+d) = (a+b)(-c-d). \qquad (3.14)$$

   The second expression is obtained from the first by distributing the minus sign (which we know from Eq. (1.28) is the same as multiplying by -1) into the $(a+b)$ factor to reverse its signs and turn it into $(-a-b)$; the $(c+d)$ factor is unchanged. Likewise, the last expression is obtained from the first by distributing the minus sign into the $(c+d)$ factor to reverse its signs and turn it into $(-c-d)$; the $(a+b)$ factor is unchanged. Note that we do *not* distribute the minus sign into *both* of the $(a+b)$ and $(c+d)$ factors. That would be the same as multiplying through by *two* factors of -1 (which would be the same as doing nothing, since $(-1)(-1) = 1$). But we have only one minus sign out front.

   Remember that when distributing a minus sign into a binomial, you need to distribute it to *both* of the terms inside the parentheses. That is, $-(a+b)$ equals $(-a-b)$ and *not* $(-a+b)$.

3. Eq. (3.14) is a special case of the more general statement obtained by replacing the minus sign out front (which is the same as a factor of -1) with a general factor of e:

$$e(a+b)(c+d) = (ea+eb)(c+d) = (a+b)(ec+ed). \tag{3.15}$$

These three expressions are all equal, but an expression that is *not* equal to them is $(ea+eb)(ec+ed)$. This contains one too many factors of e; it equals

$$e(a+b)\cdot e(c+d) = e^2(a+b)(c+d). \tag{3.16}$$

The single e out front in Eq. (3.15) is distributed to one factor *or* the other (it doesn't matter which), but not both! You should check this by plugging in some arbitrary numbers for the various letters. ♣

Example 3.3 Expand and simplify these products: $(a-3b)(5a+2b)$ and $(2n-6)(5n-3)$.

Solution Applying FOIL to the first product gives

$$\begin{aligned}(a-3b)(5a+2b) &= a(5a) + a(2b) - (3b)(5a) - (3b)(2b)\\ &= 5a^2 + 2ab - 15ab - 6b^2\\ &= 5a^2 - 13ab - 6b^2.\end{aligned} \tag{3.17}$$

The two ab-type terms are like terms, so we combined them.

If you want, you can use dots (like $a \cdot 5a$, etc.) instead of parentheses to represent the multiplications in the first line above; whichever you prefer. Actually, you don't even need any dots or parentheses because, for example, the $(3b)(5a)$ term can be written as $3b5a$. This means to multiply the four things that appear. However, it looks a little strange. People generally don't like to write numbers internally in a string of symbols, without a dot or parentheses.

You can alternatively think of the above $(a-3b)$ factor as $\big(a+(-3b)\big)$. The FOIL steps then look like

$$\begin{aligned}\big(a+(-3b)\big)(5a+2b) &= a(5a) + a(2b) + (-3b)(5a) + (-3b)(2b)\\ &= 5a^2 + 2ab + (-15ab) + (-6b^2)\\ &= 5a^2 - 13ab - 6b^2,\end{aligned} \tag{3.18}$$

in agreement with Eq. (3.17).

Applying FOIL to the second product gives

$$\begin{aligned}(2n-6)(5n-3) &= (2n)(5n) - (2n)\cdot 3 - 6(5n) + (-1)^2 6\cdot 3\\ &= 10n^2 - 6n - 30n + 18\\ &= 10n^2 - 36n + 18.\end{aligned} \tag{3.19}$$

Again you should verify that you obtain the same result if you write the initial binomials as $\big(2n + (-6)\big)$ and $\big(5n + (-3)\big)$.

For each of the above products, you can plug in some simple numbers for the letter(s), to check that the final result agrees with the initial expression.

Exercise 3.2 In Eq. (3.12) we could just as well have defined e as $a + b$ instead of $c + d$. Reproduce the result in Eq. (3.13) for the product $(a + b)(c + d)$ by defining $e \equiv a + b$.

Exercise 3.3 Expand and simplify these products: $(-4a-b)(7a-5b)$ and $(n-3)(n-5)$. For the second one, check your result for $n = 3$ and $n = 5$.

3.3 Three special cases

Let's now look at three special cases of FOIL that come up frequently.

Case 1: $(a + b)^2$

If c and d are equal to a and b, respectively, then the product $(a+b)(c+d)$ equals $(a+b)(a+b)$, which can be written more concisely as $(a + b)^2$. Applying FOIL gives

$$
\begin{aligned}
& (a + b)^2 = (a + b)(a + b) = a^2 + ab + ba + b^2 \\
\Longrightarrow \quad & \boxed{(a + b)^2 = a^2 + 2ab + b^2} \qquad (3.20)
\end{aligned}
$$

We have used the fact that ab and ba are like terms (since multiplication is commutative), which allowed us to combine (add) them together to make $2ab$. The resulting terms (a^2, $2ab$, and b^2) are all of different types since they have different powers of the letters, so the result can't be simplified any further.

As a quick check on Eq. (3.20), if $a = b = 1$ then we obtain $(1+1)^2 = 1+2+1 \implies 2^2 = 4$, which is true. And if $a = 1$ and $b = 0$ (or vice versa), we correctly obtain $1 = 1$. You should check the result for some other choices of numbers too.

For Eq. (3.20), the rectangle on the right in Fig. 3.2 now takes the form of the square shown in Fig. 3.3(a), which has two sub-squares and two sub-rectangles. From Section A.4.1 in Appendix A, the area of a rectangle is the product of the two side lengths, that is, the base times the height. (This statement is basically equivalent to counting the little squares in Fig. 3.2.) A square is a rectangle with equal sides, so the area of a square equals the square of the (single) side length. This is why we call a^2 the "square of a." It's the area of a square with side length a.

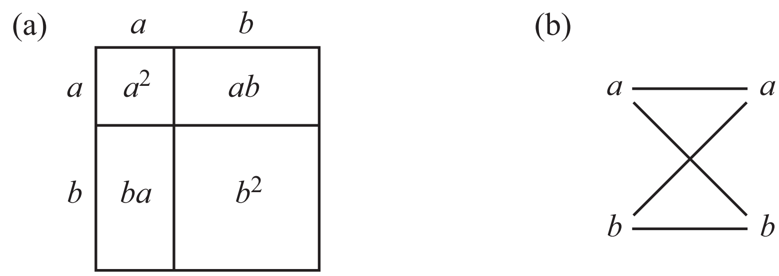

Figure 3.3: (a) Eq. (3.20) is the statement that the area of the overall square equals the sum of the areas of the two sub-squares and two sub-rectangles. a^2 and b^2 are called "diagonal" terms, and the ab's are called "off-diagonal" terms. (b) The lines represent the four product pairs in Eq. (3.20). The ab's are also called "cross" terms, because the lines connecting a and b make a cross.

Eq. (3.20) is then simply the geometric statement that in Fig. 3.3(a), the area of the overall square, which is $(a+b)^2$ since the overall side length is $a+b$, correctly equals the sum of the areas of the two sub-squares (a^2 and b^2) plus the two rectangles (each ab).

Because the two a^2 and b^2 sub-squares in Fig. 3.3(a) lie along the main diagonal line (upper left to lower right) of the overall square, the a^2 and b^2 terms in Eq. (3.20) are called the *diagonal* terms in the expansion of $(a+b)^2$. The two ab rectangle terms are called the *off-diagonal* or *cross* terms. The cross-term terminology comes from Fig. 3.3(b), where the lines indicate which pairs get multiplied. The ab and ba lines make a cross, or equivalently they travel a*cross* the figure, from one height to another. (Of course, these lines are also diagonal, but ignore that fact!)

Case 2: $(a-b)^2$

Similar to Eq. (3.20), we also have

$$(a-b)^2 = (a-b)(a-b) = a^2 - ab - ba + b^2$$
$$\Longrightarrow \quad \boxed{(a-b)^2 = a^2 - 2ab + b^2} \qquad (3.21)$$

Note that the last term here is $+b^2$ and not $-b^2$ (which is a common error), because the two minus signs in front of the b's cancel when multiplied. If you get confused about the sign, you can write $(a-b)$ as $(a+(-b))$ and then use the result from Eq. (3.20) with b replaced by $-b$. This gives

$$(a-b)^2 = (a+(-b))^2 = a^2 + 2a(-b) + (-b)^2 = a^2 - 2ab + b^2, \qquad (3.22)$$

where we have used $(-b)^2 = b^2$ since $(-1)(-1) = 1$.

As a quick check on our result, if $a = b = 1$ then we obtain $(1-1)^2 = 1-2+1 \Longrightarrow 0 = 0$, which is true. And if $a = 1$ and $b = 0$ (or vice versa), we correctly obtain $1 = 1$.

Just as with Eq. (3.20) and Fig. 3.3(a), there is a geometric interpretation of Eq. (3.21) in terms of areas. The relevant diagram is shown in Fig. 3.4. You should pause and try to think of the interpretation before reading any further. It requires more thinking than Fig. 3.3(a) did!

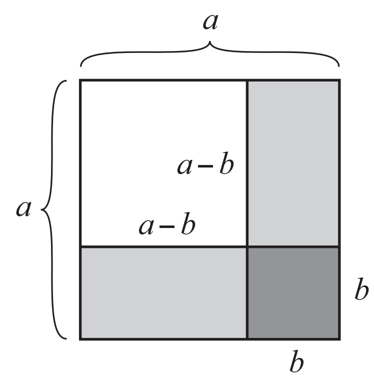

Figure 3.4: The setup for interpreting Eq. (3.21) in terms of areas.

The interpretation is as follows. The lefthand side of Eq. (3.21) is the area of the white square in Fig. 3.4 with side length $a - b$. But there is another way to obtain this $(a - b)^2$ area: The area of the white square can also be obtained by starting with the overall square (with area a^2; this is the first term on the righthand side of Eq. (3.21)) and then subtracting off the bottom a-by-b rectangle (which spans the entire width a), and then also subtracting off the right a-by-b rectangle (which spans the entire height a). These subtractions give the $-2ab$ term in Eq. (3.21).

However, we have subtracted off *too much*. We have *twice* subtracted the dark b-by-b square in the lower righthand corner, because it is part of *both* of the a-by-b rectangles. We must therefore add it back in once, so that we will have subtracted it off a net time of only once. This is the $+b^2$ term in Eq. (3.21). The summary of this process is shown in Fig. 3.5.

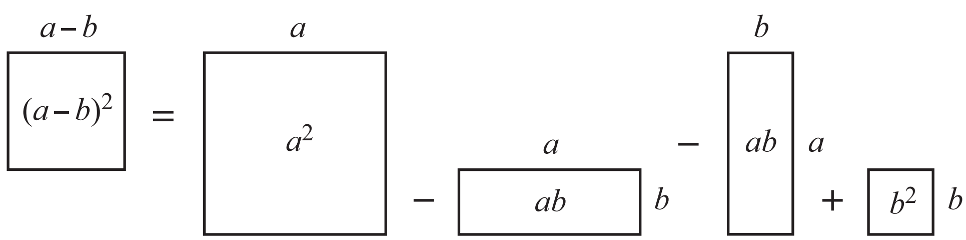

Figure 3.5: How to think about the areas in Fig. 3.4 when interpreting Eq. (3.21).

Having covered the two special cases in Eqs. (3.20) and (3.21), you now have the expertise to solve Exercises A.8 and A.9 in Section A.6.3. These exercises present two nice proofs of the Pythagorean theorem. You'll need to use the fact that the area of a triangle is half the base times the height.

Exercise 3.4 Using the fact that the area of a rectangle is the base times the height, verify that the sum of the areas of the four rectangles and the small square in Fig. 3.6 correctly equals the area of the overall square.

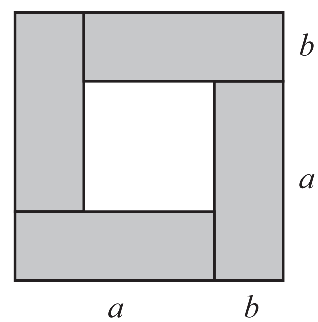

Figure 3.6: The sum of the areas of the five sub-regions equals the area of the overall square.

