Appendix B

Benefits of using letters

From *Algebra: For the Enthusiastic Beginner* (Draft version, July 2024)

As mentioned in the preface, algebra is the golden key to nearly every quantitative field, from math to physics to economics and beyond. Consistent with this, there are countless instances in this book where using letters (instead of only numbers) falls somewhere in the range between highly advantageous and absolutely necessary. A few examples are the sum formulas in Eqs. (4.55)–(4.57), solving for x in Chapter 5, and the treatment of lines in Chapter 6 and general functions in Chapter 7. You can't do much with functions if you use only numbers!

Additionally, even in situations where you're dealing with specific numbers, it's still often advantageous to work with letters instead. For example, let's say that someone asks you to find the area of a triangle whose base is 8 and height is 5. You can work this out from scratch for the specific numbers 8 and 5 (by using the technique in Section A.4.2, for example). Or better, you can first work with letters and deduce that the area of any general triangle is $bh/2$, where b is the base and h is the height (which is what we did in Section A.4.2). You can then simply plug in whatever b and h values someone might give you. Having generated the formula $bh/2$ with letters, you've done the problem once and for all.

In addition to this "once and for all" benefit, there are many other advantages to using letters, which we'll discuss in Section B.2. We'll frame that discussion in terms of an interesting and instructive problem involving the distance you can see to the horizon, which we'll solve in Section B.1. This problem is a nice application of the Pythagorean theorem, so be sure to read Section A.6 first. And we'll be solving some simple equations here, so you'll also want to read Section 5.1.

B.1 Distance to the horizon

If you've ever been high up in a tall building or on top of a mountain, you may have wondered how far you can see. Or more precisely, how far you can see to the horizon. Of course, in most scenarios like this, your view to the horizon is blocked by other buildings or mountains, and trees and such. We'll therefore assume that you're near the ocean, so that you have a clear view to the horizon.

The setup is shown in Fig. B.1. The height h that we've drawn for the building/mountain is greatly exaggerated (being about $1/5$ of the earth's radius R), to make it easier to see what's going on. In reality, there is no chance that h (for any everyday-type scenario) will be comparable to the earth's radius R. Even at the altitude of the International Space Station (which is about 250 miles), h is only $1/16$ of R (which is about 4000 miles).

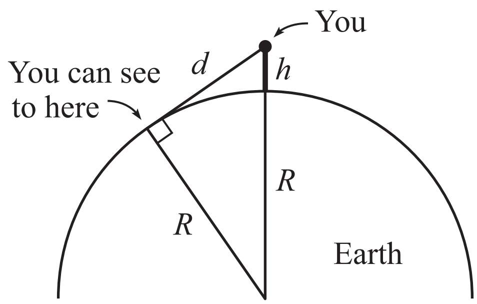

Figure B.1: The setup for finding how far you can see to the horizon.

Our goal is to find the distance d in the figure, in terms of the known quantities h and R. Now, the desired distance you can see to the horizon is slightly ambiguous. Do we mean the straight-line distance d drawn, or do we mean the curved distance along the surface of the earth? Fortunately it doesn't matter, because these two distances are essentially equal to each other for any reasonable (not excessively large) value of h. But for concreteness let's say that our goal is to find the straight-line distance d.

The straight line representing the distance d is tangent to the earth. A *tangent* line to the earth's circle is a line that just barely skims the circle. The tangent line is in fact the line of sight we're concerned with, because if you look at an angle that is slightly too high, you'll be looking at a point up in the sky. And if you look at an angle that is slightly too low, you'll be looking at a closer point on the ground (or rather the ocean, since we're assuming that's where you are), which therefore won't be the farthest point you can see. So we are indeed concerned with the tangent line, which is the transition between the ground/ocean and the sky.

You will find in Exercise B.3 that a tangent line is perpendicular to the radius at the point of contact with the circle. Let's just accept this (quite believable) fact for now. We therefore have the right angle, and hence the right triangle, shown above in Fig. B.1, which means that we can apply the Pythagorean theorem.

Numerical example

Let's do a numerical example first, and then we'll find the general answer for the distance d in terms of letters. We'll arbitrarily pick h to be

$$h = 100 \text{ meters}, \tag{B.1}$$

which is about 330 feet – a reasonably tall building. (A meter is about 39.4 inches, which is a little more than a yard, which is 3 feet = 36 inches.) We could work with any unit of length (feet, yards, meters, etc.), but we'll choose the metric system's meters because other lengths in it (like kilometers) are obtained by multiplying by simple powers of 10. A kilometer is 1000 meters. (The prefix "kilo" means 1000.) The abbreviations for meter and kilometer are "m" and "km."

