Chapter 6

Pythagorean theorem

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The *Pythagorean theorem* is one of the most beautiful theorems in mathematics. It is simple to state, easy to use, and highly accessible – it doesn't require a huge amount of mathematical machinery to prove. We'll be able to prove it (in numerous ways!) with what we've learned so far.

We'll begin by stating the basics of the Pythagorean theorem in Section 6.1, and then in Section 6.2 we'll discuss the general form of *Pythagorean triples*, which are triples of integers that satisfy the Pythagorean theorem. We'll then prove the theorem in many (seven!) ways in Section 6.3. Some of the proofs require nothing more than the $bh/2$ formula for the area of a triangle. And some don't even require that! Section 6.4 covers an interesting real-life application of the Pythagorean theorem, namely, how far you can see to the horizon from a tall building. Section 6.5 then presents many examples and exercises for practice. We'll end with a general discussion in Section 6.6 about the benefits of working with letters instead of numbers.

6.1 The theorem

The Pythagorean theorem deals with right triangles. To repeat a few things we mentioned in Chapter 5: Right triangles are ones that have a 90° angle (which is called a "right angle"). A 90° angle is simply what you have at the corner of a rectangle. The two sides that meet at the right angle are perpendicular to each other. These two perpendicular sides in a right triangle are called the *legs*. The third side (opposite the 90° angle) is called the *hypotenuse*. So in Fig. 6.1 the hypotenuse has length c, and the legs have lengths a and b. As with other words like "radius" and "circumference," the words "hypotenuse" and "leg" can refer to either the segment itself (as in the preceding sentence), or the length of the segment (as in "the hypotenuse is c"). The usage is generally clear from the context. The standard notation for a right angle is a little square, as we have drawn.

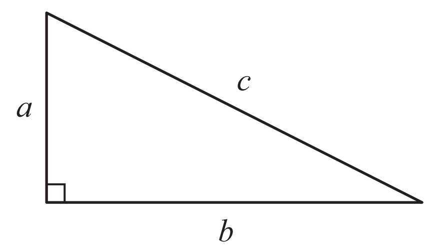

Figure 6.1

As mentioned in Section 5.4, the Pythagorean theorem states that the sides of a right triangle are related by

$$\boxed{a^2 + b^2 = c^2} \qquad \text{(Pythagorean theorem)} \tag{6.1}$$

This statement of the Pythagorean theorem was certainly known before Pythagoras' time, although it is unknown how much earlier. The date (and creator) of the first proof is also unknown. In any case, we can only wonder what Pythagoras' first encounter with the theorem looked like. . .

Pythagoras wept and despaired
As he added the legs and compared.
Then he jumped up with glee,
"Though they don't add to c,
It's a match if the lengths are all squared!"

We'll prove the theorem in Section 6.3 below, but there are a few things we should discuss first. If you draw a triangle with a random shape, the odds are that it won't be a right triangle. That is, most random sets of three numbers a, b, c don't satisfy Eq. (6.1). Only special sets do, and hence yield a right triangle. The simplest set of *integers* that satisfy the theorem is 3, 4, 5. These lengths produce a right triangle because

$$3^2 + 4^2 = 5^2 \quad \Longleftrightarrow \quad 9 + 16 = 25. \tag{6.2}$$

Most right triangles don't have integer lengths for all three sides. Or said in another way, if you pick integers for two sides of a right triangle, the third side probably won't be an integer. For example, if we pick the two legs to be 1 and 1, then the hypotenuse is given by

$$1^2 + 1^2 = c^2 \quad \Longrightarrow \quad c^2 = 2 \quad \Longrightarrow \quad c = \sqrt{2} \approx 1.414, \tag{6.3}$$

which isn't an integer. (This triangle is our old friend, the 45-45-90 right triangle.) Or if we pick the hypotenuse to be 8 and one leg to be 5, then the other leg is given by

$$a^2 + 5^2 = 8^2 \quad \Longrightarrow \quad a^2 + 25 = 64. \tag{6.4}$$

Subtracting 25 from both sides of this equation (as we learned in Section 4.5), and then taking the square root of both sides, gives

$$a^2 = 39 \quad \Longrightarrow \quad a = \sqrt{39} \approx 6.245, \tag{6.5}$$

which isn't an integer.

If all three sides of a right triangle are integers, then we call the set of these integers a *Pythagorean triple* (or just a *triple*, for short). People often list the integers of a triple inside parentheses, like "(a, b, c)." For example, in addition to the Pythagorean triple $(3, 4, 5)$ mentioned above, a few other triples are $(6, 8, 10)$, $(5, 12, 13)$, and $(8, 15, 17)$ because, as you can verify,

$$6^2 + 8^2 = 10^2, \qquad 5^2 + 12^2 = 13^2, \qquad 8^2 + 15^2 = 17^2. \tag{6.6}$$

A quick way of producing new triples from other known triples is to use the fact that any integer multiple of the three numbers in a triple yields three new numbers that are again a triple. This is true because if (a, b, c) is a triple, then we can multiple both sides of the Pythagorean theorem by s^2 (where s is an integer) to obtain another true statement. (If the two sides of an equation are equal, then multiplying these two equal quantities by the same number s^2 yields two new quantities that are again equal.) This multiplication by s^2 yields

$$a^2 + b^2 = c^2 \quad \Longrightarrow \quad s^2a^2 + s^2b^2 = s^2c^2 \quad \Longrightarrow \quad (sa)^2 + (sb)^2 = (sc)^2. \tag{6.7}$$

But this is just the statement that (sa, sb, sc) is a Pythagorean triple, as we wanted to show. For example, the $(6, 8, 10)$ triple mentioned above is the $(3, 4, 5)$ triple multiplied by $s = 2$.

This multiplication of each side of a right triangle by s and ending up with another right triangle makes intuitive sense. If you're given a right triangle, and if you scale it up uniformly by multiplying all of the sides by the same factor, then the new triangle has the same shape as the old one, so it's still a right triangle. The new triangle is *similar* to the old one (it has the same shape); recall the discussion of similarity in Section 5.4. Even if s isn't an integer, we'll still end up with a right triangle. But if the sides aren't integers, we don't call it a Pythagorean triple.

Note that the Pythagorean theorem in Eq. (6.1) is *symmetric* in a and b. That is, both a and b are raised to the same power (namely 2), and the two terms have the same coefficient (namely 1). This symmetry follows from the fact that it can't matter which leg you arbitrarily choose to label as a, and which one you label as

b. If someone claimed that the theorem took the form of, say, $a^2 + 2b^2 = c^2$, then you would get a different result for c if you switched your a and b labels. So this "theorem" can't be correct.

For example, if the two legs are 5 and 8, and if we label them as $a = 5$ and $b = 8$ (which we're free to do), then the $a^2 + 2b^2 = c^2$ "theorem" gives $5^2 + 2 \cdot 8^2 = c^2 \implies 153 = c^2 \implies c = \sqrt{153} = 12.4$. However, if we label the legs as $a = 8$ and $b = 5$ (which we're also free to do), the "theorem" gives $8^2 + 2 \cdot 5^2 = c^2 \implies 114 = c^2 \implies c = \sqrt{114} = 10.7$. But there can be only one value of c, of course. So the fact that our formula gives two different values means it can't be correct.

6.2 General form of triples

It turns out that there is a very simple and general way to produce Pythagorean triples, beyond the easy ones that are simply integer multiples of other triples. We claim that if we start with any two integers m and n, then the following three integers a, b, c are a Pythagorean triple, that is, they satisfy the Pythagorean theorem:

$$a = m^2 - n^2, \qquad b = 2mn, \qquad c = m^2 + n^2. \tag{6.8}$$

You can verify this claim by doing the following exercise.

Exercise 6.1 Show that the a, b, and c expressions in Eq. (6.8) satisfy Eq. (6.1), by calculating the sum $a^2 + b^2$ and then showing that the result equals c^2.

For integers m, n, who knew
That b is their product times 2?
And a? It's a fact:
Form the squares and subtract.
And then c? Instead add up the two.

The preceding exercise is the standard way of showing that the a, b, and c expressions in Eq. (6.8) form a Pythagorean triple. Here's another way. We want to show that $a^2 + b^2 = c^2$, and this relation is equivalent to (by subtracting a^2 from both sides) $b^2 = c^2 - a^2$. We can now invoke the handy difference-of-squares result from Eq. (3.22) to write $c^2 - a^2$ as $(c+a)(c-a)$. So our goal is to show that this product equals b^2. Plugging in the expressions for a and c from Eq. (6.8) gives

$$\begin{aligned} c^2 - a^2 &= (c+a)(c-a) \\ &= \Big((m^2 + \cancel{n^2}) + (m^2 - \cancel{n^2})\Big)\Big((\cancel{m^2} + n^2) - (\cancel{m^2} - n^2)\Big) \\ &= (2m^2)(2n^2) = 4m^2n^2 = (2mn)^2 = b^2, \end{aligned} \tag{6.9}$$

as desired.

Exercise 6.2 Show again that the a, b, and c expressions in Eq. (6.8) satisfy Eq. (6.1), by applying the difference-of-squares result like we just did, but now with the Pythagorean theorem written as $a^2 = c^2 - b^2$.

It turns out that not only does Eq. (6.8) generate Pythagorean triples, it generates *all* of them. That is, there are no triples that aren't of the form in Eq. (6.8); every triple has an associated (m, n) pair. The proof of this statement ("If three numbers are a triple, then they take the form of Eq. (6.8)") is more involved than our above proofs of the reverse statement ("If three numbers take the form of Eq. (6.8), then they are a triple"). So we'll just accept it here.

Table 6.1 lists the triples that Eq. (6.8) generates for various (m, n) pairs. Some of the triples are multiples of others. For example, (24, 10, 26) is 2 times (5, 12, 13), although in a different order.

		a	b	c
m	n	$m^2 - n^2$	$2mn$	$m^2 + n^2$
2	1	3	4	5
3	1	8	6	10
3	2	5	12	13
4	1	15	8	17
4	2	12	16	20
4	3	7	24	25
5	1	24	10	26
5	2	21	20	29
5	3	16	30	34
5	4	9	40	41

Table 6.1: Pythagorean triples

Exercise 6.3 Pick a few of the triples in Table 6.1 and verify that they do indeed satisfy the Pythagorean theorem.

If you stare at Table 6.1 long enough, a few things become clear, one of which is the following. Look at the cases where $n = m - 1$. So (m, n) takes the form of $(2, 1), (3, 2), (4, 3), (5, 4)$, etc. The a values associated with these pairs are, respectively, the odd numbers 3, 5, 7, and 9, which equal $m + n$. And in each case, you will observe that b and c differ by 1 and add up to a^2. For example, in the $(5, 4)$ case we have $40 + 41 = 9^2$. And in the $(4, 3)$ case we have $24 + 25 = 7^2$. The following example shows that this pattern holds for all of the $(m, n) = (m, m - 1)$ cases.