Case 3: $(a + b)(a - b)$

In Eq. (3.20) both binomials involved a "+," while in Eq. (3.21) they both involved a "−." What if we have one of each sign? That is, what does the product $(a + b)(a - b)$ equal? Applying FOIL gives

$$\begin{aligned} &(a + b)(a - b) = a^2 - \cancel{ab} + \cancel{ba} - b^2 \\ \Longrightarrow\quad &\boxed{(a + b)(a - b) = a^2 - b^2} \end{aligned} \tag{3.23}$$

The cross terms cancel, so we end up with the difference of two squares. You can check this result for various pairs of numbers.

Writing Eq. (3.23) in the reverse direction, we see that the difference of two squares, $a^2 - b^2$, can be *factored* (that is, written as the product of two or more things; we'll talk about factoring in Section 4.2) like this:

$$\boxed{a^2 - b^2 = (a + b)(a - b)} \tag{3.24}$$

You will find use for this relation (in either direction) many times when doing algebra. We'll also give some applications below for actual numbers. But first, you can think about the geometric interpretation of Eq. (3.24) in the following exercise.

Exercise 3.5 In Fig. 3.7 the overall square has side length a, and the sub-square in the lower-right corner has side length b. The area of the shaded region is the difference in the areas of the two squares, so it equals $a^2 - b^2$. Explain how to rearrange a piece of the figure in a way that makes it clear why the shaded area also equals the area $(a+b)(a-b)$ of a rectangle with side lengths $a+b$ and $a-b$, which is what Eq. (3.24) says.

Eqs. (3.20), (3.21), and (3.23)/(3.24) are examples of *identities*. An identity is an equation that is true no matter what values you plug in for the various letters. The two sides of the

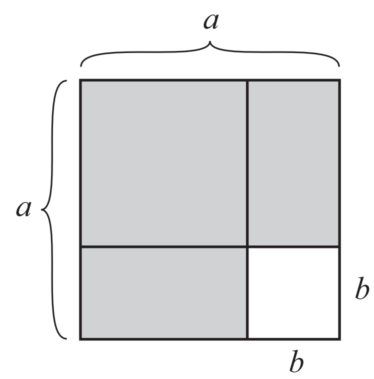

Figure 3.7: The setup for interpreting Eq. (3.24) in terms of areas.

equation are simply different ways of writing the same thing, so they are always equal. Another (very simple) example of an identity is the equation $a + a = 2a$. An even simpler identity is $1 + 1 = 2$, which involves only numbers.

An example of an equation that is *not* an identity is $3x + 1 = 7$. This equation is true only for the special value of $x = 2$. We'll learn how to solve equations like this (along with more difficult ones) in Chapter 5.

Most equations in this chapter are identities, because we're generally just rewriting expressions in different forms. Some exceptions are the $3x + 1 = 7$ equation in the preceding paragraph, the $2a^2b + 5ab = 9ab$ equation (when $a = 2$) on page 130, and simple equations of the $a = 2$, $n = 3$, etc., form.

Rationalizing the denominator

At the end of Section 2.2.5, we discussed "rationalizing the denominator." Let's now look at more complicated fractions (with two numbers in the denominator), for which Eq. (3.23) is the key to rationalizing the denominator.

Consider the fraction $5/(\sqrt{3} - 1)$. As in Eq. (2.61), we'll multiply by 1 in the form of c/c, with a wisely chosen value of c that leads to a rational number in the denominator:

$$\frac{5}{\sqrt{3}-1} = \frac{5}{\sqrt{3}-1} \cdot \frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{5(\sqrt{3}+1)}{(\sqrt{3})^2 - 1^2} = \frac{5\sqrt{3}+5}{3-1} = \frac{5\sqrt{3}+5}{2}. \tag{3.25}$$

We chose our value of c to be $\sqrt{3}+1$ because this made the overall denominator take the form of $(a-b)(a+b)$, with $a = \sqrt{3}$ and $b = 1$. And we know from Eq. (3.23) that such a product equals the difference of squares, $a^2 - b^2$. The a^2 part of this turns the $\sqrt{3}$ into the nice number 3, so we have successfully made the square root disappear from the denominator. You should verify with a calculator that the last fraction in Eq. (3.25) is indeed equal to the first.

If the given fraction had instead been $5/(\sqrt{3} + 1)$, then we would have chosen c to be $\sqrt{3} - 1$, because this would have again led to a difference of squares.

In Eq. (3.25) technically all we did was trade a square root in the denominator for one in the numerator. But as mentioned in Section 2.2.5, rationalizing the denominator is helpful because it's easier to imagine dividing some pies into a clean number of pieces each (2 here) and taking a messy number of them ($5\sqrt{3} + 5$), than to imagine dividing some pies into a messy number of pieces each ($\sqrt{3} - 1$) and taking a clean number of them (5).

Quick multiplication

Eq. (3.23) can be used in a sneaky way to quickly multiply certain pairs of numbers. Let's say someone asks you and a friend to multiply 28 and 32. You figure it will take too long to multiply out by hand, so you reach for your calculator. However, before your hand even touches it, your friend exclaims, "896!" which is in fact the right answer. How did they calculate the product so quickly?

Well, 28 is 2 less than 30, and 32 is 2 more than 30, so Eq. (3.23) gives

$$28 \cdot 32 = (30 - 2)(30 + 2) = 30^2 - 2^2 = 900 - 4 = 896. \tag{3.26}$$

The quickness of this method relies on the fact that the squares of 30 and 2 are both easy to calculate. (Since $30 = 3 \cdot 10$, just square the 3 to obtain 9, and then tack on two zeros for the $10^2 = 100$.) In the same spirit, we can quickly calculate the product $47 \cdot 53$:

$$47 \cdot 53 = (50 - 3)(50 + 3) = 50^2 - 3^2 = 2500 - 9 = 2491. \tag{3.27}$$

The "a" in these examples (the 30 or 50) is the number halfway between the two numbers you're multiplying (in other words, the average). And the "b" (the 2 or 3) is how far each of the given numbers is from the average, a.

The above strategy is speedy if you're able to do three things: (1) quickly determine a and b, (2) quickly calculate a^2 and b^2, and (3) quickly find the difference $a^2 - b^2$. The strategy isn't very useful with, say, $51 \cdot 55$, because the average is $a = 53$, and most people don't know what 53^2 is off the top of their head. Likewise for the non-integer average (namely 22.5) for the product $21 \cdot 24$. But occasionally in everyday life you'll encounter a product for which the above three conditions hold. You can then make things easy on yourself by replacing the potentially time-consuming product with a quick difference-of-squares calculation like the ones in Eqs. (3.26) and (3.27).

> If a product produces blank stares,
> Or leads you to offer up prayers,
> I might then suggest
> That it's better expressed
> As a friendly ol' difference of squares.

Example 3.4 As with Eq. (3.23), the result in Eq. (3.20) can also be used to perform quick mental math. Use Eq. (3.20) to square 41 in your head. Likewise for 20.5.

Solution With $a = 40$ and $b = 1$, Eq. (3.20) gives

$$(40 + 1)^2 = 40^2 + 2 \cdot 40 \cdot 1 + 1^2 = 1600 + 80 + 1 = 1681. \tag{3.28}$$

These calculations can easily be done in your head. Eq. (3.20) tells you to square a and b (which quickly gives 1600 and 1), and then add on twice their product (which quickly gives the cross term of 80).

For 20.5, we have $a = 20$ and $b = 0.5$. The squares of 20 and 0.5 are 400 and 0.25. (The square of $1/2$ is $1/4$, if you want to think in terms of fractions instead of decimals.) And the cross term is twice the product, which is $2(20)(0.5) = 20$. The sum of everything is then 420.25.

Example 3.5 Here's a nice example that makes use of the results for all three of the above cases: Eqs. (3.20), (3.21), and (3.23). The task will be to simplify $\big((a + b) + (a - b)\big)^2$ the hard way. The *easy* way is to note that within the larger parentheses, the b's cancel, which leaves us with just $(2a)^2 = 2^2a^2 = 4a^2$. So that's the answer. Let's now derive this the long way, which will be tedious but instructive.

Solution Our strategy will be to use the fact that $\big((a + b) + (a - b)\big)^2$ takes the same form as the lefthand side of Eq. (3.20), because it is the square of the sum of two things. In Eq. (3.20) those two things were a and b, but now they are $(a + b)$ and $(a - b)$. So in the result in Eq. (3.20), we just need to replace "a" with "$a + b$", and "b" with "$a - b$." If you don't like the double usage of the a and b letters here, you can rewrite the result in Eq. (3.20) with different letters, like $(x + y)^2 = x^2 + 2xy + y^2$, and then replace x with $a + b$, and y with $a - b$.

The $a \longrightarrow a + b$ and $b \longrightarrow a - b$ substitutions in Eq. (3.20) yield

$$\big((a + b) + (a - b)\big)^2 = (a + b)^2 + 2(a + b)(a - b) + (a - b)^2. \tag{3.29}$$

We now note that the three terms on the righthand side are exactly the three cases we discussed above. So we can use the results in Eqs. (3.20), (3.21), and (3.23) to expand the products. Collecting like terms and simplifying gives

$$\begin{aligned}\big((a + b) + (a - b)\big)^2 &= (a^2 + \cancel{2ab} + b^2) + 2(a^2 - b^2) + (a^2 - \cancel{2ab} + b^2) \\ &= (a^2 + 2a^2 + a^2) + \cancel{(b^2 - 2b^2 + b^2)} \\ &= 4a^2 + 0, \end{aligned} \tag{3.30}$$

as desired. The first line in the above equation is rather long and might initially seem disheartening. However, this example is a good lesson in pressing forward and trusting that things often (although certainly not always!) simplify when doing algebra. The ab-type terms cancel right away, and then when we collect the a^2-type and b^2-type terms, we see that the b^2's cancel too. So we're left with only the a^2 terms (4 of them, in agreement with our earlier quick method of simplification).

Remark: The parentheses in the second line of Eq. (3.30) aren't necessary, but we included them for clarity. However, if you write the second line in the following form, then the parentheses definitely *are* necessary:

$$a^2(1+2+1) + \cancel{b^2(1-2+1)}. \tag{3.31}$$

Distributing the a^2 and b^2 inside the parentheses here shows that this expression is indeed equal to the second line in Eq. (3.30). Eq. (3.31) is obtained by saying to yourself, "How many a^2's are there in the first line of Eq. (3.30)? Well, there's 1 of them there, and then 2 there, and then 1 more over there." You then just write $1+2+1$ in the parentheses. Likewise for the b^2. In obtaining Eq. (3.31), we technically did some *factoring* (see Section 4.2), although there's no need to be familiar with that term yet. ♣

Exercise 3.6 Use Eq. (3.23) to multiply 66 and 74 in your head. And use Eq. (3.20) to square 53.

Exercise 3.7 Rationalize the denominator of the fraction $6/(\sqrt{7}+2)$.

Exercise 3.8 Simplify the following expression by collecting like terms:

$$a(a+b) - b(a+c) - (a-c)^2$$

Exercise 3.9 Expand and simplify $(4a+6b)^2$.

Exercise 3.10 Simplify $(a+b)^2 - (a-b)^2$ in two ways:

(a) Expand each of the squares, and then collect like terms.

(b) Observe that the given expression is the difference of squares, and use Eq. (3.24) with the appropriate substitutions.

Exercise 3.11 In the spirit of Example 3.5, simplify $\big((1+a)-a\big)^2$ the hard way. The easy way is to note that within the larger parentheses, the a's cancel, so we simply have $(1)^2$ which is just 1. Reproduce this answer of 1 by using the result in Eq. (3.21) for the square of the difference of two things (which are $1+a$ and a here).

3.4 Trinomials, etc.

Let's now look at some ways in which we can extend the FOIL method to more complicated expressions. In Eq. (3.13) we multiplied two binomials together. That is, each set of parentheses contained two terms. But what if one set contains 3 terms, for example $c+d+e$? This is called, quite reasonably, a *trinomial*. Or how about 4 terms? Or 10? Or what if *both* sets of parentheses contain 3 terms? Or 4? How do we go about multiplying the two sets?