In addition to assuming we're near the ocean (so that we don't need to worry about trees and hills and such), we'll work in the approximation where the earth is a perfect sphere. It actually isn't; it bulges a little at the equator because it's spinning. The radius of the earth varies from about 6,357 km at the poles to 6,378 km at the equator. But there's no need for that level of accuracy here, so we'll just round these values up to 6,400 km. Hence

$$R = 6{,}400\,\text{km} \quad (\text{or equivalently } 6{,}400{,}000 \text{ meters}). \tag{B.2}$$

This equals the 4000 miles we mentioned earlier, for the following reason. There are 1609 meters in a mile, and hence about 1.609 (we'll round this to 1.6) kilometers in a mile. (A mile is the larger of the two units.) Therefore, to go from km to miles, you divide by 1.6, or equivalently multiply by 0.62. (A 10 km race is 6.2 miles.) So 6400 km equals 4000 miles, because $(6400)(0.62) \approx 4000$, and also $(4000)(1.6) = 6400$.

Let's now apply the Pythagorean theorem to the right triangle in Fig. B.1. The three sides of the triangle are: the desired distance d (one leg), the radius $R = 6{,}400{,}000$ m (the other leg), and the hypotenuse $R + h = 6{,}400{,}100$ m. The Pythagorean theorem therefore gives

$$d^2 + (6{,}400{,}000\,\text{m})^2 = (6{,}400{,}100\,\text{m})^2. \tag{B.3}$$

Subtracting $(6{,}400{,}000\,\text{m})^2$ from both sides of this equation yields

$$d^2 = (6{,}400{,}100\,\text{m})^2 - (6{,}400{,}000\,\text{m})^2 = 1{,}280{,}010{,}000\,\text{m}^2. \tag{B.4}$$

Taking the square root of both sides then gives (choosing the positive square root, since the distance d must of course be positive)

$$d = \sqrt{1{,}280{,}010{,}000\,\text{m}^2} \approx 35{,}800\,\text{m} \approx \boxed{36\,\text{km}} \tag{B.5}$$

You can therefore see about 36 kilometers (or about $(36)(0.62) = 22$ miles) from the top of a 100-meter building. That's quite far!

Using letters

The above calculation contained some large numbers, which were a bit of a pain. The numbers would have been smaller if we had chosen to work with kilometers instead of meters (the lengths would have been $R = 6400$ km and $R + h = 6400.1$ km), but these numbers are still a hassle to work with. So let's now solve the problem algebraically, that is, in terms of letters. There are significant advantages to working with letters, as we'll see in Section B.2.

In terms of letters, applying the Pythagorean theorem to the triangle in Fig. B.1 gives (replacing the two numbers in Eq. (B.3) with the two lengths R and $R + h$)

$$d^2 + R^2 = (R + h)^2 \implies d^2 + \cancel{R^2} = \cancel{R^2} + 2Rh + h^2$$
$$\implies d = \boxed{\sqrt{2Rh + h^2}} \qquad \text{(Exact result)} \tag{B.6}$$

where we have subtracted R^2 from both sides, and then taken the (positive) square root. This $\sqrt{2Rh + h^2}$ result is the general answer to the problem. For any values of R and h you're given, you can simply plug them into $\sqrt{2Rh + h^2}$ to obtain the desired distance d. You should check that if you plug in the $R = 6{,}400{,}000\,\text{m}$ and $h = 100\,\text{m}$ values we used above, you will reproduce Eq. (B.5). Eq. (B.6) is valid even if h is large, although for real-life situations h is always much less than R (that is, $h \ll R$).

Eq. (B.6) is a nice clean result. But we can go one step further to obtain an even cleaner result. In a real-life situations, the h^2 term is much smaller than the $2Rh$ term. It is smaller by the factor $(h^2)/(2Rh)$, which equals $h/2R$. And $h/2R$ is very small for any everyday value of h. If you're in a tall building with height $h = 100\,\text{m}$ (330 feet), then

$$\frac{h}{2R} = \frac{100\,\text{m}}{2(6{,}400{,}000\,\text{m})} = \frac{1}{128{,}000} \approx 8 \cdot 10^{-6}, \tag{B.7}$$

which is indeed very small. Even at the height of a commercial airplane (about 10,000 m, or 33,000 feet), the value of $h/2R$, which is 100 times larger than for 100 m (or 330 feet), is still only $1/1280$. So to a good approximation, we can simply ignore the much smaller h^2 term (when compared with the $2Rh$ term) in Eq. (B.6) and say that

$$d \approx \boxed{\sqrt{2Rh}} \qquad \text{(Approximate result)} \tag{B.8}$$

This is an extremely clean result! And here's the important point: Whenever you derive an approximate answer as we just did, you gain something and you lose something. You lose some truth, of course, because your new answer is an approximation and therefore technically not quite correct (although the error becomes very small in the appropriate limit – small h here). But you gain some simplicity. Your new answer is invariably far cleaner (often involving only one term), and this makes it much easier to see how the various letters affect the result. Nine times out of ten, the upside of the simplicity wins out over the downside of being not exactly correct. So approximations are a *good* thing!