Example 6.1 For the $n = m - 1$ cases, show that a is odd and equals $m + n$. And show that b and c differ by 1 and add up to a^2.

Solution: If we plug $n = m - 1$ into the expressions for a, b, and c in Eq. (6.8), we obtain

$$\begin{aligned} a &= m^2 - (m-1)^2 = \cancel{m^2} - (\cancel{m^2} - 2m + 1) = 2m - 1, \\ b &= 2m(m-1) = 2m^2 - 2m, \\ c &= m^2 + (m-1)^2 = m^2 + (m^2 - 2m + 1) = 2m^2 - 2m + 1. \end{aligned} \tag{6.10}$$

We want to show four things:

- a is odd: Since a takes the form of $2m - 1$ where m is an integer, we see that a is indeed odd. (Even numbers take the form of $2m$, and odd numbers take the form of $2m - 1$. Or equivalently $2m + 1$.)
- $a = m + n$: Since $m + n = m + (m - 1) = 2m - 1$, we see that a is equal to $m + n$, as desired. Alternatively, the difference-of-squares result in Eq. (3.22) tells us that if $n = m - 1$, then

$$a = m^2 - n^2 = (m + n)(m - n) = (m + n)(1) = m + n, \tag{6.11}$$

  because the difference between m and n is 1.
- b and c differ by 1: The forms we found for b and c in Eq. (6.10) tell us that $c = b + 1$, so they do in fact differ by 1.

- $b + c = a^2$: The sum of b and c is

$$b + c = (2m^2 - 2m) + (2m^2 - 2m + 1) = 4m^2 - 4m + 1. \qquad (6.12)$$

This is indeed equal to a^2 since

$$a^2 = (2m - 1)^2 = 4m^2 - 4m + 1. \qquad (6.13)$$

Here are a few exercises, two involving numbers and two involving letters.

Exercise 6.4 The size of a rectangular computer screen is generally specified by giving the length of the diagonal line. What is the size of a screen that is 11.3 inches wide and 7.0 inches tall? Or 14.4 inches wide and 9.0 inches tall?

Exercise 6.5 An American football field is 100 yards long (excluding the end zones) and 53.33 yards (160 feet) wide. If you walk from one corner to the opposite one, what distance do you save by walking diagonally instead of along two sides?

Exercise 6.6 In Table 6.1, another thing you may have noticed is that for the $n = 1$ cases, a and c are obtained by taking half of b, squaring the result, and then adding or subtracting 1. For example, in the $(5, 1)$ case, b is 10. Half of this is 5, and 5^2 is 25. Adding or subtracting 1 then gives 24 and 26, which are indeed the a and c values. The concise way of stating this result is that a and c take the form of $(b/2)^2 \pm 1$. Explain why this is true by letting $n = 1$ in Eq. (6.8).

Exercise 6.7 Another way of stating the $n = m - 1$ result discussed above in Example 6.1 is the following. Take an odd number (call it a) and square it. Then add or subtract 1 from the result. And then divide each of these two numbers by 2. Call the results c and b. (So if we start with $a = 9$, we obtain 81, and then 82 and 80, and then 41 and 40. This is the last triple in Table 6.1.)

For an (a, b, c) triplet generated this way, calculate the sum $a^2 + b^2$ and then show that it equals c^2. *Hint*: Since a is an odd number, it takes the general form of $2k - 1$ where k is an integer. (This problem gets a little messy. You'll need to square a trinomial, but stick with it!)

6.3 Seven proofs of the theorem

Let's now prove the Pythagorean theorem – in seven different ways! We'll present all of these proofs as exercises, so that you'll have the chance to do them yourself. We don't want you to miss out on any of the fun! Some of the proofs are great examples of what algebra can do for you, while others (the third and fourth ones) are purely geometric and don't require any algebra at all. You might think it's excessive to include seven proofs here, but the strategies involved are varied and instructive. And besides, it's always hard to pass up any new opportunity to prove the Pythagorean theorem! Here's the statement of the theorem:

Pythagorean theorem: *If the sides of a right triangle are a, b, and c, with c being the hypotenuse, then* $a^2 + b^2 = c^2$.

Proof: And here are the proofs:

Exercise 6.8 (Proof 1) Prove the Pythagorean theorem by using the fact that the area of the overall square in Fig. 6.2 equals the sum of the areas of the four triangles plus the area of the smaller square. (There's a bit of an optical illusion in this figure. The sides of the overall square are indeed vertical and horizontal, even if they don't look it!)

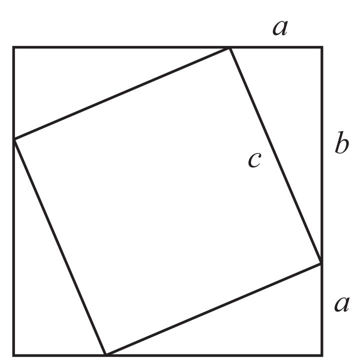

Figure 6.2

Exercise 6.9 (Proof 2) Fig. 6.3 shows a square with side length c subdivided into four right triangles with legs a and b (and hypotenuse c), along with a square in the middle with side length $b - a$. Prove the Pythagorean theorem by using the fact that (as in the preceding proof) the area of the overall square equals the sum of the areas of the four triangles plus the area of the smaller square.

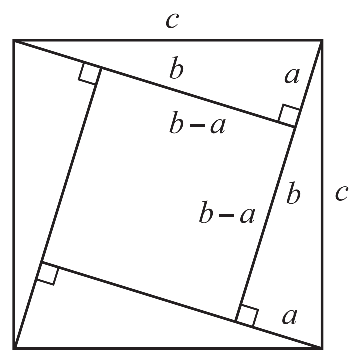

Figure 6.3

Exercise 6.10 (Proof 3) This proof requires no algebra; it's basically a geometry-only interpretation of the first proof above. Perhaps it shouldn't count as a separate proof, but it's so slick, I think it should. It's the quickest and simplest proof of them all.

Fig. 6.4 shows four shaded triangles inside a square. Show how to rearrange the triangles in a way that makes it clear that the area of the white region (which is c^2) also equals $a^2 + b^2$.

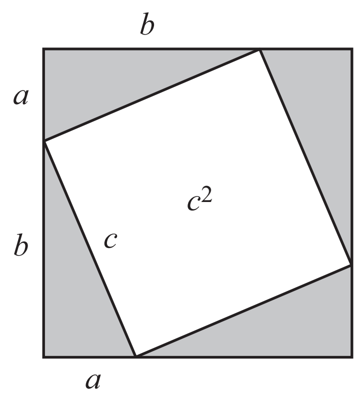

Figure 6.4

Exercise 6.11 (Proof 4) Here's another geometry-only proof. Fig. 6.5 shows a combo version of Figs. 6.2 and 6.3. Rearrange some of the shapes to show that $a^2 + b^2 = c^2$. We've given a hint by drawing two shaded squares with areas a^2 and b^2. (These shapes are indeed squares, since all sides are either a or b.)

Exercise 6.12 (Proof 5) The overall right triangle in Fig. 6.6 has side lengths a, b, and c. The altitude to the hypotenuse is drawn. Explain why the two smaller right triangles produced are similar to (that is, they have the same

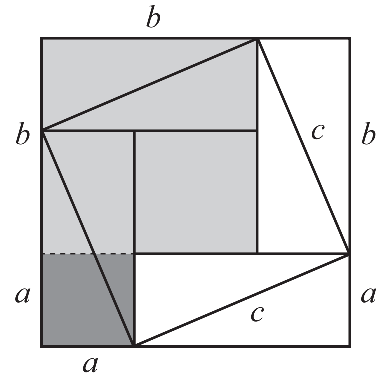

Figure 6.5

angles as) the overall right triangle. Then use this similarity to find the two lengths that c is divided into. The Pythagorean theorem will follow from the fact that these two lengths add up to c.

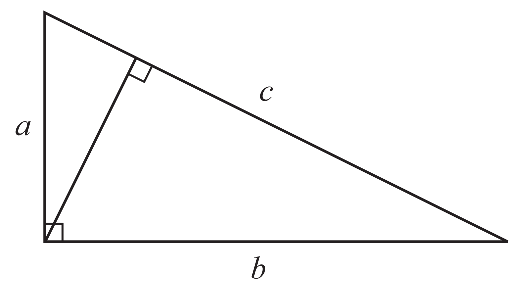

Figure 6.6

Exercise 6.13 (Proof 6) In the preceding proof, we found that the two sub-triangles in Fig. 6.6 are similar to the overall triangle. This similarity allows us to use the scaling results from Section 5.5 to determine how the areas of the sub-triangles are related to the area of the overall triangle. The Pythagorean theorem will follow from the fact that the two sub-areas add up to the overall area.

Exercise 6.14 (Proof 7) This final proof is how Euclid proved the Pythagorean theorem. It's a bit more involved than the preceding six proofs, and it relies on one fact that we haven't covered yet – the (entirely believable) "side-angle-side" (SAS) postulate, which says that if two triangles have two sides in common, along with the angle between them, then they are congruent (which is a fancy word for identical; they have the same shape and same size). If you play around with a few examples, you'll be convinced that this postulate is correct.

In Fig. 6.7, we have drawn squares on the sides of right triangle ABC. Proving the Pythagorean theorem is equivalent to showing that the sum of the areas of the two smaller squares ($a^2 + b^2$) equals the area of the big square (c^2).

Your task: First show that the triangles in each shaded pair in Fig. 6.7 are congruent, and then explain how their areas relate to the areas of the smaller squares, and also to the two rectangular sub-areas R_1 and R_2 of the large square, defined by the dashed line drawn. *Hint*: The area of a triangle is half the base times the height. Find some helpful bases and heights! For example, triangle A_1BA can be considered to have base A_1B and height A_1A_2.