The answer involves some good news and some bad news. The good news is that there's nothing new we need to do. It's the same old applications of the distributive law, but simply more of them. In the end, we just need to multiply every term in one set of the parentheses by every term in the *other* set, as we did in Eq. (3.13).

The bad news is that if you have many terms in each set of parentheses, it will take a while to do all of the multiplications. There's really no shortcut. If you have 4 terms in one set and 5 in the other, then there will be $4 \cdot 5 = 20$ terms in the answer. Some of these may be like others, in which case you can combine them.

Let's do the 2-by-3 case, $(a+b)(c+d+e)$, to see how this works. In the same manner as in Eq. (3.12), we can define the sum $c+d+e$ to be f, for convenience. We then have

$$(a+b)(c+d+e) = (a+b)f = af + bf = a(c+d+e) + b(c+d+e). \tag{3.32}$$

The first equality here is our definition of f, the second is the distributive law, and the third is again our definition of f. Using the distributive law on the two remaining sets of parentheses, in the same manner as in Eq. (3.13), yields six terms now instead of four:

$$\begin{aligned}(a+b)(c+d+e) &= a(c+d+e) + b(c+d+e) \\ &= ac + ad + ae + bc + bd + be.\end{aligned} \tag{3.33}$$

This isn't the prettiest result, but you should double check that each term is multiplied by all of the terms in the other set of parentheses, and by none of the terms in its own set. That is, the terms ab, cd, ce, and de do *not* appear. The rectangle associated with Eq. (3.33), analogous to the rectangle on the right side of Fig. 3.2, is shown in Fig. 3.8. The two sides of Eq. (3.33) represent two different ways of calculating the area of the overall rectangle.

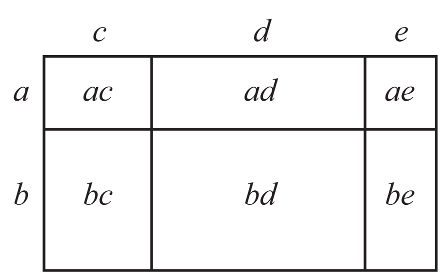

Figure 3.8: The interpretation of Eq. (3.33) in terms of areas.

No matter how many terms you have in each of the two sets of parentheses, you'll always end up with a rectangle of this sort. If there are 4 terms in the first set and 7 in the second, then you'll have a table with 4 rows and 7 columns, and hence a total of $4 \cdot 7 = 28$ entries. And very importantly:

The structure of the rectangle ensures that each term is multiplied by all of the terms in the *other* set of parentheses, and by none of the terms in its *own* set.

What if we had defined f to be $a + b$, instead of $c + d + e$, in Eq. (3.32)? We would then have

$$\begin{aligned}(a+b)(c+d+e) = f(c+d+e) &= fc + fd + fe \\ &= (a+b)c + (a+b)d + (a+b)e.\end{aligned} \tag{3.34}$$

Using the distributive law on the three remaining sets of parentheses yields six terms:

$$(a+b)(c+d+e) = ac + bc + ad + bd + ae + be. \tag{3.35}$$

We obtain the same terms as in Eq. (3.33) (as we must), but in a different order. Eq. (3.33) groups the terms according to the three a terms, and then the three b terms (corresponding to the two rows in Fig. 3.8). In contrast, Eq. (3.35) groups them according to the two c terms, then the two d terms, then the two e terms (corresponding to the three columns in Fig. 3.8).

You can get some practice with extensions of the FOIL method in the examples and exercises below. No matter how you want to look at it, when you multiply two sets of parentheses with any number of terms inside, it all comes down to a series of applications of the distributive law. The applications come in two stages, as we saw in Eqs. (3.32) and (3.33), and also in Eqs. (3.34) and (3.35). The applications can be visualized in terms of a rectangle like the one in Fig. 3.8.

There's no need to feel apprehensions
Or conjure up fancy inventions.
Just use, as we saw,
The distributive law,
And you'll conquer these FOIL extensions.

Example 3.6 In Eq. (3.20) we calculated the square of a binomial. Let's now calculate the square of a trinomial: $(a + b + c)^2$.

Solution To do this, we'll use the distributive law for three terms: $(a + b + c)d = ad + db + cd$. This relation is true no matter what the letter d is. In particular, if we let

d equal $a + b + c$, we obtain

$$\begin{aligned}(a+b+c)^2 &= (a+b+c)\overbrace{(a+b+c)}^{d}\\ &= ad + bd + cd\\ &= a(a+b+c) + b(a+b+c) + c(a+b+c)\\ &= (a^2+ab+ac) + (ba+b^2+bc) + (ca+cb+c^2)\end{aligned}$$

$$\Longrightarrow \quad \boxed{(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc} \qquad (3.36)$$

We used the commutative law to say that $ab = ba$, etc., and we collected like terms. As a quick check on our answer, if $a = b = c = 1$ then Eq. (3.36) gives $(1+1+1)^2 = 1+1+1+2+2+2 \implies 3^2 = 9$, which is correct.

Analogous to the square in Fig. 3.3(a), the square in the present case is shown in Fig. 3.9(a). Note the symmetry with respect to the main diagonal (the diagonal from upper left to lower right). If you fold the paper along the main diagonal, the corresponding rectangles (indicated by the same shading, such as the ab and ba ones) will end up on top of each other.

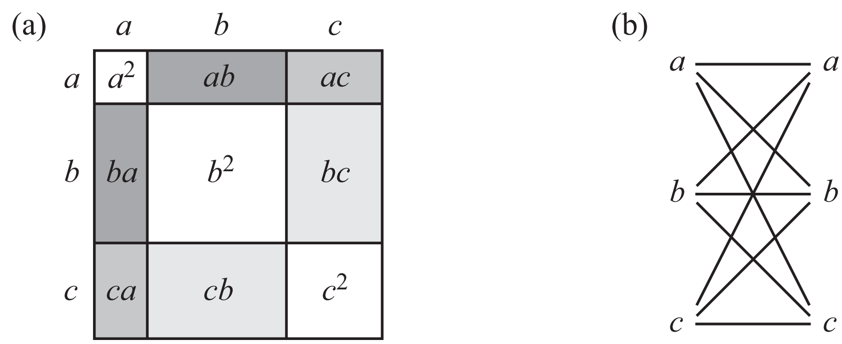

Figure 3.9: (a) The interpretation of Eq. (3.36) in terms of areas. (b) The lines represent the nine product pairs in Eq. (3.36).

As with the $(a+b)^2$ case, the a^2, b^2, and c^2 terms in the last line of Eq. (3.36) are called the "diagonal" terms, and the ab, ac, and bc terms are called the "cross" terms. The nine different products in the fourth line of Eq. (3.36) are represented by the nine lines in Fig. 3.9(b) (compared with the four lines in Fig. 3.3(b)).

Example 3.7 Expand and simplify the following two products:

$$(a-b)(a^2+ab+b^2) \qquad \text{and} \qquad (a+b)(a^2-ab+b^2)$$

Solution Using the distributive law in the same manner as in Eq. (3.32) yields

$$\begin{aligned}(a-b)(a^2+ab+b^2) &= a(a^2+ab+b^2) - b(a^2+ab+b^2)\\ &= a^3 + \cancel{a^2b} + \cancel{ab^2}\\ &\qquad - \cancel{a^2b} - \cancel{ab^2} - b^3\\ &= a^3 - b^3.\end{aligned} \tag{3.37}$$

We have vertically aligned the corresponding (like) terms in the second and third lines to make it clear which terms cancel. (You should try to align the corresponding terms whenever possible. This makes it much easier to collect like terms.) As a quick check on the result, if $a = b$ then both sides are zero.

Similarly,

$$\begin{aligned}(a+b)(a^2-ab+b^2) &= a(a^2-ab+b^2) + b(a^2-ab+b^2)\\ &= a^3 - \cancel{a^2b} + \cancel{ab^2}\\ &\qquad + \cancel{a^2b} - \cancel{ab^2} + b^3\\ &= a^3 + b^3.\end{aligned} \tag{3.38}$$

Again we have vertically aligned the corresponding terms. As a quick check on this result, if $a = -b$ then both sides are zero, because the $(a+b)$ factor on the left is zero, and on the right we have

$$a^3 + b^3 = (-b)^3 + b^3 = (-1)^3b^3 + b^3 = -b^3 + b^3 = 0, \tag{3.39}$$

since $(-1)^3 = -1$.

Exercise 3.12 Evaluate the product $(5-2)(1-3+6)$ by using the distributive law in the manner of Eq. (3.32). Then check your result by doing things the quicker way by first adding/subtracting the terms in each set of parentheses.

Exercise 3.13 Expand $(a-2b+5c)^2$ in two ways:

(a) Use the distributive law and repeat the steps in Eq. (3.36).

(b) Invoke the final result in Eq. (3.36), with b replaced with $-2b$, and c replaced with $5c$ (because $a-2b+5c$ means the same thing as $a+(-2b)+(5c)$).

Exercise 3.14 Calculate the square of a quadrinomial: $(a+b+c+d)^2$. You can either use the distributive law as we did in Eq. (3.36), or draw (or just imagine drawing) a figure like Fig. 3.9(a).

Exercise 3.15 Expand and simplify the following two products:

$$(a-b)(a^3+a^2b+ab^2+b^3) \qquad \text{and} \qquad (a+b)(a^3-a^2b+ab^2-b^3)$$

Exercise 3.16

(a) Expand and simplify $(a+b)^2(a-b)^2$ by expanding the squares and then multiplying the resulting trinomials.

(b) Expand the product again, in a quicker manner by writing it as $[(a+b)(a-b)]^2$ and making use of Eq. (3.23).

3.5 Cubes, etc.

Using the FOIL method, we found in Eq. (3.20) that $(a+b)^2 = a^2+2ab+b^2$. What if we cube $a+b$ instead of squaring it? That is, what does $(a+b)^3$ equal? We can expand this cube in two steps. We have three factors of $a+b$, so we'll first multiply two of them together, and then we'll multiply the result by the third $a+b$. Using what we know about the square of $a+b$ in Eq. (3.20), the distributive law gives (in a manner similar to Eq. (3.38))

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)^2\\ &= (a+b)(a^2+2ab+b^2)\\ &= a(a^2+2ab+b^2)+b(a^2+2ab+b^2)\\ &= a^3+2a^2b+ab^2\\ &\qquad\quad +a^2b+2ab^2+b^3\\ \Longrightarrow \quad & \boxed{(a+b)^3 = a^3+3a^2b+3ab^2+b^3}\end{aligned} \tag{3.40}$$

We have vertically aligned the corresponding terms in the fourth and fifth lines to make it clear how many terms of each kind we have. As a quick check on our answer, if $a=b=1$ then Eq. (3.40) gives $(1+1)^3 = 1+3+3+1 \implies 2^3 = 8$, which is correct.

The strategy we used in Eq. (3.40) boils down to many applications of the distributive law (since that's what FOIL can be traced to). This strategy works just as well when the three binomials aren't the same, that is, when we don't have a cube. For example, if we have the product $(a+b)(c+d)(e+f)$, we can use FOIL on the right two binomials, and then multiply the result by $a+b$. The result is (as you can verify)

$$\begin{aligned}(a+b)(c+d)(e+f) &= (a+b)(ce+cf+de+df)\\ &= (ace+acf+ade+adf)+(bce+bcf+bde+bdf).\end{aligned} \tag{3.41}$$

The parentheses in this answer aren't necessary, but we have included them for clarity. As a check on our answer, if all six letters are equal to 1, then the lefthand side of Eq. (3.41) equals $2\cdot 2\cdot 2 = 8$. And the righthand side also equals 8 since there are 8 terms, all of which are 1.

In Eq. (3.41) we multiplied the right two binomials, $c + d$ and $e + f$, first. But you can just as well start by multiplying the left two instead (or the left one and the right one). You should verify that this gives the same answer, but with the final terms in a different order.

In Eq. (3.40) we raised an expression with 2 terms (the binomial $a + b$) to the 3rd power. We'll look at higher powers of $a + b$ in Section 3.6. In Eq. (3.36) we raised an expression with 3 terms (the trinomial $a + b + c$) to the 2nd power. We can also raise trinomials (and quadrinomials, etc.) to higher powers. Things eventually get unwieldy due to the large number of terms, but you can calculate the cube of $a + b + c$ in Exercise 3.20.