> An approximate answer in terms of letters makes it easy to see how the result depends on the various letters.

For example, a quick look at Eq. (B.8) tells you that d does *not* grow linearly with h (that is, it isn't directly proportional to h), but instead grows like the *square root* of h (or at least approximately, since Eq. (B.8) is an approximation). So if you want to make d be, say, 5 times larger, then you need to make h be 25 times larger. Simply increasing h by a factor of 5 won't do it. Similarly, if you want to increase d by a factor of 10, you need to increase h by a factor of 100. Or said in a slightly different way, if you increase h by a factor of 100, you increase d by a factor of only 10. Simple relations like these between d and h aren't obvious from looking at the exactly correct (but not as simple) result in Eq. (B.6).

How close is the approximate answer in Eq. (B.8) to the exact answer in Eq. (B.6) when $R = 6{,}400{,}000$ and $h = 100$? Plugging in the numbers gives (the first result here is just a repeat of Eq. (B.5), without the rounding)

$$\begin{aligned} d_{\text{exact}} &= \sqrt{2Rh + h^2} = \sqrt{1{,}280{,}010{,}000\,\text{m}^2} = 35{,}777.23\,\text{m}, \\ d_{\text{approx}} &= \sqrt{2Rh} = \sqrt{1{,}280{,}000{,}000\,\text{m}^2} = 35{,}777.09\,\text{m}. \end{aligned} \tag{B.9}$$

We see that our $\sqrt{2Rh}$ approximation is an *extremely* good one. No one could possibly care about an error of $0.14\,\text{m} = 14\,\text{cm}$, when we're talking about distances of roughly 36 km. There's truly no harm in ignoring the comparatively tiny h^2 term. Even at the top of a tall mountain, the h^2 term will have no noticeable effect (when compared with the distance you can see), as you can verify in Exercise B.1.

When looking afar from peak,
Remember this useful technique:
In finding the distance,
Ignore the existence
Of terms whose effect is quite weak.

Exercise B.1 Repeat the calculations in Eq. (B.9) for Mt. Everest, whose height is about $h = 9000\,\text{m}$.

Exercise B.2 The height of the International Space Station is $h \approx R/16$, which equals 250 miles, or 400 km. In terms of R, find the exact and approximate answers for d in Eqs. (B.6) and (B.8). Then plug in $R = 6{,}400\,\text{km}$ to find the actual distances. By how much do they differ?

Exercise B.3 Prove that a tangent line to a circle is perpendicular to the radius at the point of contact, as shown in Fig. B.2(a).

Hint: Assume that the tangent line is *not* perpendicular to the radius, as shown in Fig. B.2(b). Then show that this assumption leads to a conclusion that you know is false (which means that the assumption couldn't have been correct in the first place). You will need to use the fact that every point on the tangent line lies outside the circle, which means that every point is at least a distance R from the center. Don't try to make too much sense of Fig. B.2(b), because your goal is to show that it's actually impossible for the setup to look that way.

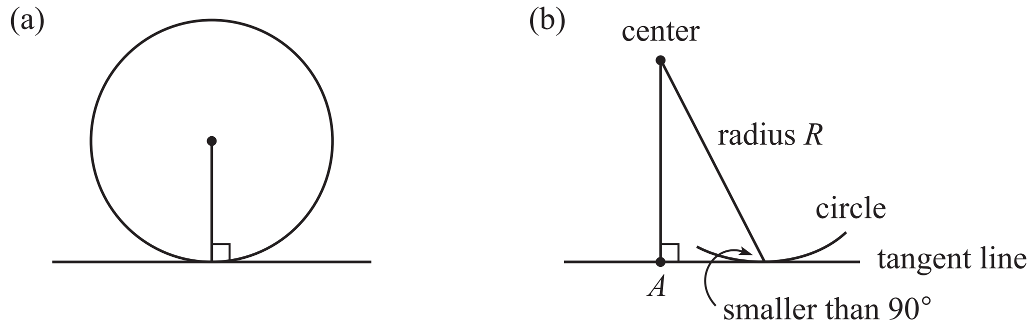

Figure B.2: Proving that a tangent line is perpendicular to the radius at the point of contact with the circle.