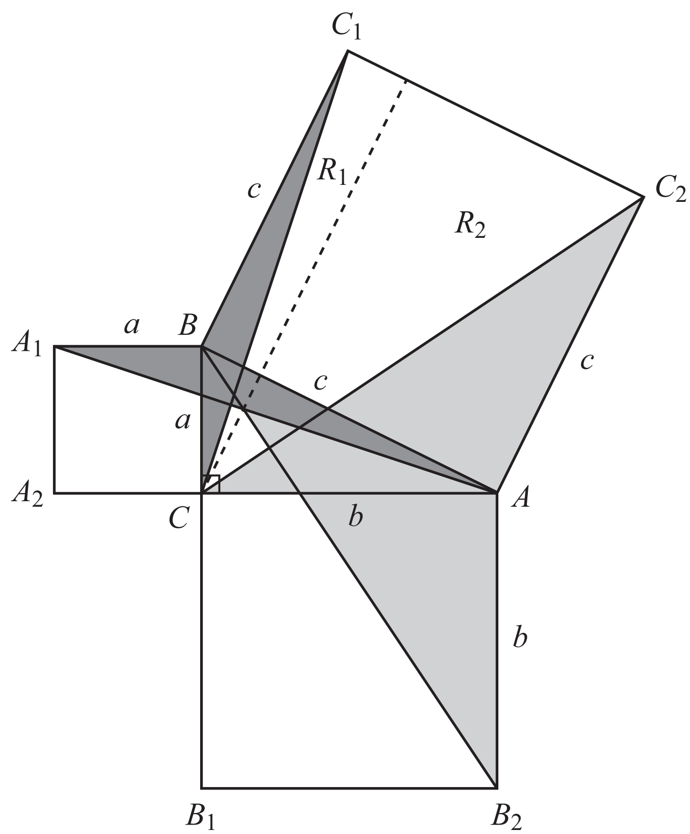

Figure 6.7

Having worked through all of these proofs, you can now say without a doubt that you understand the Pythagorean theorem! And that has its perks...

It's quick to spot which kids are cool,
As they saunter the halls of the school.
"*Who's* got the swagger? Us!
We know Pythagoras!
Sure, we're all square, but we rule!"

The converse

The Pythagorean theorem says, "If a triangle is right, then its side lengths satisfy $a^2 + b^2 = c^2$." The reverse statement (the converse) is also true:

Converse: *If a triangle's side lengths satisfy $a^2 + b^2 = c^2$, then the triangle is right.*

As with the "forward" direction of the theorem we proved above, there are many different ways to prove the converse. We'll present just one proof here.

We're starting with the assumption that $a^2 + b^2 = c^2$, and our goal is to show that the triangle is right. That is, we want to show that it *cannot* look like either of the triangles (obtuse or acute) in Fig. 6.8, where the b side is tilted. So for both of these possibilities, our goal is to show that x must be zero. That is, the x segment must in fact not exist; equivalently the top vertex is actually directly over the left end of the a side. We'll address the obtuse case here. (The acute case proceeds in the same manner, with only one small sign modification, as you can check.) This proof makes use of the "forward" direction of the Pythagorean theorem, so it assumes (quite correctly!) that we've already proved that.

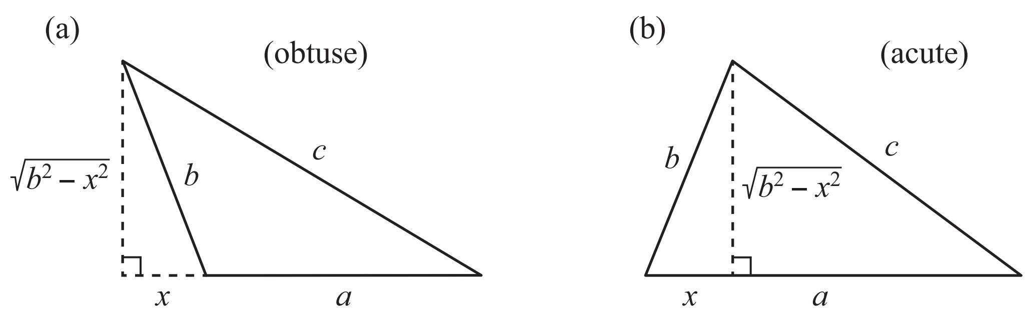

Figure 6.8

From the Pythagorean theorem, the vertical leg of the small right triangle in Fig. 6.8(a) has length $\sqrt{b^2 - x^2}$, as shown. The Pythagorean theorem applied to the overall big right triangle then gives

$$(a+x)^2 + \left(\sqrt{b^2 - x^2}\right)^2 = c^2$$
$$\implies \quad (a^2 + 2ax + \cancel{x^2}) + (b^2 - \cancel{x^2}) = c^2. \tag{6.14}$$

Subtracting c^2 from both sides yields

$$(a^2 + b^2 - c^2) + 2ax = 0 \quad \implies \quad 0 + 2ax = 0, \tag{6.15}$$

where we have used the given information that $a^2 + b^2 = c^2 \implies a^2 + b^2 - c^2 = 0$. We see that the product $2ax$ equals zero. And since neither 2 nor a is zero, it must be the case that $x = 0$. In other words, the b side is vertical, and the triangle is a right triangle, as we wanted to show.

6.4 Distance to the horizon

Here's an interesting real-life application of the Pythagorean theorem: If you're in a tall building with height h, how far can you see to the horizon? Let's assume that the building is at the ocean shore, so that we don't need to worry about trees and hills and such.

The basic setup is shown in Fig. 6.9. The height h we've drawn is very much exaggerated (being about $1/5$ of the earth's radius R), to make it easier to see what's going on. In reality, there's no chance that h (for any everyday-type scenario) would be comparable to the earth's radius R. Even at the altitude of the International Space Station (which is about 250 miles), h is only $1/16$ of R (which is about 4000 miles).

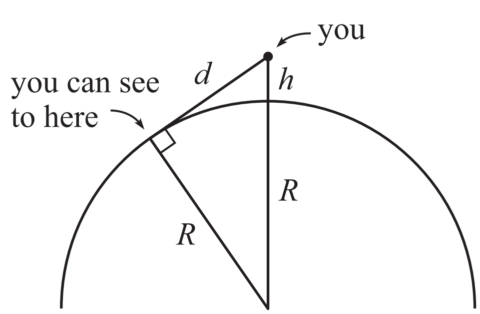

Figure 6.9

Our goal is to find the distance d in the figure, in terms of the known quantities h and R. Now, the desired distance you can see to the horizon is slightly ambiguous. Do we mean the straight-line distance d drawn, or do we mean the curved distance along the surface of the earth? Fortunately it doesn't matter, because these two distances are essentially equal for any reasonable (not excessively large) value of h. But for concreteness let's say that our goal is to find the straight-line distance d.

The straight line representing the distance d is tangent to the earth. We'll talk about "tangents" below in Exercise 6.24, but in short, a tangent line is one that just barely skims the circle. The tangent line is in fact the line of sight we're concerned with, because if you look at an angle that is slightly too high, you'll be looking at a point in the sky; and if you look at an angle that is slightly too low, you'll be looking at a nearby point on the ground (which therefore won't be the farthest point you can see). So we are indeed concerned with the tangent line – the cutoff between the ground and the sky. We'll see in Exercise 6.24 that a tangent line is always perpendicular to the radius at the point of contact with the circle. Let's just accept this (quite believable) fact for now. We therefore have the right triangle shown in Fig. 6.9, which means that we can apply the Pythagorean theorem.

Let's do a numerical example first, and then we'll find the general solution in terms of letters. We'll pick h to be

$$h = 100 \text{ meters}, \tag{6.16}$$

which is about 330 feet – a reasonably tall building. We could work with any unit of length (feet, yards, meters, etc.), but we'll choose the metric system's meters because other lengths in it (like kilometers) are obtained by multiplying by simple powers of 10. A kilometer is 1000 meters. (The prefix "kilo" means 1000.) The abbreviations for meter and kilometer are "m" and "km." A meter is about 39.4 inches, which is a little more that a yard (3 feet, or 36 inches).

In addition to assuming we're at the ocean shore (so that we don't need to worry about trees and hills), we'll work in the approximation where the earth is a perfect sphere. It actually isn't; it bulges a little at the equator because it's spinning. The radius of the earth varies from about 6,356 km at the poles to 6,378 km at the equator. There's no need for that level of accuracy here, so we'll just round these values up to 6,400 km. Hence

$$R = 6{,}400\,\text{km} \;\; \text{(or equivalently 6,400,000 meters)}, \tag{6.17}$$

which is the same as the 4000 miles we stated above. This follows from the fact that there are 1609 meters in a mile, and hence about 1.6 kilometers in a mile (a mile is the larger of the two units). So to go from km to miles, you divide by 1.609, or equivalently multiply by 0.62. (A 10 km race is 6.2 miles.) This checks: $(6400)(0.62) \approx 4000$, and $(4000)(1.6) = 6400$.

We can now apply the Pythagorean theorem. In Fig. 6.9, the three sides of our right triangle are the desired distance d, the radius $R = 6{,}400{,}000\,\text{m}$, and the hypotenuse $R + h = 6{,}400{,}100\,\text{m}$. The Pythagorean theorem therefore gives

$$d^2 + (6{,}400{,}000\,\text{m})^2 = (6{,}400{,}100\,\text{m})^2. \tag{6.18}$$

Subtracting $(6{,}400{,}000\,\text{m})^2$ from both sides of this equation gives

$$d^2 = (6{,}400{,}100\,\text{m})^2 - (6{,}400{,}000\,\text{m})^2 = 1{,}280{,}010{,}000\,\text{m}^2. \tag{6.19}$$

Taking the square root of both sides then gives

$$d = \sqrt{1{,}280{,}010{,}000\,\text{m}^2} \approx 35{,}800\,\text{m} \approx 36\,\text{km}. \tag{6.20}$$

You can therefore see about 36 kilometers (or about $(36)(0.62) = 22$ miles) from a 100-meter building. That's quite far!

Using letters

You undoubtedly noticed that the above calculation contained some large numbers, which were somewhat of a pain. The numbers would have been smaller if we had chosen to work with kilometers instead of meters (the lengths would have been $R = 6400\,\text{km}$ and $R + h = 6400.1\,\text{km}$), but numbers are still often a hassle to work with. So let's now solve the problem algebraically, that is, in terms of letters. There are significant advantages to working with letters, as we'll spell out in Section 6.6.

In terms of letters, applying the Pythagorean theorem to the triangle in Fig. 6.9 gives (in place of Eq. (6.18) with numbers)

$$\begin{aligned} d^2 + R^2 = (R+h)^2 \quad &\Longrightarrow \quad d^2 + \cancel{R^2} = \cancel{R^2} + 2Rh + h^2 \\ &\Longrightarrow \quad d = \sqrt{2Rh + h^2}, \end{aligned} \tag{6.21}$$

where we have subtracted R^2 from both sides, and then taken the square root of both sides, to obtain the last line. This $\sqrt{2Rh + h^2}$ result is the general answer to the problem. For any values of R and h we're given, we can simply plug them into $\sqrt{2Rh + h^2}$ to obtain the desired distance d. You can check that if you plug in the $R = 6{,}400{,}000\,\text{m}$ and $h = 100\,\text{m}$ values we used above, you will reproduce Eq. (6.20).