Example 3.8 Expand $(a - b)^3$ in the same manner as in Eq. (3.40). Then verify that you obtain the same result by simply replacing b with $-b$ in the last line of Eq. (3.40).

Solution Since $(a - b)^2 = a^2 - 2ab + b^2$, we have

$$\begin{aligned}(a-b)^3 &= (a-b)(a^2-2ab+b^2)\\ &= a(a^2-2ab+b^2) - b(a^2-2ab+b^2)\\ &= a^3 - 2a^2b + ab^2\\ &\qquad - a^2b + 2ab^2 - b^3\\ &= a^3 - 3a^2b + 3ab^2 - b^3.\end{aligned} \tag{3.42}$$

We obtain this same result if we replace b with $-b$ in the last line of Eq. (3.40):

$$a^3 + 3a^2(-b) + 3a(-b)^2 + (-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3, \tag{3.43}$$

since $(-1)^2 = 1$ and $(-1)^3 = -1$.

Example 3.9 Expand $(n + 2)(n + 3)(n + 4)$, and check your result for $n = 0$, 1, and 2.

Solution We'll first apply FOIL to the product $(n + 3)(n + 4)$:

$$(n+3)(n+4) = n^2 + 4n + 3n + 12 = n^2 + 7n + 12. \tag{3.44}$$

We must now multiply this by $n + 2$:

$$\begin{aligned}(n+2)(n^2+7n+12) &= n(n^2+7n+12) + 2(n^2+7n+12)\\ &= (n^3+7n^2+12n) + (2n^2+14n+24)\\ &= n^3 + 9n^2 + 26n + 24.\end{aligned} \tag{3.45}$$

In the second line here, we could have vertically aligned the corresponding terms in the two sets of parentheses, as we did in various earlier calculations. But since this example involves only one letter, it's easy to see which terms involve the same power of n, even if they're all in the same line.

The suggested checks on the result in Eq. (3.45) yield the following relations, which are all true:

$$\begin{aligned}
n = 0: &\quad (0+2)(0+3)(0+4) \stackrel{?}{=} 0^3 + 9\cdot 0^2 + 26\cdot 0 + 24 \implies 2\cdot 3\cdot 4 = 24,\\
n = 1: &\quad (1+2)(1+3)(1+4) \stackrel{?}{=} 1^3 + 9\cdot 1^2 + 26\cdot 1 + 24 \implies 3\cdot 4\cdot 5 = 60,\\
n = 2: &\quad (2+2)(2+3)(2+4) \stackrel{?}{=} 2^3 + 9\cdot 2^2 + 26\cdot 2 + 24 \implies 4\cdot 5\cdot 6 = 120.
\end{aligned}$$

Exercise 3.17 Expand $(n-1)^3$, and then check your result against Eq. (3.42).

Exercise 3.18 Expand $(n-1)(n-2)(n-3)$, and check your result for $n = 1$, 2, 3, and 4.

Exercise 3.19 In the spirit of Exercise 3.11, simplify $\big((1+a)-a\big)^3$ the hard way. The easy way is to note that within the larger parentheses, the a's cancel, so we simply have $(1)^3$ which is just 1. Reproduce this answer of 1 by using the result in Eq. (3.42) with the replacements $a \to 1+a$ and $b \to a$.

Exercise 3.20 Use the result in Eq. (3.36) to expand and simplify $(a+b+c)^3$. There are many terms to deal with, so be systematic and take it slowly!

Geometric interpretation

Returning to our result for $(a+b)^3$ in Eq. (3.40), we can interpret the four terms on the righthand side geometrically as *volumes*, in the same way we interpreted the three terms in Eq. (3.20) as *areas* in Fig. 3.3(a). The following discussion (which might stretch your visualization skills!) is by no means necessary for an understanding of Eq. (3.40). Expanding $(a+b)^3$ involves FOIL plus a couple more applications of the distributive law, and we successfully did this in Eq. (3.40). So we're done. But if you're wondering what the geometric volume interpretations are for the four terms on the righthand side of Eq. (3.40), they can be understood as follows.

The reasoning below uses the fact that (as noted in Section A.5 in Appendix A) the volume of a rectangular block is the product of its three side lengths. So the volume of the block in Fig. 3.10 is abc. This is the base area ab times the height c. In the special case where the rectangular block is a cube, the three side lengths are all equal (that is, $a = b = c$). If we label the common side length as d, then the volume of the cube is $d \cdot d \cdot d = d^3$.

The cube in Fig. 3.11 gives the geometric interpretation of Eq. (3.40). The visualization of volumes here is a little trickier than the areas in Fig. 3.3(a), since it's generally harder to visualize things in three dimensions than in two. But the interpretation goes like this: The

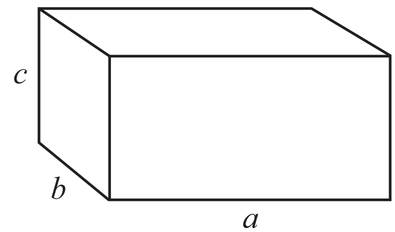

Figure 3.10: The volume of a rectangular block is abc.

lefthand side of Eq. (3.40) is the volume of the whole cube, since its side length is $a + b$. We claim that the whole cube can be built up from eight sub-blocks whose volumes are given by the terms on the righthand side of Eq. (3.40).

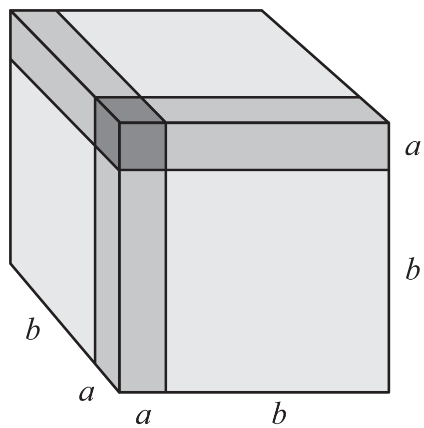

Figure 3.11: The setup for interpreting Eq. (3.40) in terms of volumes.

There are four types of terms:

- a^3: This is the volume of the small dark-shaded cube in Fig. 3.11 with side length a.
- a^2b: This is the volume of each of the medium-shaded blocks, since their dimensions are a-by-a-by-b. And there are 3 of these in the figure, consistent with the coefficient of 3 in Eq. (3.40).
- ab^2: This is the volume of each of the light-shaded blocks, since their dimensions are a-by-b-by-b. Again there are 3 of these in the figure, consistent with the coefficient of 3 in Eq. (3.40).
- b^3: This is the volume of the cube (which you can't see in Fig. 3.11) with side length b. This cube is diagonally opposite to the a-sided cube, in the same way that the b-sided square is diagonally opposite to the a-sided square in Fig. 3.3(a).

You can think about the full cube in the following way. If you start with a cubical b-sided unfrosted cake, and if you then put frosting with thickness a on only three of its faces (sharing a common corner), then your frosted cake will look like Fig. 3.11 (equivalently, like the cake on the front cover!), where the $1 + 3 + 3 = 7$ shaded blocks represent frosting. (Assume that you extend the frosting past the edges of the b-sided cake, as you would do in real life, so that the medium and dark-shaded blocks are included.) The other three faces of the cubical b-sided cake (which you can't see in Fig. 3.11) are exposed. Fig. 3.12 shows a rotated view where you can see some of the exposed cake (the unshaded part).

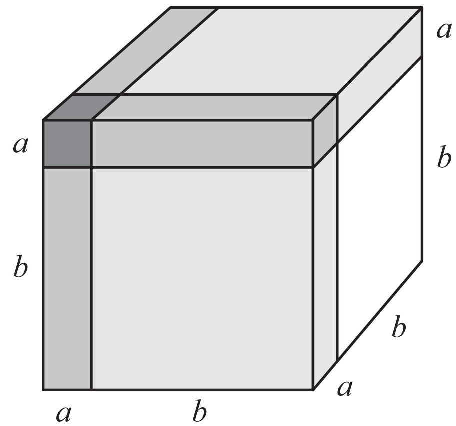

Figure 3.12: A rotated view of the cube in Fig. 3.11, showing the unfrosted b^3 sub-cube.

Putting it all together, we see that Eq. (3.40) represents the statement that the volume of the entire cube in Fig. 3.11 equals the sum of the volumes of the $1+3+3+1 = 8$ sub-blocks.

> If cubing a sum makes you ache,
> And the meaning of terms is opaque,
> Just picture some frosting,
> It's not so exhausting,
> It's really just pieces of cake!

For the more general case in Eq. (3.41) involving the product of three different binomials, the generalization of the 2-D (two-dimensional) figure on the right side of Fig. 3.2 is a 3-D rectangular block with unequal side lengths and unequal sub-lengths. You can test your visualization skills by drawing a picture that indicates which sub-block corresponds to each of the 8 terms on the righthand side of Eq. (3.41).

Exercise 3.21 Give the geometric interpretation of Eq. (3.42) in terms of volumes in the cube in Fig. 3.13, along the lines of what we did in Fig. 3.4 for $(a - b)^2$. The interpretation is a bit tricky, because you'll need to be careful about double (and triple) counting, in the same way we needed to be careful about double counting (double subtraction of the dark square) in Fig. 3.4. There's more to think about in the present setup, so you'll need to put on your 3-D thinking cap. But at least you know that Eq. (3.42) (with minus signs in front of two of the terms) is what you're aiming for.

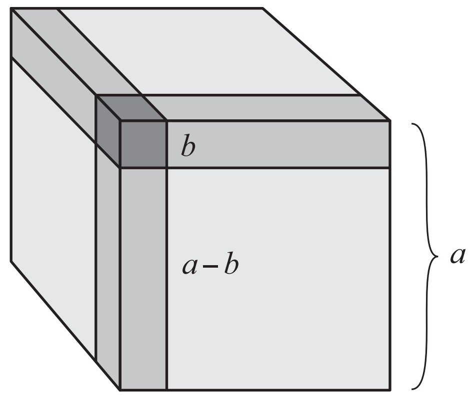

Figure 3.13: The setup for interpreting Eq. (3.42) in terms of volumes.

3.6 Binomial expansion

In Eq. (3.20) we calculated the square of the binomial $a + b$, and in Eq. (3.40) we calculated the cube. Let's now look at the fourth and higher powers. Using the result for $(a + b)^3$ in Eq. (3.40) and applying the same strategy as in that equation, we have

$$\begin{aligned}
(a+b)^4 &= (a+b)(a+b)^3 \\
&= (a+b)(a^3+3a^2b+3ab^2+b^3) \\
&= a(a^3+3a^2b+3ab^2+b^3)+b(a^3+3a^2b+3ab^2+b^3) \\
&= a^4+3a^3b+3a^2b^2+ab^3 \\
&\qquad +a^3b+3a^2b^2+3ab^3+b^4
\end{aligned}$$

$$\Longrightarrow \quad \boxed{(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4} \tag{3.46}$$

As in Eq. (3.40), we have vertically aligned the corresponding terms in the fourth and fifth lines. As a quick check on our answer, if $a = b = 1$ then Eq. (3.46) gives $(1 + 1)^4 = 1 + 4 + 6 + 4 + 1 \implies 2^4 = 16$, which is correct.

The term *binomial expansion* refers the expansion of $(a + b)^n$ for any (positive integer) value of n. So the expansion in Eq. (3.46) is the binomial expansion for $n = 4$.

Exercise 3.22 Calculate $(a + b)^5$ by using the above result for $(a + b)^4$ and applying the same distributive-law strategy as in Eq. (3.46).

We can continue the above strategy for larger exponents. Or we can be more efficient and look for a pattern. Table 3.1 shows what we know so far, including the result for $(a+b)^5$ from the above exercise. Note that the binomial expansion for a given n contains $n+1$ terms. For example, $(a+b)^5$ has 6 terms. This is true because the powers of b start at 0 and run up to 5, and there are 6 numbers in this set. Likewise for the powers of a.

$$
\begin{aligned}
(a+b)^1 &= a+b \\
(a+b)^2 &= a^2+2ab+b^2 \\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3 \\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\
(a+b)^5 &= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5
\end{aligned}
$$

Table 3.1: The binomial expansion up to $n = 5$.