Other values of h

What does Eq. (B.8) give for some other values of h? If you're standing at the shore of the ocean, let's say that your eyes are at a height of $h \approx 2\,\text{m}$. We then have

$$d = \sqrt{2Rh} = \sqrt{2(6{,}400{,}000\,\text{m})(2\,\text{m})} = 5060\,\text{m} \approx 5\,\text{km} \approx 3\text{ miles}. \tag{B.10}$$

On one hand, this might seem like a large distance, given that your eyes are only 2 meters above the ground. But on the other hand, this distance is much smaller than it would be if the earth were flat!

The values of $d \approx \sqrt{2Rh}$ for a few other values of h are listed in Table B.1. The h's are given in meters, and the d's are given in both kilometers and miles. Remember that to go from kilometers to miles, you multiply by 0.62. Are the various values of d larger or smaller than what you expected? Note: In some cases we've kept more significant figures in d than we're entitled to. We've done this so that you can check your calculations if you want to reproduce the numbers yourself. We've used 6,400,000 m for R, and 0.62 for the conversion from kilometers to miles, although these are just rounded figures.

Location	h (in m)	d (in km)	d (in miles)
Standing ant	0.01	0.36	0.2
Your eye near ground	0.1	1.1	0.7
Person standing	2	5	3
Somewhat tall building	100	36	22
Burj Khalifa observatory	550	84	52
Pike's Peak, Colorado	4300	235	146
Commercial airplane	10,000	358	222
Space Station	400,000	2263	1403

Table B.1: The $d \approx \sqrt{2Rh}$ distances to the horizon, for various heights h.

Concerning the first entry in Table B.1, the standing ant would need to be at the shore of *very* still water. Even the tiniest ripples (near where the tangent line in Fig. B.1 touches the earth) would ruin our perfect-sphere assumption for the earth. As h gets larger, ripples (and eventually waves) can be ignored.

Mt. Everest is 8849 m tall, which is roughly the same as the 10,000 m commercial-airplane entry in the table. So you can see about 200 miles from the top of Mt. Everest, as you found in Exercise B.1. Or rather, you *could* see that far if there weren't other mountains around.

Depending on where the measurement is taken, the distance between the east and west coasts of the US is about 2500 miles. So when the Space Station is over the middle of the US, it is about 1250 miles from each coast. (Or actually a little more, since it is high up and needs to look diagonally downward.) Therefore, since the Space Station can see 1400 miles in either direction, it can (just barely) see both coasts at the same time.

Here's an easy formula to remember if you want to determine the distance d associated with a given height h. Let h be N meters. (So N is just a pure number without any units.) Then

$$d = \sqrt{2Rh} = \sqrt{2(6{,}400{,}000\,\text{m})(N\,\text{m})} \approx (3600\,\text{m})\sqrt{N}. \tag{B.11}$$

So we can write d as

$$d \approx \boxed{(3.6\,\text{km})\sqrt{N}} \quad \text{or} \quad \boxed{(2.2\,\text{miles})\sqrt{N}} \tag{B.12}$$

where we have multiplied by 0.62 to obtain the number of miles. Therefore, if the height h is N meters, you simply need to take the square root of that N value and then multiply by either 3.6 km or 2.2 miles, depending on how you want to express your answer. But remember that in both cases, N is the number of *meters*.

Note that if we square both sides of Eq. (B.8), we obtain $d^2 = 2Rh$. Dividing both sides by $2R$ then gives h in terms of d:

$$h = \boxed{\frac{d^2}{2R}} \tag{B.13}$$

This equation provides the answer to the question: If you want to see a given distance d, what does your height h need to be? (This is the opposite of our original question of finding d in terms of h.) If you want to see $d = 160$ km (100 miles), then Eq. (B.13) gives the required h as (we'll work with kilometers here, although you could very well use meters; there would just be some additional 0's in the numbers)

$$h = \frac{(160\,\text{km})^2}{2(6{,}400\,\text{km})} = 2\,\text{km} = 2000\,\text{m}. \tag{B.14}$$

This translates to about 6,600 feet, which is a fairly tall mountain. This case lies between the Burj Khalifa and Pike's Peak entries in Table B.1.

Exercise B.4 If you dig a straight tunnel from Boston to New York City (about 300 km apart), what is the depth h of the tunnel at its deepest point? (Make a guess before solving the problem.) *Hint*: Draw a picture and look for a useful right triangle. Ignore the h^2 term you encounter, as we did above. Assume the earth is a perfect sphere.