Eq. (6.21) is a nice clean result. But we can go one step further to obtain an even cleaner result. In a real-life situations, the h^2 term is *much* smaller than the $2Rh$ term. It is smaller by the factor $h/2R$ (since $2Rh \cdot h/2R = h^2$), which is very small for any everyday value of h. If you're in a tall building with height $h = 100\,\text{m}$ (330 feet), then

$$\frac{h}{2R} = \frac{100\,\text{m}}{2(6{,}400{,}000\,\text{m})} = \frac{1}{128{,}000} \approx 8 \cdot 10^{-6}. \tag{6.22}$$

Even at the height of a commercial airplane (about 10,000 m, or 33,000 feet), the value of $h/2R$, which is 100 times larger than for 100 m (or 330 feet), is still only $1/1280$. So to a good approximation we can simply ignore the h^2 term in Eq. (6.21) and say that

$$d \approx \sqrt{2Rh}. \tag{6.23}$$

This is an extremely clean result! Now, whenever you derive an approximate answer like we just did, you gain something and you lose something. You lose some truth, of course, because your new answer is an approximation and therefore technically not correct (although the error becomes very small in the appropriate limit – small h here). But you gain some aesthetics. Your new answer is invariably much cleaner (often involving only one term), which makes it much easier to see what's going on.

For example, a quick look at Eq. (6.23) tells you that d does *not* grow linearly with h (that is, it isn't directly proportional to h), but instead grows like the *square root* of h. So if you want to make d be, say, 5 times larger, then you need to make h be 25 times larger. Simply increasing h by a factor of 5 won't do it. Similarly, if you want to increase d by a factor of 10, you need to increase h by a factor of 100. Or said in a slightly different way, if you increase h by a factor of 100, you increase d by a factor of only 10. Simple relations like these between d and h aren't obvious from looking at the correct (but not as simple) result in Eq. (6.21).

How close is the approximate answer in Eq. (6.23) to the exact answer in Eq. (6.21) when $R = 6{,}400{,}000$ and $h = 100$? Plugging in the numbers gives (the first result here is just a repeat of Eq. (6.20), without the rounding)

$$\begin{aligned} d_{\text{exact}} &= \sqrt{2Rh + h^2} = \sqrt{1{,}280{,}010{,}000\,\text{m}^2} = 35{,}777.23\,\text{m}, \\ d_{\text{approx}} &= \sqrt{2Rh} = \sqrt{1{,}280{,}000{,}000\,\text{m}^2} = 35{,}777.09\,\text{m}. \end{aligned} \tag{6.24}$$

We see that our $\sqrt{2Rh}$ approximation is a *very* good one. No one could possibly care about an error of $0.14\,\text{m} = 14\,\text{cm}$, when we're talking about distances of roughly 36 km. There's truly no harm in ignoring the comparatively tiny h^2 term. Even at the top of a tall mountain, the h^2 term will have no noticeable effect (when compared with the distance you can see).

When looking afar from peak,
Remember this useful technique:
In finding the distance,
Ignore the existence
Of terms whose effect is quite weak.

Exercise 6.15 The height of the International Space Station is $h \approx R/16$, which equals 250 miles, or 400 km. In terms of R, find the exact and approximate answers for d in Eqs. (6.21) and (6.23). Then plug in $R = 6{,}400\,\text{km}$ to find the actual distances. By how much do they differ?

What does Eq. (6.23) give for some other values of h? If you're standing at the shore of the ocean, let's say that your eyes are at a height of $h = 2$ m. We then have

$$d = \sqrt{2Rh} = \sqrt{2(6{,}400{,}000\,\text{m})(2\,\text{m})} = 5060\,\text{m} \approx 5\,\text{km} \approx 3\text{ miles}. \qquad (6.25)$$

On one hand, this might seem like a large distance, given that your eyes are only 2 meters above the ground. But on the other hand, this distance is much smaller than it would be if the earth were flat!

The values of $d \approx \sqrt{2Rh}$ for a few other values of h are listed in Table 6.2. The h's are given in meters, and the d's are given in both kilometers and miles. Remember, to go from kilometers to miles, we multiply by 0.62. Are the various values of d larger or smaller than what you expected? (In some cases we've kept more significant figures than we're entitled to. We've done this so that you can check your calculations if you want to reproduce the numbers yourself. We've used 6,400,000 m for R, and 0.62 for the conversion from kilometers to miles, although these are just rounded figures.)

Location	h (in m)	d (in km)	d (in miles)
Standing ant	0.01	0.36	0.2 (1200 ft)
Your eye near ground	0.1	1.1	0.7
Person standing	2	5	3
Somewhat tall building	100	36	22
Burj Khalifa observatory	550	84	52
Pike's Peak, Colorado	4300	235	145
Commercial airplane	10,000	358	222
Space Station	400,000	2263	1403

Table 6.2: Distances to the horizon

Concerning the first entry in Table 6.2, the standing ant would need to be at the shore of *very* still water. Even the tiniest ripples (near where the tangent line in Fig. 6.9 touches the earth) would ruin our perfect-sphere assumption for the earth. As h gets larger, ripples (and eventually big waves) can be ignored.

Mt. Everest is about 8850 m tall, which is roughly the same as the 10,000 m commercial-airplane entry in the table. So you can see about 200 miles from the top of Mt. Everest. Or rather, you *could* see that far if there weren't other mountains around.

Depending on where the measurement is taken, the straight-line distance between the east and west coasts of the US is about 2500 miles. Therefore, since the Space Station can see 1400 miles in either direction, for a total of 2800 miles, it can (just barely) see both coasts at the same time.

Here's an easy formula to remember if you want to determine the distance d associated with a given height h. Let h be N meters. (So N is just a pure number without any units.) Then

$$d = \sqrt{2Rh} = \sqrt{2(6{,}400{,}000\,\text{m})(N\,\text{m})} \approx (3600\,\text{m})\sqrt{N}. \tag{6.26}$$

So we can write d as

$$d \approx (3.6\,\text{km})\sqrt{N} \quad \text{or} \quad (2.2\,\text{miles})\sqrt{N}. \tag{6.27}$$

So whatever the height h is in meters, you simply need to take the square root of that and then multiply by either 3.6 km or 2.2 miles, depending on how you want to express your answer. But remember that in either case, N is the number of *meters*.

Note that if we square both sides of Eq. (6.23), we obtain $d^2 = 2Rh$. Dividing both sides by $2R$ (and switching sides) then gives h in terms of d:

$$h = \frac{d^2}{2R}. \tag{6.28}$$

This equation gives the answer to the question: If you want to see a given distance d, what does your height h need to be? (This is the opposite of our original question of finding d in terms of h.) If you want to see $d = 160$ km (100 miles), then Eq. (6.28) gives the required h as (we'll work entirely with kilometers here, although you could very well use meters; there would just be some additional 0's in the numbers)

$$h = \frac{(160\,\text{km})^2}{2(6{,}400\,\text{km})} = 2\,\text{km} = 2000\,\text{m}. \tag{6.29}$$

This translates to about 6,600 feet, which is a fairly tall mountain. This case lies between the Burj Khalifa and Pike's Peak entries in Table 6.2.

Exercise 6.16 If you dig a straight tunnel from Boston to New York City (about 300 km apart), what is the depth h of the tunnel at its deepest point? (Make a guess before solving the problem.) *Hint*: Draw a picture and look for a useful right triangle. Ignore the h^2 term you will encounter, as we did above. Assume the earth is a perfect sphere.

6.5 Many examples and exercises

Let's now do a number of examples and exercises. As always, you are encouraged to treat the examples as exercises and try them on your own first.

Example 6.2 In Example 5.4 we used the area of an octagon to produce an estimate of π. Let's obtain another estimate here, now using the perimeter. We'll need to find the octagon's side length; the altitude we drew in Fig. 5.46 is helpful for this.

Solution: Fig. 6.10 shows a 45° pie piece (just the triangle, without the rounded end). Letting the radius be 1 as usual, the 45-45-90 triangle in the left part of the pie piece has legs with lengths $1/\sqrt{2}$ (from Section 5.4), as shown.

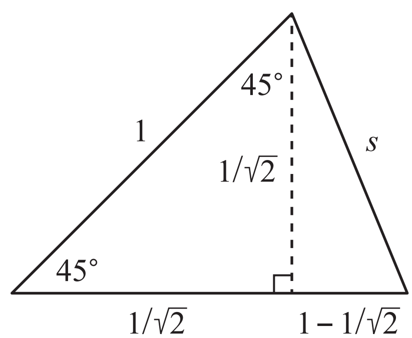

Figure 6.10

The important point to realize now is that the bottom side of the pie piece has length 1, because it's also a radius. So a length $1 - 1/\sqrt{2}$ is left for the short segment on the right side, as shown. The Pythagorean theorem applied to the right triangle in the right part of the pie piece then gives the octagon's side length s as

$$s^2 = \left(1/\sqrt{2}\right)^2 + \left(1 - 1/\sqrt{2}\right)^2 = 1/2 + \left(1 - 2/\sqrt{2} + 1/2\right) = 2 - \sqrt{2}. \quad (6.30)$$

Taking the square root of both sides yields $s = \sqrt{2 - \sqrt{2}} \approx 0.765$, and then multiplying this by 8 to find the perimeter of the octagon gives $P_{\text{oct}} \approx 6.12$. The $C_{\text{circ}} > P_{\text{oct}}$ statement that the circumference of the circle is greater than the perimeter of the octagon is then $2\pi > 6.12$, or equivalently $\pi > 3.06$, after dividing by 2. This value is about 97% of the true $\pi \approx 3.14$ value, so the approximation is a very good one.

Example 6.3 Show that if the side lengths of a right triangle are equally spaced (that is, if the hypotenuse exceeds the longer leg by the same amount that the longer leg exceeds the shorter leg), then the sides are in the ratio of 3 : 4 : 5.

Solution: Let d be the difference between successive sides, and let the longer leg be a. Then the side lengths are $a - d$, a, and $a + d$. So the Pythagorean theorem gives

$$\begin{aligned} a^2 + (a-d)^2 = (a+d)^2 \implies & a^2 = (a+d)^2 - (a-d)^2 \qquad (6.31) \\ \implies & a^2 = (\cancel{a^2} + 2ad + \cancel{d^2}) - (\cancel{a^2} - 2ad + \cancel{d^2}) \\ \implies & a^2 = 4ad \implies a = 4d \implies d = a/4. \end{aligned}$$

The second-to-last equation is obtained by dividing both sides of the previous one by a, and then the last equation is obtained by further dividing by 4, and then switching sides. (Or we could have simply divided by $4a$ in a single step.)

The $a - d$ leg of the triangle is therefore $a - d = a - a/4 = 3a/4$. And the hypotenuse is $a + d = a + a/4 = 5a/4$. So the three sides are $3a/4$, a, and $5a/4$. Scaling all of these up by a factor of 4 gives sides of $3a$, $4a$, and $5a$. These are indeed in the ratio of 3 : 4 : 5.

REMARKS:

1. Before doing any work on this problem, we already knew that a $(3, 4, 5)$ right triangle has equally spaced lengths, since $5 - 4 = 4 - 3$. What we did in this solution was show that there are no other ratios (that is, no other shapes) that also have this property.