To see more easily what's going on, Table 3.2 shows just the coefficients (called, quite reasonably, *binomial coefficients*) without all the a's and b's. The letter n signifies the exponent. We have included the trivial $(a+b)^0 = 1$ result for completeness. The triangle of numbers in the table is known as *Pascal's triangle*. It gives the coefficients in the binomial expansion. As we have observed on various occasions, the sum of the entries in each row is 2^n (obtained from setting $a = b = 1$ in Table 3.1).

$n = 0$:						1					
$n = 1$:					1		1				
$n = 2$:				1		2		1			
$n = 3$:			1		3		3		1		
$n = 4$:		1		4		6		4		1	
$n = 5$:	1		5		10		10		5		1

Table 3.2: Pascal's triangle up to $n = 5$.

Pascal's triangle gives the coefficients in the binomial expansion.

What is the pattern in Pascal's triangle? In particular, how is each row obtained from the one above it? Can you figure out where the "6" and "10" entries come from? If you want more data, Eq. (3.47) below gives the 1, 6, 15, etc., entries in the $n = 6$ row.

After a little fiddling, you will probably notice that each entry is the sum of the two entries above it (or just the single 1 if you're at the end of a line). For example, $6 = 3 + 3$ and $10 = 4 + 6$. The reason for this addition is evident in Eq. (3.46) above. Consider the

$4a^3b$ term, which corresponds to the left 4 in the $n = 4$ row in Table 3.2. This $4a^3b$ term comes from $3a^3b$ plus a^3b in Eq. (3.46), and these in turn come from $a(3a^2b)$ and $b(a^3)$, respectively. The $3a^2b$ here corresponds to the left 3 in the $n = 3$ row, and the a^3 corresponds to the left 1 in that row. Therefore, the left 4 in the $n = 4$ row correctly equals the sum of the 1 and 3 above it, as we wanted to show.

Each entry in Pascal's triangle is the sum of the two entries above it (or just the single 1 if you're at the end of a line).

If you want to calculate $(a + b)^6$, then even if you don't know the above sum rule, there's still no need to write out the long equation analogous to Eq. (3.46) (or Eq. (3.94) in the solution to Exercise 3.22). You just need to *imagine* multiplying (but not actually doing so) the $(a + b)$ factor by the $(a + b)^5$ expression in the last line in Table 3.1. Consider, for example, the a^4b^2-type term in $(a + b)^6$. This term is generated by (see Fig. 3.14) taking the $5a^4b$ term in the last line of Table 3.1 and multiplying it by b to obtain $5a^4b^2$, and also taking the $10a^3b^2$ term and multiplying it by a to obtain $10a^4b^2$.

$$(a + b)^6 = (a + b)(a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5)$$

Figure 3.14: The two products that generate the a^4b^2-type term in $(a + b)^6$.

Putting these two terms together yields $5a^4b^2 + 10a^4b^2 = 15a^4b^2$. So in the end we're basically just adding the left 5 and 10 in the last row in Table 3.2 to obtain 15. Without much work, we can therefore add up the other pairs of adjacent entries in the $n = 5$ row to obtain the coefficients in expansion for $n = 6$:

$$(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6. \tag{3.47}$$

If you want to check that this result is correct, you can draw for each term the two arrows analogous to the ones we drew in Fig. 3.14.

We see that when expanding $(a + b)^n$, there is never any need to multiply out lengthy and messy expressions. To obtain the coefficients in the expansion, all you need to do is use Pascal's triangle (evaluated down to the nth row).

It might *seem* like expansions will lead
To a mess that will make you concede.
But it doesn't take hours
To calculate powers;
A triangle's all that you need!

Note that in all of the above expansions, the sum of the exponents in each term equals the power n. For example, the sum of the exponents in each of the seven terms on the righthand side of Eq. (3.47) is 6. This is true because whenever we increase n by 1 and tack on another factor of $(a+b)$, each term in the previous expansion gets multiplied by an additional a or b (as in Fig. 3.14), which increases the sum of the exponents by 1.

In the above discussion, we raised a binomial (an expression with two terms) to the power of 3, 4, 5, etc., and we found that the coefficients in the results are given by Pascal's triangle. This should be contrasted with Example 3.6 and Exercise 3.14, which involved expressions with more than two (namely three and four) terms, but which we raised only to the power of 2. The results of those expansions are fairly simple. The diagonal terms all have a coefficient of 1, and the cross terms all have a coefficient of 2. In contrast, the coefficients in Pascal's triangle aren't just simple 1's and 2's. But they're still tractable. We'll produce a general formula for them in Appendix C.

Exercise 3.20 (the cube of a trinomial) is the one problem we've done that involved an expression with more than two terms, raised to a power that was larger than 2. We won't do any more problems of that type, because things can get a bit messy when both the number of terms and the power are large (by which we mean larger than 2).

Exercise 3.23 The coefficients in Eq. (3.47) give the $n = 6$ row of Pascal's triangle. Write down the $n = 7, 8, 9$ rows, using the fact that each entry is the sum of the two entries above it.

Exercise 3.24 Expand $(a+b)^4$ by writing it as $(a+b)^2(a+b)^2$, instead of the $(a+b)(a+b)^3$ form we used in Eq. (3.46).

Exercise 3.25 By looking at the first two terms in the binomial expansions of $(a+b)^3$ and $(a+b)^4$, determine the first two terms in the expansion of $(a+b)^7$.

3.7 Approximations

[*This section is more abstract and can be skipped on a first reading.*]

What, approximately, is the value of 1.001^4? One way to answer this is to plug 1.001^4 into your calculator. But let's instead take the following non-calculator route, which we'll be able to extend to cases involving letters instead of numbers.

The strategy will be to write 1.001 as $1 + 0.001$ and then use the expansion of $(a+b)^4$ in Eq. (3.46), with $a = 1$ and $b = 0.001$:

$$\begin{aligned}(1+0.001)^4 &= 1^4 + 4\cdot 1^3\cdot(0.001) + 6\cdot 1^2\cdot(0.001)^2 + 4\cdot 1\cdot(0.001)^3 + (0.001)^4 \\ &= 1.004\,006\,004\,001. \end{aligned} \tag{3.48}$$

We've put small gaps in this long decimal after each set of three digits to make it easier to read. You can check the result with a calculator, depending on how many digits your calculator displays.

Eq. (3.48) gives the exact value of $(1+0.001)^4$. As far as an approximate value goes, there are many possible answers, depending on what accuracy you want. The simplest approximation is to just say that the value is about 1. A better approximation is to say that the value is about 1.004. An even better one is 1.004 006. And so on. Which of these values you want to use depends on what accuracy you need, and this in turn depends on what you're using the approximation for. There's no one right answer. Different purposes require different degrees of accuracy.

No one can argue much with a statement like "1.001^4 exceeds 1 by approximately 0.004." This follows from the next-to-worst approximation (1.004) in the preceding paragraph. As we'll see in Chapter 12, this degree of accuracy is plenty good for some very important purposes.

Let's now be more general. The 4 in the above result comes from the second entry in the $n=4$ row in Pascal's triangle in Table 3.2 . In general, the second entry in each row in Pascal's triangle is always n. This means that the expansion of $(a+b)^n$ always starts with the two terms,

$$\boxed{(a+b)^n = a^n + na^{n-1}b + \cdots} \tag{3.49}$$

All of the expansions in Table 3.1 start this way, and likewise all of the Pascal's-triangle rows in Exercise 3.23 start with 1 and n. We'll prove this formally in Exercise 11.5 in Chapter 11, but it's quite clear from all of the results we've seen. The sum rule in Pascal's triangle means that you just keep adding 1 to obtain the second term in each successive row.

The coefficient of the second term in the binomial expansion is always n.

This fact is particularly useful when you have expressions like $(a+\epsilon)^n$, where ϵ is very small, like the 0.001 number above. (The Greek letter ϵ (epsilon) is a common one to use for a small quantity. The letter δ (delta) is also common.) For example, if $n=8$, we have

$$(a+\epsilon)^8 = a^8 + 8a^7\epsilon + (?)a^6\epsilon^2 + \cdots . \tag{3.50}$$

We could figure out what the "?" is by continuing Pascal's triangle in Table 3.2 down to the 8th row (as we did in Exercise 3.23; the "?" is 28). But the point here is that there's no need to bother, provided that (a) ϵ is very small, and (b) we're concerned only with an *approximate* result (in particular, the next-to-worst approximate result, like the 1.004 above).

The assumption that ϵ is very small means that the term with ϵ^2 in Eq. (3.50) is much smaller than the term with ϵ, just like the $(0.001)^2$ term in Eq. (3.48) is much smaller than the (0.001) term. (This statement assumes that the a in Eq. (3.50) isn't too small.) We can therefore ignore the ϵ^2 term, provided that we're concerned only with a (next-to-worst) approximate result.

The other terms in Eq. (3.50) with even higher powers of ϵ (the ϵ^3 through ϵ^8 terms represented by the "$\cdots$" dots) are even smaller, just like the higher powers of 0.001 in Eq. (3.48) are successively smaller. So we can ignore all the ϵ^3-and-beyond terms, as we did with the ϵ^2. We therefore have

$$(a+\epsilon)^8 \approx a^8 + 8a^7\epsilon \qquad \text{(valid when } \epsilon \ll a\text{)} \tag{3.51}$$

(The "$\ll$" means "much less than.") If the 8 here is replaced with a general n, then Eq. (3.51) becomes

$$\boxed{(a+\epsilon)^n \approx a^n + na^{n-1}\epsilon} \qquad \text{(valid when } \epsilon \ll a\text{)} \tag{3.52}$$

Of course, if ϵ is very small, the $8a^7\epsilon$ term in Eq. (3.51) is much smaller than the a^8 term. So by the same logic as above, we could say that we should ignore the $8a^7\epsilon$ term too, in which case we would simply have $(a+\epsilon)^8 \approx a^8$. This is indeed an approximately correct expression when ϵ is very small, in the same way that "1" was an approximate answer to 1.001^4 above. But the approximation in Eq. (3.51) is a slightly better one, in the same way that 1.004 was a slightly better approximation to 1.001^4.

So just as we did with the 1.004 approximation, we can use Eq. (3.51) to say that if ϵ is very small, then $(a+\epsilon)^8$ exceeds a^8 by approximately $8a^7\epsilon$. In this $8a^7\epsilon$ result, if we make the replacements $8 \to 4$, $7 \to 3$, $a \to 1$, and $\epsilon \to 0.001$, we reproduce the above 0.004 number.

If ϵ is *exactly* equal to zero, then the $8a^7\epsilon$ term (the second one in the expansion) is zero, as are all the other terms (except of course the first one, the a^8). But when we're dealing with approximations, we never set the small ϵ exactly equal to zero. We simply make it be *very small* but *nonzero*. The "very small" part of this means that the second term in the expansion is much larger than all of the subsequent terms (which have higher powers of ϵ). And the "nonzero" part means that the second term is *something*, which means that it actually matters.

When approximate answers were sought,
The students all had the same thought:
The term that is second
Is crucial, they reckoned –
It's small, but it isn't quite naught.

The approximation in Eq. (3.52) is just one of countless approximations that prove useful in math and other subjects. Why are they useful? Perhaps their most important benefit is that they allow you to write certain expressions in much simpler forms when you're working with letters. Examples of this are presented in Appendix B, Section 11.6, and Section 12.5. Another important benefit is that if you don't have a calculator handy, you can still get a good idea of the value of a numerical result, as we saw in the 1.001^4 example above.

Example 3.10 With an exponent of $n = 3$, and with ϵ replaced by b, the approximation in Eq. (3.52) becomes $(a+b)^3 \approx a^3 + 3a^2b$. These are the first two terms in Eq. (3.40). What is the geometric interpretation of this approximation (valid when $b \ll a$), in terms of the cube in Fig. 3.15? (Note: The a and b in this figure are reversed relative to Fig. 3.11.)