B.2 Advantages of letters

In the preceding section, we solved the distance-to-horizon problem twice, first by using numbers, and then by using letters (plugging in the numbers only at the end). The logic behind the solutions was the same, but they looked a bit different on paper. The technique of using letters instead of numbers is called solving a problem *symbolically*, which basically just means you're doing algebra (working with letters) instead of arithmetic (working with numbers).

If you're given a problem where the quantities are specified numerically, it is often advantageous to immediately change the numbers to letters, like replacing 6,400 km with R, and 100 m with h, in the horizon problem. You can then solve the problem in terms of the letters. After you obtain a symbolic answer (that is, one that involves letters), you can plug in the actual numerical values to obtain a numerical answer. There are many benefits of solving problems symbolically. Let's list them out:

1. Solving problems symbolically is quicker. It's much easier to multiply an R by an h by writing them down on a piece of paper next to each other, than it is to multiply their numerical values on a calculator. If solving a problem involves five or ten such operations, the time would add up if you performed all the operations on a calculator.

2. You are less likely to make a mistake. Numbers can get messy. It's very easy to mistype an 8 for a 9 in a calculator, but you're probably not going to miswrite a k for an h on a piece of paper. And even if you do, you'll quickly realize that it should be an h. You certainly won't just give up on the problem and deem it unsolvable because no one gave you the value of k!

3. You can do the problem once and for all. (We noted this benefit in the introduction to this appendix.) If someone comes along and says, oops, the value of h is actually 90 m instead of 100 m, then you don't need to solve the whole problem again. You can simply plug the new value of h into your symbolic answer. That's the beauty of working with letters. A symbolic answer is valid for *any* value of the letters you might want to plug in (well, subject to your assumptions, like the $h \ll R$ one for the $d \approx \sqrt{2Rh}$ result in Eq. (B.8)).

4. You can see the general dependence of your answer on the various parameters (letters). For example, you can see that the $d = \sqrt{2Rh}$ result in Eq. (B.8) increases as either R or h increases. (For short, we say in this case that "d grows with R and h.") Furthermore, you can see *how* d grows with R and h: It grows in a square-root manner for both (as we noted for h on page 402). So if you increase h by a factor of 100, you can see only 10 times as far. Equivalently, the square (instead of square root) behavior in the $h = d^2/2R$ result in Eq. (B.13) tells you that if you increase the distance d by a factor of 10, then you need to increase h by a factor of 100. There is *much* more information contained in a symbolic answer like Eq. (B.8) or (B.13) than in a numerical answer like Eq. (B.5).

As a bonus, symbolic answers nearly always look nice and pretty. Even in cases like Eq. (B.6) where the symbolic answer isn't super pretty, there is still a huge amount of information. Under the $h \ll R$ approximation, you can say that the h^2 term is very small compared with the $2Rh$ term and can therefore be ignored. This leaves you with the result in Eq. (B.8), which is in fact super pretty.

5. YOU CAN CHECK UNITS. Symbolic answers allow you to easily check units. In the $d = \sqrt{2Rh}$ result in Eq. (B.8), both R and h have units of meters (or feet, or whatever unit of length you're working with). So the units of the distance d are $\sqrt{\text{m} \cdot \text{m}} = \sqrt{\text{m}^2} = \text{m}$, which is correct. (In determining the units, we can ignore the numerical factor of 2, since it doesn't have any units.)

   If you make a mistake and obtain an answer of, say, $d = \sqrt{2R/h}$ with the h in the denominator, then you know it can't be correct, because the units are $\sqrt{\text{m}/\text{m}} = \sqrt{1} = 1$, where the 1 here simply means that $\sqrt{2R/h}$ doesn't have any units. This is incorrect, since the distance d must have units of meters. Similarly, if you accidentally dropped the R in $d = \sqrt{2Rh}$ and obtained an answer of $d = \sqrt{2h}$, this has the incorrect units of $\sqrt{\text{m}}$. If you ever end up with the wrong units, you'll know that you need to check over your work.

   Of course, the units will also work out (assuming you haven't made a mistake) if you solve a problem in terms of numbers instead of letters, as we did in Eqs. (B.3)–(B.5). You therefore can (and always should) also check the units of your answer when working with numbers. But again, solving a problem in terms of numbers instead of letters can often be a pain.