2. Let's be more precise about how we solved Eq. (6.31). When we arrived at the $a^2 = 4ad$ equation, what we technically should have done was to get everything on one side of the equation, and then factor. So subtracting $4ad$ from both sides gives $a^2 - 4ad = 0$, and then factoring this gives $a(a - 4d) = 0$. There are two ways the lefthand side can be zero. One is for a to be zero. However, although $a = 0$ is a solution to our mathematical equation, it isn't a solution to our problem, because the side lengths $a - d$, a, and $a + d$ are then $-d$, 0, and d. And since side lengths must be positive, we therefore reject this solution, even though it does mathematically satisfy $a^2 + b^2 = c^2$.

The other solution is the one we're concerned with. The binomial $a - 4d$ is zero when $a = 4d$, or equivalently when $d = a/4$, as we found above. When we divided by a above in Eq. (6.31), we were tacitly (and correctly) assuming that a couldn't be zero.

3. In the above solution, we let the longer leg be a. What if we instead let the shorter leg be a? If d is again the common spacing, the sides are now a, $a + d$, and $a + 2d$. So the Pythagorean theorem gives

$$\begin{aligned} a^2 + (a+d)^2 &= (a+2d)^2 \\ \Longrightarrow \quad a^2 + (a^2 + 2ad + d^2) &= a^2 + 4ad + 4d^2. \end{aligned} \tag{6.32}$$

Getting all of the terms over on the righthand side by subtracting $2a^2$, $2ad$, and d^2 from both sides gives

$$0 = 3d^2 + 2ad - a^2. \tag{6.33}$$

To solve this equation for d in terms of a, we can use the factoring method we learned in Section 4.2.3. After a little guessing and checking with FOIL, we find that Eq. (6.33) can be factored into

$$0 = (3d - a)(d + a). \tag{6.34}$$

The first binomial on the right side is zero when $3d = a \Longrightarrow d = a/3$, and the second is zero when $d = -a$. This second root isn't allowed, because the side lengths a, $a + d$, and $a + 2d$ are then a, 0, and $-a$. And as we noted in the preceding remark, the side lengths must be positive.

The $d = a/3$ solution is the one we're concerned with. The longer leg is then $a + d = a + a/3 = 4a/3$, and the hypotenuse is $a + 2d = a + 2 \cdot a/3 = 5a/3$. So the three sides are a, $4a/3$, and $5a/3$. Scaling all of these up by a factor of 3 gives sides of $3a$, $4a$, and $5a$, which are in the ratio of $3 : 4 : 5$. ♣

Example 6.4 In Fig. 6.11 circles with radii a and b lie on top of a line (that is, they are tangent to it) and touch each other at a single point. What is the distance BA between the points of contact on the line? (Use the fact that a tangent line is perpendicular to the radius at the point of contact; see Exercise 6.24 below.)

Solution: The key is the shaded right triangle in the figure. The hypotenuse is the sum of the radii, so it equals $a + b$, as shown. And the vertical leg is

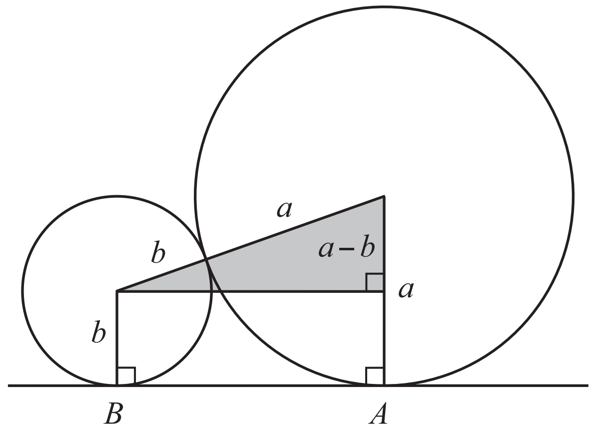

Figure 6.11

the difference of the radii, so it equals $a - b$. (This is how much higher one center is than the other.) The horizontal leg is the desired distance BA, so the Pythagorean theorem gives

$$\begin{aligned} (BA)^2 + (a-b)^2 &= (a+b)^2 \\ \Longrightarrow \quad (BA)^2 &= (a+b)^2 - (a-b)^2 \\ &= (\cancel{a^2} + 2ab + \cancel{b^2}) - (\cancel{a^2} - 2ab + \cancel{b^2}) \\ &= 4ab \\ \Longrightarrow \quad BA &= \sqrt{4ab} = 2\sqrt{ab}. \end{aligned} \tag{6.35}$$

Note that the algebraic steps here are the same as in Exercise 3.9.

In the special case where $a = b$, Eq. (6.35) gives $BA = 2\sqrt{a^2} = 2a$. This makes sense, because we have two equally sized circles sitting next to each other, so BA spans the sum of the two (equal) radii.

Remark: The quantity $\sqrt{ab}$ is called the *geometric mean* (GM) of a and b. Another type of mean is the *arithmetic mean* (AM), which is just the average $(a+b)/2$. (We'll talk about these means in Chapter 12.) Now, the hypotenuse of a right triangle, which is $a+b$ in the present setup, is always greater than or equal to each leg, in particular the horizontal leg which we just found equals $2\sqrt{ab}$ here. (They are equal if the other leg $a - b$ is zero, which happens if $a = b$. The "triangle" is then just a flat line, which admittedly isn't much of a triangle.) So we have

$$\text{hypotenuse} \geq \text{leg} \quad \Longrightarrow \quad a + b \geq 2\sqrt{ab} \quad \Longrightarrow \quad \frac{a+b}{2} \geq \sqrt{ab}, \tag{6.36}$$

where we have divided both sides by 2. This result tells us that the arithmetic mean is always greater than or equal to the geometric mean (with equality occurring if $a = b$). This statement is called the "AM-GM inequality." You can test it for various values of a and b. For example, if $a = 12$ and $b = 3$, then the AM is $(12+3)/2 = 7.5$, and the GM is $\sqrt{12 \cdot 3} = \sqrt{36} = 6$. And 7.5 is indeed greater than 6.

With no mention of triangles, the above proof of the AM-GM inequality essentially boils down to the following sequence of inequalities:

$$(a+b)^2 \geq (a+b)^2 - (a-b)^2 \quad \Longrightarrow \quad (a+b)^2 \geq 4ab \quad \Longrightarrow \quad \frac{a+b}{2} \geq \sqrt{ab}. \tag{6.37}$$

The first of these inequalities follows from the fact that $(a-b)^2$, being a square, is always greater than or equal to zero (independent of which of a or b is larger). And since we're subtracting it from $(a+b)^2$ on the righthand side, the result must be less than or equal to $(a+b)^2$. The second inequality comes from the cancelations in Eq. (6.35). And the third inequality is obtained by taking the square root of both sides and then dividing both sides by 2.

In terms of the right triangle in Fig. 6.11, the lefthand side of the first inequality in Eq. (6.37) is the square of the hypotenuse, and the righthand side is the square of the BA leg (from the Pythagorean theorem usage in Eq. (6.35)). ♣

Example 6.5 Fig. 6.12 shows a right triangle with sides a, b, and c. The inscribed circle is drawn, and the contact points with the three sides divide them into lengths m, n, and r, as shown. (The lower-right lengths on sides a and b are in fact equal to the radius r, because of the square that arises from the fact that a radius is always perpendicular to a tangent; see Exercise 6.24 below. Also, the two m lengths shown are indeed equal, as are the two n lengths, because the two tangents drawn from a given point have the same length; again see Exercise 6.24. We'll just accept these facts here.)

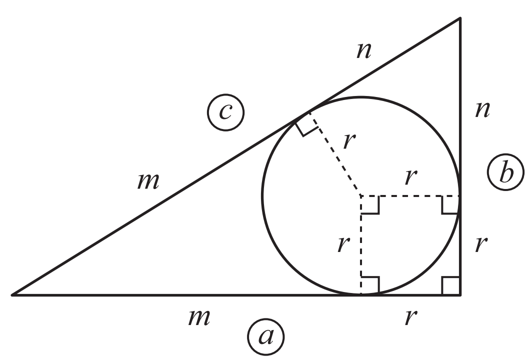

Figure 6.12

(a) Write down and simplify the statement of the Pythagorean theorem, when written in terms of m, n, and r.

(b) Find the area of the triangle in terms of m, n, and r.

(c) Use your result from part (a) to rewrite the area, and show that it equals mn (the product of the lengths into which the hypotenuse is divided).

Solution:

(a) The side lengths are $m + r$, $n + r$, and $m + n$, so the Pythagorean theorem gives

$$\begin{aligned} (m+r)^2 + (n+r)^2 &= (m+n)^2 \\ \Longrightarrow \quad (\cancel{m^2} + 2mr + r^2) + (\cancel{n^2} + 2nr + r^2) &= (\cancel{m^2} + 2mn + \cancel{n^2}) \\ \Longrightarrow \quad 2mr + 2nr + 2r^2 &= 2mn \\ \Longrightarrow \quad mr + nr + r^2 &= mn. \end{aligned} \tag{6.38}$$

(b) Since the legs (the base and height of the triangle) have lengths $a = m + r$ and $b = n + r$, the area of the triangle is

$$A = \frac{ab}{2} = \frac{(m+r)(n+r)}{2} = \frac{mn + mr + nr + r^2}{2}. \tag{6.39}$$

(c) From Eq. (6.38), the sum of the last three terms in the numerator of Eq. (6.39) is mn. Substituting this in, the area in Eq. (6.39) becomes

$$A = \frac{mn + (mr + nr + r^2)}{2} = \frac{mn + mn}{2} = \frac{\cancel{2}mn}{\cancel{2}} = mn, \tag{6.40}$$

as desired.

The above solution to this problem wasn't too long, but let's now spend some time discussing various things further.

REMARKS:

1. The simplicity of the above mn result for the area suggests that there might be a clean geometric way of seeing why it's true, without needing to do any algebra. And indeed, if we consider the m and n segments on the legs (instead of on the hypotenuse) we can draw a suggestive figure. Fig. 6.13 shows the idea.

   The goal is to show that the mn area of the m-by-n rectangle in the upper-left part of the figure equals the area of triangle ABC, or equivalently

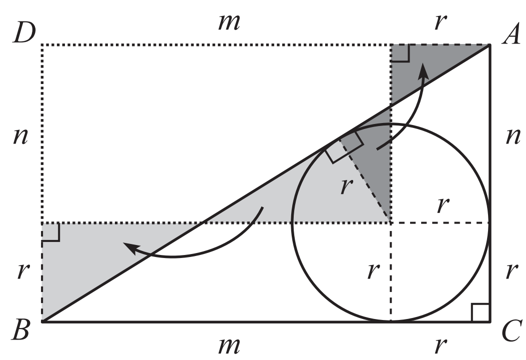

Figure 6.13

triangle ABD. And this is indeed the case, because the m-by-n rectangle can be transformed into triangle ABD by moving the shaded triangles as shown. (The like-shaded right triangles are congruent, because they are (1) similar due to the common angle where they touch, and also the common right angle, and hence the common third angle, and (2) they have the same size due to the common r leg.)