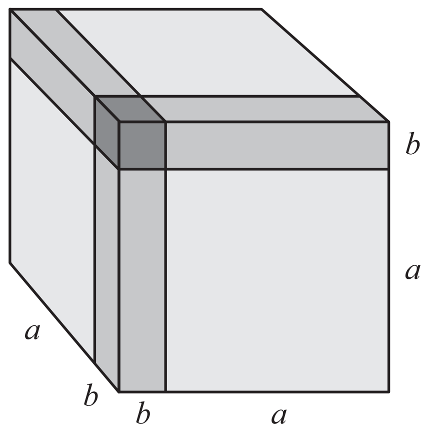

Figure 3.15: The setup for the geometric interpretation of the $(a + b)^3 \approx a^3 + 3a^2b$ approximation.

Solution a^2b is the volume of each of the three light-shaded pieces in Fig. 3.15. So $3a^2b$ is the total volume of the light-shaded pieces. The $(a + b)^3 \approx a^3 + 3a^2b$ approximation is therefore the statement that the $(a+b)^3$ volume of the overall cube is approximately equal to the a^3 volume of the sub-cube (which we can't see in the figure) plus the $3a^2b$ volume of the three light-shaded pieces. That is, we're ignoring the three medium-shaded pieces and the dark-shaded one, because they're small compared with the light-shaded pieces if b is much smaller than a.

Example 3.11 If ϵ is very small, find an approximate expression for

$$\frac{(x+\epsilon)^6 - x^6}{\epsilon}. \tag{3.53}$$

Solution With an exponent of 6, the approximation in Eq. (3.52) is

$$(x+\epsilon)^6 \approx x^6 + 6x^5\epsilon. \tag{3.54}$$

We therefore have

$$\frac{(x+\epsilon)^6 - x^6}{\epsilon} \approx \frac{(\cancel{x^6} + 6x^5\epsilon) - \cancel{x^6}}{\epsilon} = \frac{6x^5\cancel{\epsilon}}{\cancel{\epsilon}} = 6x^5. \tag{3.55}$$

So $6x^5$ is our approximate answer. You are encouraged to check this result by plugging some numbers into a calculator, such as $\epsilon = 0.0001$ along with $x = 1$ or $x = 2$.

Eq. (3.54) is the next-to-worst approximation of $(x+\epsilon)^6$, and it got the job done here. The worst approximation, namely $(x+\epsilon)^6 \approx x^6$, would *not* be sufficient, because it would turn Eq. (3.53) into $(x^6 - x^6)/\epsilon = 0/\epsilon$, which equals 0 for any nonzero value of ϵ. x^6 is too coarse an approximation.

We'll revisit expressions like the one in Eq. (3.53) later on in Section 12.5 when we talk about general slopes. The seemingly innocuous quotient in Eq. (3.53) is one of the most useful expressions in mathematics!

Exercise 3.26 Approximately how much larger is $(1.000\,001)^7$ than 1? Approximately how much larger is $(2.000\,001)^7$ than 2^7?

Exercise 3.27 If ϵ is very small, find an approximate expression for

$$\frac{(x+\epsilon)^5 - x^5}{\epsilon}\,.$$

3.8 Exercise solutions

1. (a) Since the commutative law tells us that the order of the letters within a product doesn't matter, we can make the following groupings of like terms:

$$\begin{aligned}
&6ac + 4b - 2a^2b + 3abc - ca + 7ba^2 - 9bca - 3aba - 2ac \\
&\quad = (6ac - ca - 2ac) + (-2a^2b + 7ba^2 - 3aba) + (3abc - 9bca) + 4b \\
&\quad = 3ac + 2a^2b - 6abc + 4b. \qquad (3.56)
\end{aligned}$$

Note that the $4b$ term is the only one of its type.

(b) The distributive law gives

$$\begin{aligned}
&a(a-b) - b(b-a) - a(a+c) \\
&\quad = (\cancel{a^2} - \cancel{ab}) - (b^2 - \cancel{ba}) - (\cancel{a^2} + ac) \\
&\quad = -b^2 - ac. \qquad (3.57)
\end{aligned}$$

The ab terms do indeed cancel, because the second one has two minus signs in front of it, so it's the same as $+ab$. If you want to take care of all the minus signs right away and write things without parentheses, then the second line above would look like

$$\cancel{a^2} - \cancel{ab} - b^2 + \cancel{ba} - \cancel{a^2} - ac. \qquad (3.58)$$

(c) Breaking up the first fraction (and then canceling some factors), and using the distributive law on the term with the parentheses, gives

$$\begin{aligned}
&\frac{a^3 + 3a^2b}{ab^2} - \frac{a}{b}\left(\frac{a}{b} + 1\right) \\
&\quad = \frac{a^3}{ab^2} + \frac{3a^2b}{ab^2} - \left(\frac{a^2}{b^2} + \frac{a}{b}\right) \\
&\quad = \cancel{\frac{a^2}{b^2}} + \frac{3a}{b} - \left(\cancel{\frac{a^2}{b^2}} + \frac{a}{b}\right) \\
&\quad = \frac{2a}{b}. \qquad (3.59)
\end{aligned}$$

In the first fraction in the second line above, we canceled an a in the numerator and denominator. And in the second fraction we canceled an ab.

(d) The distributive law gives

$$\begin{aligned}
&\sqrt{a}\big(\sqrt{a} - \sqrt{b}\big) - 4a + \sqrt{ab} \\
&\quad = \big(\sqrt{a}\sqrt{a} - \sqrt{a}\sqrt{b}\big) - 4a + \sqrt{ab} \\
&\quad = a - \cancel{\sqrt{ab}} - 4a + \cancel{\sqrt{ab}} \\
&\quad = -3a. \qquad (3.60)
\end{aligned}$$

We used the $\sqrt{a}\sqrt{b} = \sqrt{ab}$ result from Exercise 2.13 to obtain the third line.

2. Following the same steps as in Eqs. (3.12) and (3.13), but now with $e \equiv a + b$, gives

$$(a+b)(c+d) = e(c+d) = ec + ed = (a+b)c + (a+b)d, \tag{3.61}$$

which becomes (applying the distributive law to the terms on the righthand side)

$$\begin{aligned}(a+b)(c+d) &= (a+b)c + (a+b)d \\ &= (ac+bc) + (ad+bd) \\ &= ac + bc + ad + bd.\end{aligned} \tag{3.62}$$

These are the same four terms as in Eq. (3.13), but in a different order. The acronym for the result in Eq. (3.62) is FIOL, which isn't quite as catchy as FOIL!

The two sets of parentheses in the second line in Eq. (3.62) correspond in Fig. 3.2 to the left and right *columns* (the F and I one, and the O and L one), respectively. In contrast, the two sets of parentheses in the second line in Eq. (3.13) correspond to the top and bottom *rows* (the F and O one, and the I and L one), respectively.

3. Applying FOIL to the first product, and remembering that two minus signs make a plus sign (since they're the same as the product $(-1)(-1) = 1$), gives

$$\begin{aligned}(-4a-b)(7a-5b) &= -(4a)(7a) + (-1)^2(4a)(5b) - b(7a) + (-1)^2 b(5b) \\ &= -28a^2 + 20ab - 7ab + 5b^2 \\ &= -28a^2 + 13ab + 5b^2.\end{aligned} \tag{3.63}$$

If you don't like dealing with so many minus signs, you can multiply the given product through by $(-1)(-1)$ (which equals 1, so it doesn't change anything). If you then distribute a -1 into each of the two sets of parentheses (which you can do since $(-1)(-1)$ contains *two* -1's), this changes the sign of every term and yields the product $(4a+b)(-7a+5b)$, which has only one minus sign to deal with.

Applying FOIL to the second product gives

$$\begin{aligned}(n-3)(n-5) &= n^2 - 5n - 3n + 15 \\ &= n^2 - 8n + 15.\end{aligned} \tag{3.64}$$

If $n = 3$, the $(n-3)$ factor on the lefthand side is zero, so the product is therefore also zero. And the righthand side correctly also equals zero since $3^2 - 8\cdot 3 + 15 = 0$. Similarly, if $n = 5$, the lefthand side is again zero since the $(n-5)$ factor is zero. And the righthand side correctly also equals zero since $5^2 - 8\cdot 5 + 15 = 0$.

4. The four shaded rectangles each have an area of ab. The side length of the small (white) square is $a - b$, so its area is $(a-b)^2$. And the overall square has side length $a + b$, so its area is $(a+b)^2$. Our goal is therefore to show that

$$(a-b)^2 + 4\cdot ab = (a+b)^2. \tag{3.65}$$

Using the results in Eq. (3.20) and Eq. (3.21), this equation becomes

$$(a^2 - 2ab + b^2) + 4ab = a^2 + 2ab + b^2. \tag{3.66}$$

The two sides here are indeed equal, because the $-2ab$ and $+4ab$ terms on the lefthand side combine to make $+2ab$, in agreement with the middle term on the righthand side.

5. All you need to do is move the dark-shaded rectangle in Fig. 3.16 from the bottom of the figure up to the right. This produces a rectangle with the indicated side lengths of $a - b$ and $a + b$, as desired. The total initial shaded area (namely $a^2 - b^2$) equals the total final shaded area (namely $(a + b)(a - b)$), so this geometrically demonstrates the $a^2 - b^2 = (a + b)(a - b)$ relation.

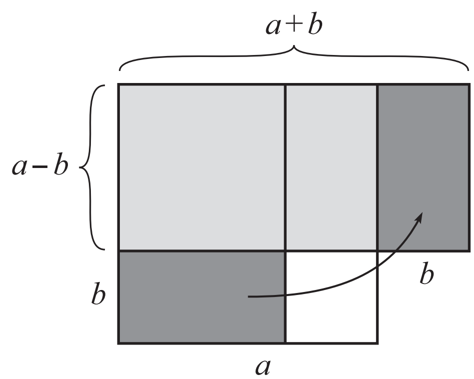

Figure 3.16: The geometric explanation of $a^2 - b^2 = (a + b)(a - b)$.

6. Eq. (3.23) gives

$$66 \cdot 74 = (70 - 4)(70 + 4) = 70^2 - 4^2 = 4900 - 16 = 4884. \tag{3.67}$$

The hardest part of this calculation is probably the subtraction of 16.

Eq. (3.20) gives

$$(50 + 3)^2 = 50^2 + 2\cdot 50\cdot 3 + 3^2 = 2500 + 300 + 9 = 2809. \tag{3.68}$$

As mentioned in Example 3.4, we just need to square 50 and 3 (these are easy), and then add on twice their product (this takes a little thought, although the $2 \cdot 50$ part of the cross term yields a nice simple 100). Adding the results is then fairly quick.

7. We want to multiply the given fraction by 1 in the form of c/c, where $c = \sqrt{7} - 2$ since that will produce a difference of squares in the denominator. This yields

$$\frac{6}{\sqrt{7}+2} = \frac{6}{\sqrt{7}+2} \cdot \frac{\sqrt{7}-2}{\sqrt{7}-2} = \frac{6(\sqrt{7}-2)}{(\sqrt{7})^2 - 2^2} = \frac{6\sqrt{7}-12}{7-4} = \frac{6\sqrt{7}-12}{3} = 2\sqrt{7} - 4. \tag{3.69}$$

We were able to cancel the 3 in the denominator, so we ended up with no denominator at all (or equivalently a denominator of 1). You should verify with a calculator that $2\sqrt{7} - 4$ is indeed equal to the original fraction.

8. Using the distributive law along with FOIL (which itself is just a few applications of the distributive law), we obtain

$$(\cancel{a^2}+\cancel{ab})-(\cancel{ba}+bc)-(\cancel{a^2}-2ac+c^2). \tag{3.70}$$

In this exercise, combining terms reduces to canceling them; the a^2 terms cancel, as do the ab terms. We're therefore left with

$$-bc+2ac-c^2. \tag{3.71}$$

Note that the sign in front of the $2ac$ term is in fact a plus sign, since the two minus signs in front of it in Eq. (3.70) cancel.

REMARK: If you want, you can factor out a c in the above result and write it as $c(-b+2a-c)$, which you can verify with the distributive law. (We'll talk about factoring in Section 4.2.) It's a matter of opinion as to whether this expression is simpler than the one in Eq. (3.71).