6. YOU CAN CHECK SPECIAL/EXTREME/LIMITING CASES. This benefit is so important that it gets its own subsection:

Checking special/extreme/limiting cases

This benefit goes hand-in-hand with the above "general dependence" benefit. Since symbolic answers allow you to see the dependence on the various letters, you can easily determine what your answer is (or at least how it behaves) in various special or extreme or limiting cases. For example, perhaps you can determine what your answer is when a particular letter equals zero (a special case). Or when it is very large (an extreme case). Or when two letters are equal to each other (another special case). Or when they are very close to each other but not exactly equal (a limiting case). And so on. For convenience, we'll often refer to the process of checking special/extreme/limiting cases simply as "checking limits" since it's shorter to say.

> Checking special/extreme/limiting cases is often referred to as "checking limits."

The usefulness of checking limits boils down to the following fact:

> It is generally the case that your intuition gives you information about what an answer should be in special/extreme/limiting cases, even if you don't have any intuition about general values of the letters. You should take advantage of this.

For example, I have no clue how far I can see to the horizon from a height of, say, 500 meters. But I *do* know for sure that I can see zero distance from zero height. (It's up to you whether you want to call this "intuition" or just an obvious fact.) And indeed, the $d = \sqrt{2Rh}$ result in Eq. (B.8) correctly equals zero when $h = 0$. If you accidentally replaced the h here with R and ended up with an answer of $\sqrt{2R^2}$, or if you simply forgot the h and ended up with $\sqrt{2R}$, then these answers don't equal zero when $h = 0$. ($\sqrt{2R}$ also has the wrong units.) So you'd know that you should go back and check over your work. Likewise if you accidentally wrote the result in Eq. (B.6) as, say, $\sqrt{2Rh + R^2}$. This doesn't equal zero when $h = 0$.

Checking limits is a quick and easy process that is readily available at your fingertips. All you need to do is think intuitively about what's going on in some limiting cases.

> You don't need to be a magician,
> Or cough up a hefty tuition.
> When you check an extreme,
> There's a nice simple scheme:
> Test your answer against intuition.

In the other extreme where h gets large, the $d = \sqrt{2Rh + h^2}$ result in Eq. (B.6) also gets large, which makes sense; d correctly grows as h grows. Note that for large h, the approximate $\sqrt{2Rh}$ result in Eq. (B.8) isn't valid, because it was derived under the assumption that h is much smaller than R. So we need to use the exactly correct $\sqrt{2Rh + h^2}$ expression here.

What if we assume that h is not only large, but also much larger than R? That is, $h \gg R$. Imagine that you're on the moon, looking at the earth. So this definitely isn't an "everyday" situation. In this $h \gg R$ case, the $2Rh$ term in Eq. (B.6) is small compared with the h^2 term. It is smaller by the factor $(2Rh)/(h^2)$, which equals $2R/h$. And this is very small if $h \gg R$. So to a reasonable approximation, we can ignore the $2Rh$ term in $\sqrt{2Rh + h^2}$, and we're left with just $d \approx \sqrt{h^2} = h$. (Ignoring the $2Rh$ term in the present $h \gg R$ case is the opposite of what we did when we ignored the h^2 term in the $h \ll R$ case in the derivation of Eq. (B.8).) This $d \approx h$ result makes sense, because from the moon, the distance d to the earth's horizon is approximately equal to the distance h to the nearest point on the earth; see Fig. B.3.

Actually, a better intuitive approximation for d in the $h \gg R$ case in Fig. B.3 is $h + R$, due to the extra distance of roughly R to reach the horizon, as opposed to just reaching the nearest point on the earth. Equivalently, the long leg d in the right triangle in Fig. B.3 is approximately equal to the hypotenuse $h + R$ since the triangle is very thin. However, we're assuming that R is much smaller than h here, so the distinction between h and $h + R$ isn't too important. But if you're interested, Exercise 11.23 shows how the $h + R$ approximation follows algebraically from the exact $\sqrt{2Rh + h^2}$ result when $h \gg R$.

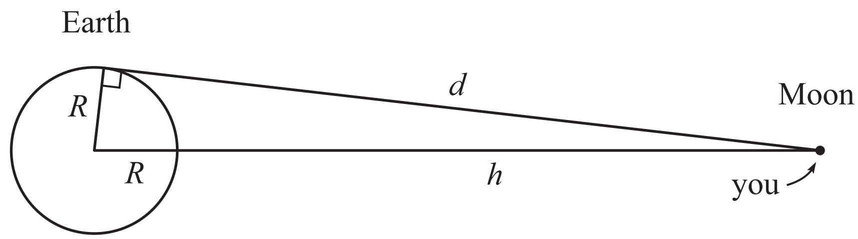

Figure B.3: If you are very far from the earth, the distance d you can see to the horizon is approximately equal to your height h above the earth.