2. Since Eq. (6.38) tells us that $mn = mr + nr + r^2$, the above $A = mn$ result can also be written as

$$A = mn = mr + nr + r^2 = (m + n + r)r = Sr, \tag{6.41}$$

where S is the "semiperimeter" (half the perimeter). That is, $S = P/2 = (2m + 2n + 2r)/2 = m + n + r$. This $A = Sr$ result actually holds for all triangles, not just right triangles. To see why, consider Fig. 6.14, where triangle ABC is divided into sub-triangles BCD, CAD, and ABD. These triangles have bases a, b, and c, and they all have the same altitude r. So the sum of their areas (which is the area of triangle ABC) is

$$\frac{ar}{2} + \frac{br}{2} + \frac{cr}{2} = \frac{a + b + c}{2} \cdot r = \frac{P}{2}r = Sr, \tag{6.42}$$

as desired.

She found a new way to express
A triangle's A with success.
The method prescribed:
Take the circle inscribed,
And then multiply r by the S.

Going one step further, this $A = Sr$ result actually holds for all polygons (not just triangles) that have an inscribed circle, that is, one that tangentially touches all sides. (A randomly drawn polygon with four or more

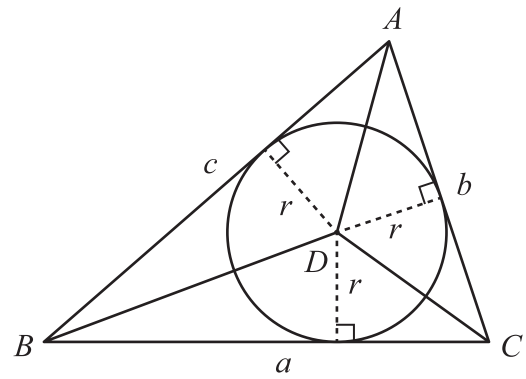

Figure 6.14

sides doesn't in general have this inscribed-circle property. Only special polygons do.) The triangle result in Eq. (6.42) is a special case of the more general $A = (P/2)r = Sr$ result for polygons in Eq. (5.51) in the solution to Exercise 5.24 (where r was equal to 1).

3. Working backwards through the steps of this exercise, we can actually produce another (an 8th!) proof of the Pythagorean theorem. We'll start with two expressions for the area (the standard $bh/2$ one in Eq. (6.39), and the Sr semiperimeter one in Eq. (6.42)) and equate them:

$$Sr = \frac{bh}{2} \quad \Longrightarrow \quad (m+n+r)r = \frac{(m+r)(n+r)}{2}. \tag{6.43}$$

We'll now proceed through a series of steps that will turn this equation into the Pythagorean theorem. Expanding the products and multiplying both sides by 2 gives

$$2mr + 2nr + 2r^2 = mn + mr + nr + r^2. \tag{6.44}$$

Subtracting $mr + nr + r^2$ from both sides then yields

$$mr + nr + r^2 = mn. \tag{6.45}$$

Multiplying both sides by 2 and then adding $m^2 + n^2$ to both sides gives

$$2mr + 2nr + 2r^2 + m^2 + n^2 = 2mn + m^2 + n^2. \tag{6.46}$$

Finally, grouping the terms in a helpful manner yields

$$\begin{aligned}(m^2 + 2mr + r^2) + (n^2 + 2nr + r^2) &= m^2 + 2mn + n^2 \\ \Longrightarrow \quad (m+r)^2 + (n+r)^2 &= (m+n)^2 \\ \Longrightarrow \quad a^2 + b^2 &= c^2,\end{aligned} \tag{6.47}$$

which is the statement of the Pythagorean theorem, as desired.

Admittedly, the above sequence of steps would be a bit out of the blue if we hadn't already solved the exercise in the "forward" direction. But having already performed those steps, along with knowing what we were aiming for (the second line in Eq. (6.47)), things were reasonably predictable. Once we produced Eq. (6.45), we just needed to proceed through Eq. (6.38) in reverse.

In addition to the $bh/2$ and Sr expressions for the area, there is also the mn one, which is justified via Fig. 6.13. Equating any two of these three expressions will produce Eq. (6.45) (as you can quickly verify) and therefore likewise produce the Pythagorean theorem. In other words, the equalities $bh/2 = mn$ and $Sr = mn$ are the starting points of two more proofs. If you want to count these as distinct new proofs, we're now up to 10 of them! ♣

Exercise 6.17 A rectangular box has length a, width b, and height c. What is the length of the diagonal between two opposite corners? (You will need to use the Pythagorean theorem *twice*.)

Exercise 6.18 A large cylindrical storage tank with diameter d and height h has a spiral staircase, as shown in Fig. 6.15, which runs once around the cylinder as it climbs the height h. (The slope of the staircase is uniform.) What is the length of the staircase? In the special case where $d = h$, what is the length in terms of h? *Hint*: A cylinder is a "flat" space, in the sense that you can make one out of a piece of paper without ripping the paper.

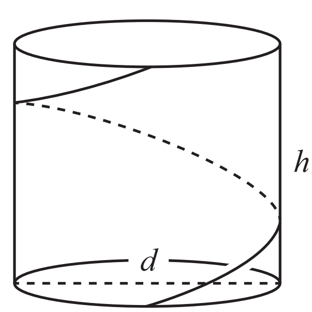

Figure 6.15

Exercise 6.19 A pendulum with length R swings back and forth, moving in the arc of a circle. When it is a distance x off to the side, as shown in Fig. 6.16, what is its height y above the lowest point, in terms of x and R? Assume that y is much smaller than R (even though we haven't drawn it that way), and make an appropriate approximation. Some helpful lines are drawn.

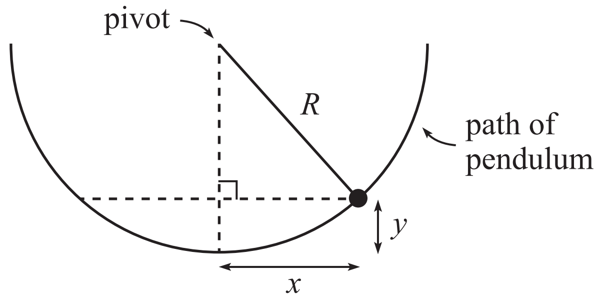

Figure 6.16

Exercise 6.20 In addition to being the smallest Pythagorean triple, a 3-4-5 right triangle shows up in a very simple geometric setup. Fig. 6.17 shows two identical large circles, both with radius 1, tangent to each other and to a line. A small circle is then drawn tangent to the two large circles and the line. Find the radius r of the small circle, and then show that the right triangle drawn has a 3-4-5 shape.

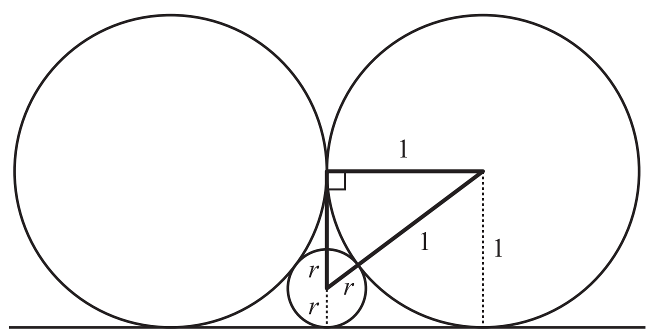

Figure 6.17

Exercise 6.21 In Exercise 5.22 we used the area of a dodecagon to produce an estimate of π. Obtain another estimate here, now using the perimeter. You will need to find the dodecagon's side length; the altitude we drew in Fig. 5.53 is helpful for this.

Exercise 6.22 The goal of this exercise is to generalize the procedure in Example 6.2 and Exercise 6.21.

(a) In Fig. 6.18, AC is a side of an n-gon inscribed in a circle with radius 1, and AB and BC are sides of a $2n$-gon. Assume that you somehow already know the $AC = a_n$ side length of the n-gon. Your task: Show that the $AB = BC = a_{2n}$ side length of the $2n$-gon is given in terms of a_n by

$$a_{2n} = \sqrt{2 - 2\sqrt{1 - a_n^2/4}}\,. \tag{6.48}$$

(You will need to find the x and y lengths shown.)

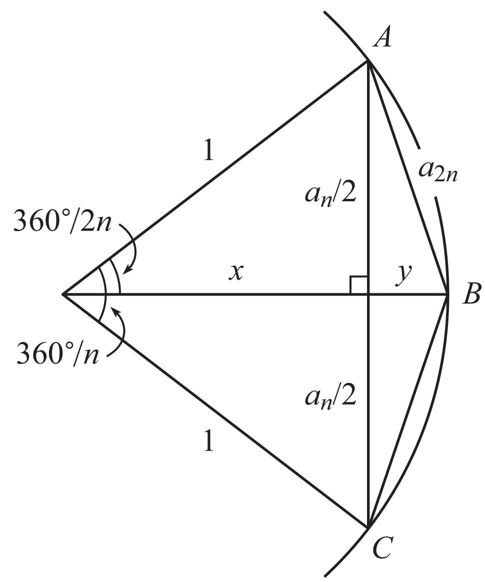

Figure 6.18

(b) Using the known $a_4 = \sqrt{2}$ side length of a square inscribed in a circle with radius 1, show that the above formula reproduces the result in Example 6.2 for the a_8 side of an octagon.

Going one step further, what is a_{16}? And what is the resulting estimate of π? (It's easier to work in terms of decimals here, so just plug your square roots into a calculator.) You can keep going and obtain estimates of π by using a 32-gon and a 64-gon, etc., if you wish!

Exercise 6.23

(a) Show that if you inscribe a triangle in a circle, with a diameter being one of the sides, then the triangle is a right triangle (with the diameter as the hypotenuse). *Hint*: Make use of the isosceles triangles in Fig. 6.19(a), after explaining why they are in fact isosceles.