Interestingly, since $c(-b+2a-c)$ takes the form of c times something (which happens to be $-b+2a-c$), this expression is zero if $c=0$, no matter what the values of a and b are. The same must therefore be true for the expression we started with. Let's check this explicitly. If we set $c=0$ in the original expression, it becomes

$$a(a+b)-b(a+0)-(a-0)^2=(\cancel{a^2}+\cancel{ab})-\cancel{ba}-\cancel{a^2}=0, \tag{3.72}$$

as desired. The a^2 and ab terms both cancel, as we saw in Eq. (3.70). ♣

9. Using Eq. (3.20) with the replacements $a\to 4a$ and $b\to 6b$ gives

$$(4a+6b)^2=(4a)^2+2(4a)(6b)+(6b)^2=16a^2+48ab+36b^2. \tag{3.73}$$

Alternatively, we can factor out a 2 and write the given binomial as $2(2a+3b)$. (As we've mentioned, we'll talk about factoring in Section 4.2.) We then have

$$\begin{aligned}\big(2(2a+3b)\big)^2&=2^2(2a+3b)^2=4\big((2a)^2+2(2a)(3b)+(3b)^2\big)\\&=4(4a^2+12ab+9b^2)=16a^2+48ab+36b^2.\end{aligned} \tag{3.74}$$

Not much was gained here by factoring out the common factor of 2. And in fact, this second method involved a larger number of steps. However, in more complicated problems it is often a good idea to factor out any common factors at the start.

10. (a) Expanding the squares gives

$$\begin{aligned}(a+b)^2-(a-b)^2&=(\cancel{a^2}+2ab+\cancel{b^2})-(\cancel{a^2}-2ab+\cancel{b^2})\\&=2ab-(-2ab)\\&=4ab.\end{aligned} \tag{3.75}$$

(b) Using Eq. (3.24) with the replacements $a \rightarrow a + b$ and $b \rightarrow a - b$ gives

$$\begin{aligned}(a+b)^2 - (a-b)^2 &= \Big((a+\cancel{b}) + (a-\cancel{b})\Big)\Big((\cancel{a}+b) - (\cancel{a}-b)\Big)\\ &= (2a)(2b)\\ &= 4ab,\end{aligned} \tag{3.76}$$

in agreement with the result in part (a).

11. Using Eq. (3.21) with the replacements $a \rightarrow 1 + a$ and $b \rightarrow a$ gives

$$\begin{aligned}\big((1+a) - a\big)^2 &= (1+a)^2 - 2(1+a)a + a^2\\ &= (1 + \cancel{2a} + a^2) - (\cancel{2a} + 2a^2) + a^2\\ &= 1 + \cancel{(a^2 - 2a^2 + a^2)}\\ &= 1.\end{aligned} \tag{3.77}$$

12. The distributive law gives

$$\begin{aligned}(5-2)(1-3+6) &= 5(1-3+6) - 2(1-3+6)\\ &= (5\cdot 1 - 5\cdot 3 + 5\cdot 6) - (2\cdot 1 - 2\cdot 3 + 2\cdot 6)\\ &= 5 - 15 + 30 - 2 + 6 - 12\\ &= 12.\end{aligned} \tag{3.78}$$

For the quick way, if you add/subtract the terms in each set of parentheses first, you obtain $(3)(4) = 12$. Much quicker indeed! This method is of course far superior when you're working with actual numbers. But if you're working with letters, as in Eq. (3.32), there's no way to add the letters. You can combine many numbers into a single number, but you can't combine many letters into a single letter. Well, you can *define* a group of letters to be a single letter, as in $f \equiv c + d + e$, but that's only a temporary action.

You are encouraged to make up and solve a few more exercises like this one, with some randomly chosen numbers. The answer you obtain from the quick method will allow you to check the answer you obtain from the more lengthy distributive method.

13. (a) Repeating the steps in Eq. (3.36) (being careful with the minus signs) and collecting like terms gives

$$\begin{aligned}&(a - 2b + 5c)^2 \\ &\quad = a(a-2b+5c) - 2b(a-2b+5c) + 5c(a-2b+5c)\\ &\quad = (a^2 - 2ab + 5ac) + (-2ba + 4b^2 - 10bc) + (5ca - 10cb + 25c^2)\\ &\quad = a^2 + 4b^2 + 25c^2 - 4ab + 10ac - 20bc.\end{aligned} \tag{3.79}$$

As a quick check on this answer, if $a = b = c = 1$, we have $(1 - 2 + 5)^2 \stackrel{?}{=} 1 + 4 + 25 - 4 + 10 - 20 \implies 4^2 = 16$, which is correct.

(b) Invoking the final result in Eq. (3.36) with the given replacements yields

$$\begin{aligned} &a^2 + (-2b)^2 + (5c)^2 + 2a(-2b) + 2a(5c) + 2(-2b)(5c) \\ &\quad = a^2 + 4b^2 + 25c^2 - 4ab + 10ac - 20bc. \end{aligned} \tag{3.80}$$

This method was quicker than the one in part (a), but it was good to do it the long way in part (a) for practice. The quicker method using the final result in Eq. (3.36) makes it clear where the various coefficients come from. The diagonal coefficients (1, 4, 25) come from squaring the given coefficients (1, −2, 5), and the cross-term coefficients (−4, 10, −20) are twice the products of the two relevant given coefficients (with minus signs included).

14. The distributive law yields a straightforward but lengthy calculation:

$$\begin{aligned} (a+b+c+d)^2 &= (a+b+c+d)(a+b+c+d) \\ &= a(a+b+c+d) + b(a+b+c+d) \\ &\qquad + c(a+b+c+d) + d(a+b+c+d) \\ &= (a^2 + ab + ac + ad) + (ba + b^2 + bc + bd) \\ &\qquad + (ca + cb + c^2 + cd) + (da + db + dc + d^2) \\ &= a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd. \end{aligned} \tag{3.81}$$

It's quicker to use a figure, because we just have to write down each term once, instead of multiple times in the various lines in the above calculation. The figure that corresponds to Fig. 3.9(a) is shown in Fig. 3.17. We obtain one of each of the terms along the main diagonal, plus two of each of the terms off the diagonal (symmetrically located with respect to the diagonal, indicated by the shading). So when we add everything together, we correctly end up with the last line of Eq. (3.81).

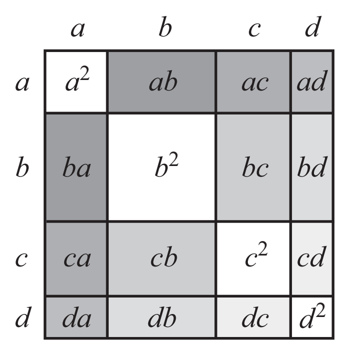

Figure 3.17: The interpretation of Eq. (3.81) in terms of areas.

From looking at Eqs. (3.36) and (3.81), the pattern is clear for the five-letter case $(a+b+c+d+e)^2$ and beyond. The expansion consists of the diagonal terms like a^2, plus the off-diagonal terms (the cross terms) like $2ab$. Every possible pair of letters appears in the off-diagonal terms. For example, with four letters, all of ab, ac, ad, bc, bd, and cd appear in Eq. (3.81).

REMARK: If we let the number of letters in the parentheses be a general number n, how many different types of pairs are there in the off-diagonal (cross) terms? That is, how many terms of the form $2ab$ do we have in the final result? For example, if $n = 3$, Eq. (3.36) tells us that there are three different types of pairs. And for $n = 4$, Eq. (3.81) says there are six pairs.

To answer this, note that there are n^2 sub-squares/rectangles in the figure analogous to Fig. 3.17, because there are n letters on each side. Of these n^2 subregions, n of them are the unshaded squares along the main diagonal. So that leaves $n^2 - n$ other sub-regions (the shaded rectangles). Since each pair of letters (like ab) appears twice in the overall square, we must divide by 2 to obtain the number of distinct pairs. (Each shaded rectangle above the main diagonal is a repeat of the corresponding shaded rectangle below it.) So there are $(n^2 - n)/2$ distinct pairs. Equivalently, there are $(n^2 - n)/2$ shaded rectangles above the main diagonal. Likewise for below. Note that this can be written as $n(n - 1)/2$.

Let's check this $n(n-1)/2$ result for the specific cases we've done: If $n = 2$, then $n(n-1)/2$ equals $2 \cdot 1/2 = 1$, in agreement with the one pair of letters (ab) in Eq. (3.20). If $n = 3$, we obtain $3 \cdot 2/2 = 3$, in agreement with the three types of pairs of letters (ab, ac, bc) in Eq. (3.36). And if $n = 4$, we obtain $4 \cdot 3/2 = 6$, in agreement with the six types of pairs of letters (ab, ac, ad, bc, bd, cd) in Eq. (3.81). ♣

15. Using the distributive law in the same manner as in Eq. (3.37) yields

$$\begin{aligned}
&(a-b)(a^3 + a^2b + ab^2 + b^3) \\
&\quad = a(a^3 + a^2b + ab^2 + b^3) - b(a^3 + a^2b + ab^2 + b^3) \\
&\quad = a^4 + \cancel{a^3b} + \cancel{a^2b^2} + \cancel{ab^3} \\
&\qquad\quad - \cancel{a^3b} - \cancel{a^2b^2} - \cancel{ab^3} - b^4 \\
&\quad = a^4 - b^4.
\end{aligned} \tag{3.82}$$

We have vertically aligned the corresponding terms in the third and fourth lines to make it clear which terms cancel. As a quick check on this result, if $a = b$ then both sides are zero.

Similarly,

$$\begin{aligned}
&(a+b)(a^3 - a^2b + ab^2 - b^3) \\
&\quad = a(a^3 - a^2b + ab^2 - b^3) + b(a^3 - a^2b + ab^2 - b^3) \\
&\quad = a^4 - \cancel{a^3b} + \cancel{a^2b^2} - \cancel{ab^3} \\
&\qquad\quad + \cancel{a^3b} - \cancel{a^2b^2} + \cancel{ab^3} - b^4 \\
&\quad = a^4 - b^4.
\end{aligned} \tag{3.83}$$

As a quick check on this result, if $a = -b$ then both sides are zero, because the $(a + b)$ factor on the left is zero, and on the right we have

$$a^4 - b^4 = (-b)^4 - b^4 = (-1)^4b^4 - b^4 = b^4 - b^4 = 0, \tag{3.84}$$

since $(-1)^4 = 1$.

In comparing this exercise with Example 3.7, we see that the two answers here are the same (they're both $a^4 - b^4$), whereas they were different in Example 3.7 (namely $a^3 - b^3$ and $a^3 + b^3$). In general, the even/oddness of the value of the exponents in the final result is what determines if the answers are the same or different. We'll see why when we talk about factoring in Section 4.2.

16. (a) Following the suggested steps gives

$$\begin{aligned}
&(a+b)^2(a-b)^2\\
&= (a^2+2ab+b^2)(a^2-2ab+b^2)\\
&= a^2(a^2-2ab+b^2) + 2ab(a^2-2ab+b^2) + b^2(a^2-2ab+b^2)\\
&= a^4 - \cancel{2a^3b} + a^2b^2\\
&\qquad + \cancel{2a^3b} - 4a^2b^2 + \cancel{2ab^3}\\
&\qquad\qquad + a^2b^2 - \cancel{2ab^3} + b^4\\
&= a^4 - 2a^2b^2 + b^4.
\end{aligned} \tag{3.85}$$

(b) Using Eq. (3.23) and then expanding the square (either from scratch by using FOIL, or by using Eq. (3.21) with the replacements $a \to a^2$ and $b \to b^2$) gives

$$\begin{aligned}
[(a+b)(a-b)]^2 &= [a^2-b^2]^2\\
&= (a^2)^2 - 2(a^2)(b^2) + (b^2)^2\\
&= a^4 - 2a^2b^2 + b^4,
\end{aligned} \tag{3.86}$$

in agreement with the result in part (a).

17. Expanding $(n-1)^2$ with FOIL yields $n^2 - 2n + 1$. Multiplying this by the third factor of $n-1$ then gives

$$\begin{aligned}
(n-1)^3 &= (n-1)(n^2-2n+1)\\
&= n(n^2-2n+1) - 1\cdot(n^2-2n+1)\\
&= (n^3-2n^2+n) + (-n^2+2n-1)\\
&= n^3 - 3n^2 + 3n - 1.
\end{aligned} \tag{3.87}$$

This correctly agrees with Eq. (3.42) with the replacements $a \to n$ and $b \to 1$.