As another example of checking limiting cases, consider the $2\pi r\epsilon + \pi\epsilon^2$ result in Eq. (A.53) in Exercise A.3, for the area of a ring between two circles. The answer must equal zero when the thickness ϵ of the ring is zero. And the above expression correctly has this property. Furthermore, when ϵ is very small, but not exactly zero, we can ignore the very small ϵ^2 term, in which case the expression reduces to $2\pi r\epsilon$. This is consistent with the fact that the area of the ring is (when ϵ is very small) essentially equal to the circumference $2\pi r$ times the thickness ϵ, as we saw in the solution to Exercise A.3. If you made a mistake and dropped the 2 and obtained an answer of $\pi r\epsilon + \pi\epsilon^2$, then although the $\epsilon = 0$ special case correctly yields zero, the ϵ-small-but-not-exactly-zero case doesn't yield $2\pi r$ times ϵ. So you'd know to go back and check over your work. No matter how many limiting cases you check that are correct, if you obtain even just one that isn't correct, then you know that your answer must be wrong.

Another example of checking a special case is the $a^8 - b^8 = (a^4+b^4)(a^2+b^2)(a+b)(a-b)$ factoring result in Eq. (4.42) in Example 4.6. In the special case where $a = b$, the lefthand side is zero. And the righthand side is correctly also zero. (Likewise for the $a = -b$ special case; both sides are zero.) If you forgot the $a - b$ factor on the right, then the righthand side wouldn't be zero, so you'd know you made a mistake in your factoring. Special cases are very useful for checking your answers, so please use them!

> When you arrive at a symbolic answer after solving a problem, you should *always* look for special/extreme/limiting cases to check.

Note well: You should look for cases to check, not because I'm telling you to(!), but rather because they will either (a) give you the definite information that your answer is incorrect (in which case you now know that you need to go fix it), or (b) allow you to feel a little more confident about your answer if you've checked a number of cases and they all agree with what you know must be true. Such is the case with the sum formulas in Eqs. (4.55)–(4.57). After checking those formulas for a number of small values of n, you will certainly feel more confident that they're actually correct.

Of course, checking special/extreme/limiting cases will never tell you that your answer is *definitely* correct. It's quite possible that you've produced an incorrect answer that just happens by luck to give the correct result for a number of special cases (although the more cases you check, the smaller the probability is that this happens). However, as we've noted, looking at a special case might very well tell you that your answer is *definitely incorrect*. If plugging in a special value for a letter gives an answer that doesn't agree with you intuition, then (assuming that your intuition is correct) you have obtained the irrefutable information that your answer is wrong. This seemingly dispiriting revelation is actually a *good* thing, because as mentioned above, at least you now know that you should go back and check over your work. This outcome certainly beats pressing onward in blissful ignorance, thinking that you have the correct answer!

In summary, there are many significant benefits to using letters instead of numbers. Numbers are messy and constraining to work with (just a single answer pops out), whereas letters are clean and liberating – you're free to make them be whatever you want in your symbolic answer. You should therefore embrace all of the wonderful benefits of letters and relish every opportunity to use them!

They strove to be mighty trend setters
And be free from numerical fetters.
Their motto on numbers?
"Reject what encumbers!
And bask in the glory of letters!"

B.3 Exercise solutions

1. Plugging $R = 6{,}400{,}000$ and $h = 9000$ into Eqs. (B.6) and (B.8) gives (using scientific notation; see the entry for that in the Glossary)

$$
\begin{aligned}
d_{\text{exact}} &= \sqrt{2Rh + h^2} = \sqrt{1.15281 \cdot 10^{11}\,\text{m}^2} = 339{,}531\,\text{m}, \\
d_{\text{approx}} &= \sqrt{2Rh} = \sqrt{1.152 \cdot 10^{11}\,\text{m}^2} = 339{,}411\,\text{m}.
\end{aligned}
\tag{B.15}
$$

The $\sqrt{2Rh}$ approximation is again a very good one. The error is only 120 m (about 1/8 of a kilometer), which is negligible compared with the 339 km (210 miles) that you can see (assuming that Mt. Everest is hypothetically next to an ocean).