(b) Now prove the "reverse" statement: Show that if you circumscribe a circle around a right triangle, then the hypotenuse is a diameter of the circle. *Hint*: Your goal is to show that the midpoint of the hypotenuse is

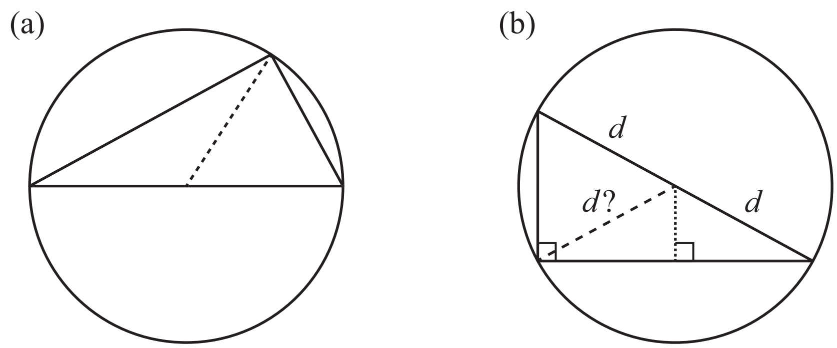

Figure 6.19

the center of the circle, which is equivalent to showing that its distance to the right-angled vertex (the long-dashed line drawn in Fig. 6.19(b)) is the same as the (common) distance d to the two other vertices. A helpful vertical (short-dashed) line is drawn. Look for some similar triangles.

Exercise 6.24 A *tangent* is a line that touches a circle at exactly one point. That is, it just barely touches the circle. The word "tangent" can be used as a noun, as in the preceding sentence, and also as an adjective, as in "This line is tangent to the circle."

We've encountered tangents on a few occasions already. This exercise contains two theorems about them. You might consider these theorems to be fairly obvious, but it's still good to formally prove them.

(a) Prove that a tangent to a circle is perpendicular to the radius at the point of contact, as shown in Fig. 6.20(a). *Hint*: You can prove this by making use of the left/right symmetry in the picture. Or you can see what incorrect result the Pythagorean theorem would imply if the radius and tangent *weren't* perpendicular.

(b) From a given point P outside a circle, draw the two tangent lines to the circle. Prove that these two tangents have the same length, as shown in Fig. 6.20(b).

Exercise 6.25 (This exercise is an extension of Example 6.4.) If we add a third circle to the setup in Fig. 6.11, as shown in Fig. 6.21, what is its radius r, in terms of the other two radii a and b? *Hint*: Apply the result from Example 6.4 multiple times, and use the fact that the segments along BA must add up properly. Solving for r in the equation you obtain will involve a few algebraic steps.

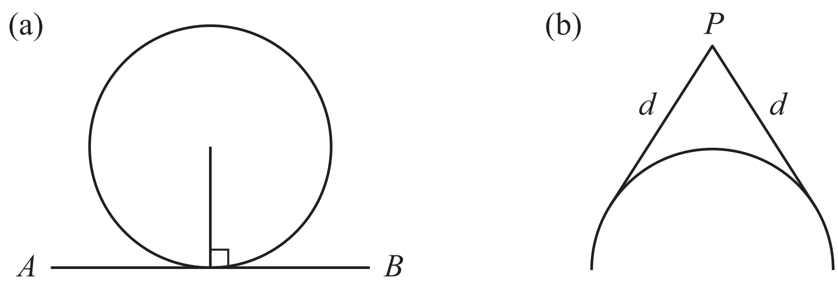

Figure 6.20

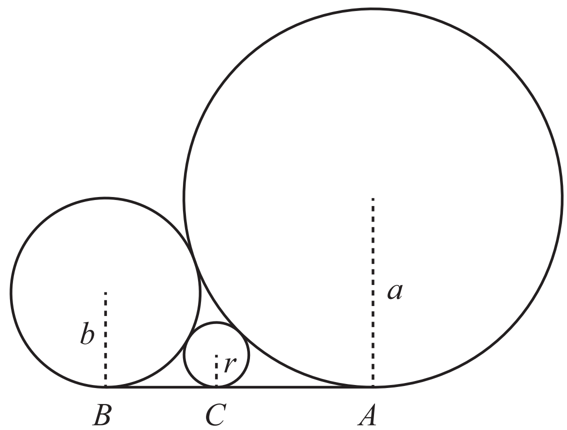

Figure 6.21

Exercise 6.26 (This exercise is very similar to Example 6.4.) Fig. 6.22 shows a circle with radius b centered at B. From an arbitrary point A outside the circle, the tangents (the dashed lines) are drawn; let their length be a. A circle with radius a is then drawn, centered at A. Find the length d of the common tangent to the two circles. *Hint:* You can quickly determine two sides of the shaded triangle.

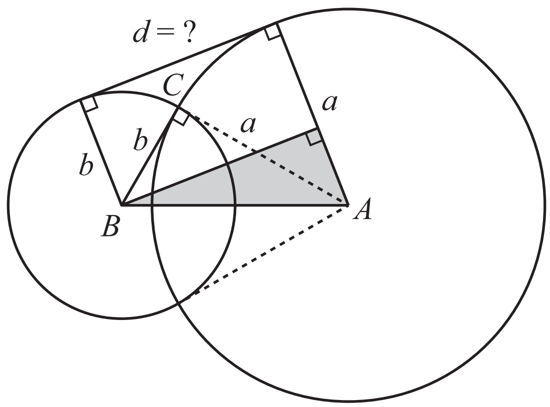

Figure 6.22

6.6 Benefits of using letters

In Section 6.4 we solved the distance-to-horizon problem twice, first using numbers, and then using letters (plugging in the numbers only at the end). The logic behind the solutions was the same, but they looked a bit different on paper. The technique of using letters instead of numbers is called solving a problem *symbolically*, which basically just means you're doing algebra (with letters) instead of arithmetic (with numbers).

If you're solving a problem where the quantities are specified numerically, it is often advantageous to immediately change the numbers to letters (like replacing 6,400 m with R, and 100 m with h in the horizon example). You can then solve the problem in terms of the letters. After you obtain a symbolic answer, you can plug in the actual numerical values to obtain a numerical answer. There are many benefits of solving problems symbolically. And now that you have algebra at your fingertips, you should take advantage of these benefits. Let's list them out.

- IT IS QUICKER. It's much easier to multiply an R by an h by writing them down on a piece of paper next to each other, than it is to multiply their numerical values on a calculator. If solving a problem involves five or ten such operations, the time would add up if you performed all the operations on a calculator.

- YOU ARE LESS LIKELY TO MAKE A MISTAKE. Numbers can get messy. It's very easy to mistype an 8 for a 9 in a calculator, but you're probably not going to miswrite a k for an h on a piece of paper. But even if you do, you'll quickly realize that it should be an h. You certainly won't just give up on the problem and deem it unsolvable because no one gave you the value of k!

- YOU CAN DO THE PROBLEM ONCE AND FOR ALL. If someone comes along and says, oops, the value of h is actually 90 m instead of 100 m, then you won't need to do the whole problem again. You can simply plug the new value of h into your symbolic answer. That's the beauty of working with letters. A symbolic answer is valid for *any* value of the letter you might want to plug in (well, subject to any approximations, like the $h \ll R$ one in the horizon example).

- YOU CAN SEE THE GENERAL DEPENDENCE OF YOUR ANSWER ON THE VARIOUS PARAMETERS (LETTERS). For example, you can see that the $d = \sqrt{2Rh}$ result in Eq. (6.23) increases as either R or h increases. (For short, we say in this case that "d grows with R and h.") Furthermore, you can see *how* d grows with R and h: It grows in a square-root manner for both. So if you increase h by a factor of 100, you can see only 10 times as far. Equivalently, the square (instead of square root) behavior in the $h = d^2/2R$ result in Eq. (6.28) tells us that if we increase the distance d by a factor of 10, then we need to increase h by a factor of 100. There is *much* more information contained in the symbolic answer in Eq. (6.23) than in the numerical answer in Eq. (6.20).

As a bonus, symbolic answers nearly always look nice and pretty. Even in cases like in Eq. (6.21) where the symbolic answer isn't super pretty, there is still a huge amount of information. Under the $h \lll R$ approximation, you can say that the h^2 term is very small compared with the $2Rh$ term, which means that you can ignore it. This leaves you with the result in Eq. (6.23), which is in fact super pretty.

- YOU CAN CHECK SPECIAL/EXTREME CASES. (This is a long bullet point, since it's so important.) This benefit goes hand-in-hand with the previous "general dependence" advantage. Since symbolic answers allow you to see the dependence on the various letters, you can easily determine what your answer is (or at least how it behaves) in various special or extreme cases. For example, perhaps you can determine what your answer is when a particular letter equals zero. Or when it is very large. Or when two letters are equal to each other. And so on.

  It is often the case that your intuition gives you information about what the answer should be in special/extreme cases, even if you don't have any intuition about general values of the letters. You should take advantage of this. For example, I have no clue how far I can see to the horizon from a height of, say, 500 meters. But I *do* know for sure that I can see zero distance from zero height. (It's up to you whether you want to call this "intuition" or just an obvious fact.) And indeed, the $d = \sqrt{2Rh}$ result in Eq. (6.23) correctly equals zero when $h = 0$. If you accidentally replaced the h here with R and ended up with an answer of $\sqrt{2R^2}$, or if you simply forgot the h and ended up with $\sqrt{2R}$, then these answers don't equal zero when $h = 0$. So you'd know that you needed to go back and check over your work. Likewise if you accidentally wrote the result in Eq. (6.21) as, say, $\sqrt{2Rh + R^2}$.

  You don't need to be a magician,
  Or cough up a hefty tuition.
  When you check an extreme,
  There's a nice simple scheme:
  Test your answer against intuition!

In the other extreme, where h gets very large, the $d = \sqrt{2Rh + h^2}$ result in Eq. (6.21) also gets very large, which makes sense; d correctly grows as h grows. (For large h, the $\sqrt{2Rh}$ result in Eq. (6.23) isn't valid, since it was derived under the assumption that h is much smaller than R. So we need to use the $\sqrt{2Rh + h^2}$ expression here.) In particular, if h is much larger than R (imagine that you're on the moon, looking at the earth), then the $2Rh$ term is small compared with the h^2 term (it is smaller by the factor $2R/h$). So to a reasonable approximation, we can ignore the $2Rh$ in $\sqrt{2Rh + h^2}$, in which case we're left with $d \approx \sqrt{h^2} = h$. This makes sense; from the moon, the distance d to the earth's horizon is approximately equal to the distance h to the nearest point on the earth; see Fig. 6.23.

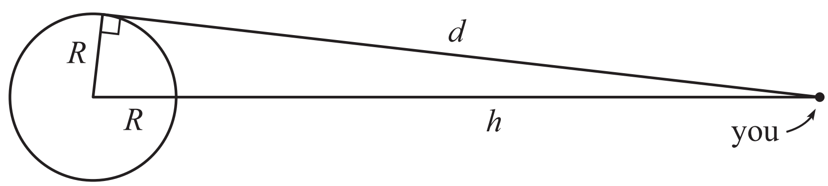

Figure 6.23

A better intuitive approximation for d is $h + R$, due to the extra distance of roughly R to reach the horizon, as opposed to just the nearest point on the earth. Equivalently, the long leg d in Fig. 6.23 is approximately equal to the hypotenuse $h + R$ since the right triangle is very thin. (However, we're assuming R is much smaller than h, so the distinction between h and $h+R$ isn't too important.) If you made a mistake and obtained a d of, say, $\sqrt{2Rh + 2h^2}$, then ignoring the $2Rh$ term would yield an approximate answer of $\sqrt{2}h$. When h is large, this answer is much larger than either of our approximate intuitive answers (h or $h + R$). It therefore can't be correct. So you'd know to go back and check over your work.