18. We'll first apply FOIL to the product $(n-2)(n-3)$:

$$(n-2)(n-3) = n^2 - 3n - 2n + 6 = n^2 - 5n + 6. \tag{3.88}$$

We must now multiply this by $n-1$:

$$\begin{aligned}
(n-1)(n^2-5n+6) &= n(n^2-5n+6) - 1\cdot(n^2-5n+6)\\
&= (n^3-5n^2+6n) + (-n^2+5n-6)\\
&= n^3 - 6n^2 + 11n - 6.
\end{aligned} \tag{3.89}$$

For the checks, we quickly see that $n = 1$, 2, and 3 each make one of the factors in the given expression $(n-1)(n-2)(n-3)$ equal to zero, which means that the whole product is zero. So the checks for these three numbers are

$$\begin{aligned}
n = 1: &\quad 0 \stackrel{?}{=} 1^3 - 6\cdot 1^2 + 11\cdot 1 - 6,\\
n = 2: &\quad 0 \stackrel{?}{=} 2^3 - 6\cdot 2^2 + 11\cdot 2 - 6,\\
n = 3: &\quad 0 \stackrel{?}{=} 3^3 - 6\cdot 3^2 + 11\cdot 3 - 6.
\end{aligned} \tag{3.90}$$

These statements are all true, as you can verify. For $n = 4$, the check is

$$n = 4: \quad (4-1)(4-2)(4-3) \stackrel{?}{=} 4^3 - 6\cdot 4^2 + 11\cdot 4 - 6 \implies 3\cdot 2\cdot 1 = 6,$$

which is true.

19. With $a \to 1 + a$ and $b \to a$, Eq. (3.42) gives

$$\big((1+a) - a\big)^3 = (1+a)^3 - 3(1+a)^2 a + 3(1+a)a^2 - a^3. \tag{3.91}$$

The $(1+a)^3$ and $(1+a)^2$ terms here can be evaluated by letting $a \to 1$ and $b \to a$ in Eqs. (3.40) and (3.20). Or you can just work them out from scratch. The result is

$$\begin{aligned}
\big((1+a) - a\big)^3 &= (1 + 3a + 3a^2 + a^3) - 3(1 + 2a + a^2)a + 3(1+a)a^2 - a^3\\
&= (1 + 3a + 3a^2 + a^3) - (3a + 6a^2 + 3a^3) + (3a^2 + 3a^3) - a^3\\
&= 1 + a(3-3) + a^2(3 - 6 + 3) + a^3(1 - 3 + 3 - 1)\\
&= 1.
\end{aligned} \tag{3.92}$$

You can check that the third line here is correct, by applying the distributive law and verifying that you produce all the terms in the second line. We chose to write the third line in the form analogous to Eq. (3.31), instead of the form analogous to the second line in Eq. (3.30), because there is a large number of terms here, and the form in Eq. (3.31) is a slightly more efficient way of organizing everything.

20. Writing $(a+b+c)^3$ as $(a+b+c)(a+b+c)^2 \equiv (a+b+c)d$, the distributive law turns this into $ad + bd + cd$. Using the result for $(a+b+c)^2 \equiv d$ in Eq. (3.36), we then have (in all of the terms below, we'll always keep the letters in the order of a, b, c):

$$\begin{aligned}
(a+b+c)^3 &= (a+b+c)(a+b+c)^2 \qquad (3.93)\\
&= (a+b+c)(a^2 + b^2 + c^2 + 2ab + 2ac + 2bc)\\
&= a^3 + ab^2 + ac^2 + 2a^2b + 2a^2c + 2abc \qquad \text{(the } ad \text{ term)}\\
&\quad + a^2b + b^3 + bc^2 + 2ab^2 + 2abc + 2b^2c \qquad \text{(the } bd \text{ term)}\\
&\quad + a^2c + b^2c + c^3 + 2abc + 2ac^2 + 2bc^2 \qquad \text{(the } cd \text{ term)}\\
&= (a^3 + b^3 + c^3) + 3(a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2) + 6abc.
\end{aligned}$$

As a quick check, if all three letters are equal to 1, then $(a + b + c)^3 = 3^3 = 27$. And the final result correctly also equals $3 + 3 \cdot 6 + 6 = 27$.

In the final result, the terms in the first set of parentheses are the cubes of the letters, the terms in the second set involve all the possible products of the square of one letter times another, and the last term contains the product of all three letters.

The 2-D (two-dimensional) square in Fig. 3.9(a), with its $3^2 = 9$ sub-squares/rectangles, gives the geometric interpretation of $(a + b + c)^2$. For the present $(a + b + c)^3$ case, the geometric interpretation involves a 3-D cube with $3^3 = 27$ sub-blocks. This 3-D object is a bit harder to visualize, but you are encouraged to try to draw a picture of it! However, you may want to first read the "Geometric interpretation" subsection beginning on page 152, which covers the $(a + b)^3$ case.

21. In terms of cake and frosting, the overall cube with side length a in Fig. 3.13 consists of a cubical cake with side length $a - b$, covered with frosting with thickness b on three of its faces. (We can't see any of the unfrosted cake in the figure.) The result in Eq. (3.42) is the statement that the $(a - b)^3$ volume of the unfrosted cake can be obtained by starting with the a^3 volume of the entire frosted cake, and then subtracting off all of the frosting. This can be done by subtracting (and adding) various pieces in the following manner.

We'll start with the entire a^3 volume in Fig. 3.13 and then slice off the top slab of frosting with volume a^2b. This volume includes one light-shaded region, two medium-shaded regions, and the one dark-shaded region. We'll then do the same thing with the front a^2b slab, and also the left a^2b slab. (Pretend that it's possible to remove a given sub-region again, even if we've already removed it. We did this in Fig. 3.4 with the dark-shaded square; we removed it twice.) So at this point we have subtracted a total volume of $3a^2b$. This is the $-3a^2b$ term in Eq. (3.42).

Everything is fine with the light-shaded regions; we have subtracted each of them once. However, we have an issue with the medium-shaded regions. We have subtracted each of them *twice*, because each one is part of two different a^2b slabs. So we must add each one back in once, since we want to subtract each of them only once. We'll therefore add on a volume of ab^2 three times. This is the $3ab^2$ term in Eq. (3.42). Note that each of these ab^2 volumes includes a medium-shaded region *and* the dark-shaded region.

Everything is now fine with the medium-shaded regions; each of them has been subtracted a net time of once. However, we have an issue with the dark-shaded region. We subtracted it three times in the three a^2b slabs, since it is part of each of those. And then we added it back in three times in the three ab^2 volumes, since it is also part of each of those. We have therefore subtracted it a net of $3 - 3 = 0$ times. But we need to subtract it once. So we must now subtract the dark-shaded cube. This is the $-b^3$ term in Eq. (3.42).

Every piece of the frosting has now been subtracted a net time of once. And we have successfully accounted for all of the terms on the righthand side of Eq. (3.42).

REMARK: To obtain the unfrosted volume $(a - b)^3$ from the entire volume a^3, we could technically just start with the entire a^3 volume in Fig. 3.13 and then subtract the three light-shaded regions, and also the three medium-shaded regions, and also the one dark-shaded region. However, this way of looking at things simply brings us back to the interpretation in Fig. 3.11, but now with some of the relevant lengths being $a - b$. That's not the interpretation we're looking for here, because Eq. (3.42) involves only a's and b's, and no $a - b$ terms. So the geometric interpretation shouldn't involve any $a - b$ lengths. ♣

22. Using the result for $(a + b)^4$ from Eq. (3.46), we have

$$\begin{aligned}
(a+b)^5 &= (a+b)(a+b)^4 \\
&= (a+b)(a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4) \\
&= a(a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4) \\
&\qquad + b(a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4) \\
&= a^5 + 4a^4b + 6a^3b^2 + 4a^2b^3 + ab^4 \\
&\qquad + a^4b + 4a^3b^2 + 6a^2b^3 + 4ab^4 + b^5 \\
&= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5. \qquad (3.94)
\end{aligned}$$

As a quick check on our answer, if $a = b = 1$ then Eq. (3.94) gives $(1 + 1)^5 = 1 + 5 + 10 + 10 + 5 + 1 \implies 2^5 = 32$, which is correct.

23. Using the fact that each entry is the sum of the two entries above it, the desired rows are shown in Fig. 3.18 (including the $n = 6$ row for good measure).

$n = 6$:			1	6	15	20	15	6	1	
$n = 7$:		1	7	21	35	35	21	7	1	
$n = 8$:	1	8	28	56	70	56	28	8	1	
$n = 9$:	1	9	36	84	126	126	84	36	9	1

Figure 3.18: The $n = 6$ through $n = 9$ rows of Pascal's triangle.

In Appendix C we'll find a general formula for the entries in terms of n.

24. Using the standard result for $(a + b)^2$, we have

$$\begin{aligned}
&(a+b)^2(a+b)^2 \\
&= (a^2 + 2ab + b^2)(a^2 + 2ab + b^2) \\
&= a^2(a^2 + 2ab + b^2) + 2ab(a^2 + 2ab + b^2) + b^2(a^2 + 2ab + b^2) \\
&= a^4 + 2a^3b + a^2b^2 \\
&\qquad + 2a^3b + 4a^2b^2 + 2ab^3 \\
&\qquad\qquad + a^2b^2 + 2ab^3 + b^4 \\
&= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4, \qquad (3.95)
\end{aligned}$$

in agreement with Eq. (3.46). In the fourth, fifth, and sixth lines, we have vertically aligned terms with the same powers of a and b.

For additional practice along these lines, you can expand $(a+b)^5$ by writing it as $(a+b)^2(a+b)^3$, instead of the $(a+b)(a+b)^4$ form in Exercise 3.22.

25. From Table 3.1, the expansion of $(a+b)^3$ starts off like $a^3+3a^2b+\cdots$, and the expansion of $(a+b)^4$ starts off like $a^4+4a^3b+\cdots$. So the product is

$$\begin{aligned}(a+b)^7 &= (a+b)^3(a+b)^4\\ &= (a^3+3a^2b+\cdots)(a^4+4a^3b+\cdots)\\ &= a^7+a^3\cdot 4a^3b+3a^2b\cdot a^4+\cdots\\ &= a^7+7a^6b+\cdots\,. \end{aligned}\tag{3.96}$$

In going from the second to third line, we needed only the FOI part of FOIL, because the L term involves b^2. That's relevant for the third term in the expansion (the one of the form a^5b^2), but we're concerned only with the first two terms here. None of the terms represented by the "$\cdots$" dots above will affect the first two terms we produced, because they all contain at least two powers of b.

The 7 in Eq. (3.96) agrees with what we found in Exercise 3.23. That exercise and Table 3.2 make it clear that in general the coefficient of the second term in the expansion of $(a+b)^n$ is always n.

26. The approximation analogous to Eq. (3.51) is

$$(a+\epsilon)^7 \approx a^7+7a^6\epsilon. \tag{3.97}$$

So $(a+\epsilon)^7$ is larger than a^7 by approximately $7a^6\epsilon$. If $a=1$ and $\epsilon=0.000\,001$, this yields $7(1^6)(0.000\,001)=0.000\,007$. And if $a=2$ and $\epsilon=0.000\,001$, it yields $7(2^6)(0.000\,001)=0.000\,448$. You can check both of these results with a calculator. For the latter, you will indeed see a 448 in the display.

27. This solution requires only a slight modification to the solution in Example 3.11. The approximation analogous to Eq. (3.54) is

$$(x+\epsilon)^5 \approx x^5+5x^4\epsilon. \tag{3.98}$$

We therefore have

$$\frac{(x+\epsilon)^5-x^5}{\epsilon} \approx \frac{(\cancel{x^5}+5x^4\epsilon)-\cancel{x^5}}{\epsilon} = \frac{5x^4\cancel{\epsilon}}{\cancel{\epsilon}} = 5x^4. \tag{3.99}$$

So $5x^4$ is our approximate answer. You can check this result by plugging some numbers into your calculator.