2. In terms of R, plugging $h = R/16$ into Eqs. (B.6) and (B.8) gives

$$
\begin{aligned}
d_{\text{exact}} &= \sqrt{2Rh + h^2} = \sqrt{2R(R/16) + (R/16)^2} \\
&= \sqrt{R^2(1/8 + 1/256)} = R\sqrt{0.1289} = (0.3590)R, \\
d_{\text{approx}} &= \sqrt{2Rh} = \sqrt{2R(R/16)} \\
&= \sqrt{R^2(1/8)} = R\sqrt{0.125} = (0.3536)R.
\end{aligned}
\tag{B.16}
$$

With $R = 6{,}400$ km, these results become

$$
\begin{aligned}
d_{\text{exact}} &= (0.3590)(6{,}400\,\text{km}) = 2298\,\text{km}, \\
d_{\text{approx}} &= (0.3536)(6{,}400\,\text{km}) = 2263\,\text{km}.
\end{aligned}
\tag{B.17}
$$

The difference in these answers is only 35 km, which is about 0.015 (equivalently, 1.5%) of the exact 2298 km distance. So the approximation in Eq. (B.8) is still very good, even for the large (but still small compared with R) h value of the Space Station.

3. Consider the two angles that the radius makes with the tangent line. Under the assumption (which we'll end up showing is incorrect) that the tangent line is *not* perpendicular to the radius at the point of contact, one of the two angles must be larger than 90°, and one must be smaller than 90°, as we drew in Fig. B.2(b).

   Consider the angle that is smaller than 90°, and draw a right triangle containing that angle, as we drew in Fig. B.2(b). Since the leg of a right triangle is (due to the Pythagorean theorem) always shorter than the hypotenuse, which is R here, the vertical leg of the right triangle is smaller than R. This implies that point A must be *inside* the circle (because its distance from the center is less than the radius R). This contradicts the fact that every point on a tangent line lies *outside* the circle (except for the single point that lies on the circle). Therefore, since our assumption of non-perpendicularity leads to a false statement (that A is inside the circle), we conclude that the assumption must have been incorrect. The radius and tangent line must therefore in fact be perpendicular.

4. The setup is shown in Fig. B.4. The desired maximum depth is h, and the total distance from Boston to NYC is $d = 300$ km.

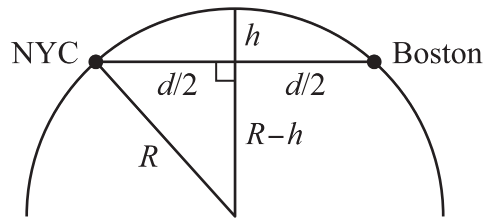

Figure B.4: The setup for determining the maximum depth h of a tunnel. The distance between Boston and NYC is exaggerated for clarity.

There is technically an ambiguity about whether 300 km is the straight-line distance or the curved distance along the surface of the earth. But it doesn't matter since these two distances are essentially the same. If the two cities were significantly farther apart, then we would have to worry about this issue. The straight-line distance is the useful one in Fig. B.4, so if (as would be the case in real life) we're given the curved distance, the first thing we'd have to do is somehow calculate the straight-line distance. But there's no need to do that in the present setup, since the two distances are basically the same, because 300 km is small compared with the 6,400 km radius of the earth. It's possible to show (by using trigonometry) that the two distances differ by only about 0.01%.

The right triangle in Fig. B.4 has legs $d/2$ and $R-h$, and hypotenuse R. So the Pythagorean theorem gives

$$(d/2)^2 + (R-h)^2 = R^2 \implies (d/2)^2 = R^2 - (R-h)^2 \implies d^2/4 = \cancel{R^2} - (\cancel{R^2} - 2Rh + h^2). \tag{B.18}$$

As we did in the distance-to-horizon problem in the text, we can ignore the h^2 term, because it is negligible compared with the $2Rh$ term. We're then left with

$$\frac{d^2}{4} = 2Rh \implies h = \boxed{\frac{d^2}{8R}} \tag{B.19}$$

Plugging in the Boston-NYC distance of 300 km gives

$$h = \frac{(300\,\text{km})^2}{8(6{,}400\,\text{km})} \approx 1.75\,\text{km} \approx 1.1\text{ miles}. \tag{B.20}$$

Is this answer larger or smaller than you expected? Personally, my first guess was that the tunnel would be deeper than this. But in retrospect, the earth is nearly flat on the scale of 300 km, so this small value of h is quite believable.

Note that our earlier result for the tower height h in Eq. (B.13) is 4 times the result for the depth h in Eq. (B.19). So if you build a tower in Boston tall enough to see NYC (the ground there, not just the skyline), and if you also dig a tunnel between the two cities, then the tower will be 4 times as tall as the tunnel is deep. Since we just found that the tunnel will be 1.75 km deep, the tower will be $4(1.75\,\text{km}) = 7\,\text{km}$ (or 4.3 miles) tall.