As another example, consider the $2La + 2Wa + 4a^2$ result in Eq. (5.5) in Example 5.1. In the special case where $a = 0$, the frame has no width, so the area clearly has to be zero. And the above answer is indeed zero when $a = 0$. If you made a mistake and accidentally replaced the first a with an L, yielding $2L^2 + 2Wa + 4a^2$, then you'd know this couldn't be correct, because it equals $2L^2$, instead of zero, when $a = 0$.

Likewise for the $2\pi ra + \pi a^2$ result in Eq. (5.17) for the area of a ring between two circles. The answer must equal zero when the thickness a of the ring is zero, and the above expression correctly has this property. Furthermore, when a is very small, but not exactly zero, we can ignore the very small a^2 term, in which case the expression reduces to $2\pi ra$. This is consistent with the fact that the area of the ring is (when a is very small) essentially equal to the circumference $2\pi r$ times the thickness a, as we saw in Eq. (5.19). If you made a mistake and dropped the 2 and obtained an answer of $\pi ra + \pi a^2$, then although the $a = 0$ special case correctly yields zero, the a-small-but-not-exactly-zero case doesn't yield $2\pi r$ times a. So you'd know to go back and check over your work. No matter how many special cases you check that are correct, if you obtain even just one that isn't correct, then you know that your answer must be wrong.

Another example of checking a special case is the $a^8 - b^8 = (a^4 + b^4)(a^2 + b^2)(a+b)(a-b)$ factoring result in Eq. (4.38) in Example 4.6. In the special case where $a = b$, the lefthand side is zero. And the righthand side is correctly also zero. (Likewise for the $a = -b$ special case; both sides are zero.) If you forgot the $a - b$ factor on the right, then the righthand side wouldn't be zero, so you'd know you made a mistake in your factoring. As another example, in the solution to Example 6.4 we noted that the $BA = 2\sqrt{ab}$ result in Eq. (6.35) correctly reduces to $2a$ in the special case where $a = b$.

Bottom line: When you arrive at an answer after solving a problem, you should *always* look for special/extreme cases to check. And you should do this not because I'm telling you to(!), but rather because it will either (a) give you the definite information that your answer is incorrect (in which case you now know you need to go fix it), or (b) allow you to feel a little more confident about your answer if you've checked a number of special/extreme cases and they all agree with what you know must be true. Such is the case with the sum formulas in Exercises 4.16, 4.17, and 4.18. After checking those formulas for a number of small values of n, you will certainly be more confident that they're actually correct.

Of course, checking special/extreme cases will never tell you that your answer is *definitely* correct. It's quite possible that you've produced an incorrect answer that just happens by luck to give the correct answer for a number of special cases. However, as we've noted, looking at a special/extreme case might very well tell you that your answer is *definitely incorrect*. If plugging in a special value for a letter gives an answer that doesn't agree with you intuition, then (assuming that your intuition is correct) you have obtained the irrefutable information that your answer is wrong. This seemingly dispiriting information is actually a *good* thing, because as mentioned above, at least you now know that you should go back and check over your work. This outcome certainly beats pressing onward in blissful ignorance, thinking that you have the correct answer, when in fact you don't!

- You can check units. In addition to checking special/extreme cases, symbolic answers also allow you to easily check units. In the $d = \sqrt{2Rh}$ result in Eq. (6.23), both R and h have units of meters (or feet, or whatever unit of length you're working with). So the units of the d are $\sqrt{\text{m} \cdot \text{m}} = \sqrt{\text{m}^2} = \text{m}$, which is correct. (In determining the units, we can ignore the numerical factor 2, since it doesn't have any units.)

  If you made a mistake and obtained an answer of $\sqrt{2R/h}$, with the h in the denominator, then you'd know it had to be wrong, because the units are $\sqrt{\text{m}/\text{m}} = \sqrt{1} = 1$, where the 1 here simply means that $\sqrt{2R/h}$ doesn't have any units. This is incorrect, since d must have units of meters. Similarly, if you accidentally dropped the R and obtained an answer of $\sqrt{2h}$, this has units of $\sqrt{\text{m}}$, which is incorrect.

  Of course, the units will also work out (assuming you don't make a mistake) if you solve a problem in terms of numbers instead of letters, as we saw in Eqs. (6.18)–(6.20). You therefore can (and should) also check the units of your answer when working with numbers. But again, solving a problem in terms of numbers instead of letters can often be a pain.

In summary, there are many significant benefits of using letters instead of numbers. And now that you've had lots of practice with algebra, you're able to work with letters at will. So take advantage of all of their wonderful benefits – they'll make your life much more pleasant!

They strove to be mighty trend setters,
And be free from numerical fetters.
Their motto on numbers?
"Reject what encumbers!
And bask in the glory of letters!"

6.7 Exercise solutions

1. Plugging the a and b expressions from Eq. (6.8) into Eq. (6.1) gives

$$\begin{aligned} a^2 + b^2 &= (m^2 - n^2) + (2mn)^2 \\ &= (m^4 - 2m^2n^2 + n^4) + 4m^2n^2 \\ &= m^4 + 2m^2n^2 + n^4 \\ &= (m^2 + n^2)^2 \\ &= c^2, \end{aligned} \tag{6.49}$$

   as desired.

2. Since $c^2 - b^2$ equals $(c + b)(c - b)$, our goal is to show that this product equals a^2. Plugging in the expressions for b and c from Eq. (6.8) gives

$$\begin{aligned} c^2 - b^2 &= (c + b)(c - b) \qquad (6.50) \\ &= \Big((m^2 + n^2) + 2mn\Big)\Big((m^2 + n^2) - 2mn\Big) \\ &= (m + n)^2(m - n)^2 = \big((m + n)(m - n)\big)^2 = (m^2 - n^2)^2 = a^2, \end{aligned}$$

   as desired. In the last line, we used the difference-of-squares result in reverse.

3. We'll just check a few of the triples here. The following relations are all true:

$$\begin{aligned} 7^2 + 24^2 = 25^2 &\iff 49 + 576 = 625, \\ 21^2 + 20^2 = 29^2 &\iff 441 + 400 = 841, \\ 9^2 + 40^2 = 41^2 &\iff 81 + 1600 = 1681. \end{aligned} \tag{6.51}$$

4. From the Pythagorean theorem, the diagonal of 11.3-by-7 rectangle is

$$\sqrt{11.3^2 + 7^2} = \sqrt{176.7} = 13.3. \tag{6.52}$$

   And the diagonal of a 14.4-by-9.0 rectangle is

$$\sqrt{14.4^2 + 9^2} = \sqrt{288.4} = 17. \tag{6.53}$$

   For the first of these, people usually just call it a 13-inch screen, although technically it's 13.3.

   There are many (an infinite number of) other shapes of rectangles, with different width-to-height ratios (called the "aspect ratio"), that also have diagonals of 13.3 and 17. However, computers generally stick to a few common aspect ratios, like 16 : 10 (equivalently 8 : 5), 16 : 9, and 4 : 3.

5. If you walk along two sides, the distance (in yards) is simply 100+53.33 = 153.33. From the Pythagorean theorem, the length of the diagonal is

$$\sqrt{100^2 + 53.33^2} = \sqrt{12{,}844} = 113.33. \tag{6.54}$$

So you save 153.33 − 113.33 = 40 yards by walking diagonally. The diagonal isn't that much longer than the long side (113 vs. 100), so you save almost as much as the short side (40 vs. 53).

Interestingly, if you've ever wondered how big an acre is, it's a little less than a football field. This follows from the fact that an acre is defined to be 43,560 square feet (1/640 of a square mile, as you can verify, since 1 mile = 5,280 feet), and a football field (300 feet by 160 feet) is 48,000 square feet.

6. If $n = 1$, the expressions in Eq. (6.8) become

$$a = m^2 - 1, \qquad b = 2m, \qquad c = m^2 + 1. \tag{6.55}$$

Since $b/2$ is simply m, we see that a and c do indeed take the form of $(b/2)^2 \pm 1 = m^2 \pm 1$.

7. Starting with $a = 2k - 1$ and following the given recipe, we first square a to obtain

$$a^2 = (2k-1)^2 = 4k^2 - 4k + 1. \tag{6.56}$$

We then add or subtract 1 to obtain $4k^2 - 4k + 2$ and $4k^2 - 4k$. Finally, we divide by 2 and label the results as c and b:

$$c = 2k^2 - 2k + 1, \qquad b = 2k^2 - 2k. \tag{6.57}$$

Our goal is to show that (a, b, c) is a Pythagorean triple, that is, that $a^2 + b^2 = c^2$. The sum $a^2 + b^2$ equals

$$\begin{aligned} a^2 + b^2 &= (2k-1)^2 + (2k^2 - 2k)^2 \\ &= (4k^2 - 4k + 1) + (4k^4 - 8k^3 + 4k^2) \\ &= 4k^4 - 8k^3 + 8k^2 - 4k + 1. \end{aligned} \tag{6.58}$$

We want to show that this equals c^2, which is obtained by squaring the trinomial for c in Eq. (6.57). Using the trinomial result from Example 3.6, we obtain

$$\begin{aligned} (2k^2 - 2k + 1)^2 &= \Big((2k^2)^2 + (-2k)^2 + 1^2\Big) \\ &\quad + 2\Big((2k^2)(-2k) + (2k^2)(1) + (-2k)(1)\Big) \\ &= \big(4k^4 + 4k^2 + 1\big) + \big(-8k^3 + 4k^2 - 4k\big) \\ &= 4k^4 - 8k^3 + 8k^2 - 4k + 1, \end{aligned} \tag{6.59}$$

in agreement with the $a^2 + b^2$ sum in Eq. (6.58), as desired.

This calculation was a bit long, and there were many places where we could have made a mistake. But we were careful, and it all worked out. The reward for the effort was two long expressions that were exactly equal to each other. Perhaps the most likely error in the calculation is forgetting to distribute the 2 into all three terms in the parentheses in the second line of Eq. (6.59).

The purpose of this exercise was to get some algebra practice. If the goal were instead to show as quickly as possible that $a^2+b^2 = c^2$ (once b and c are obtained in Eq. (6.57)), then the best route would be to write this as $a^2 = c^2 - b^2$, and then use the difference-of-squares result. This method is much quicker, as you can verify!

Note that our starting $a = 2k - 1$ expression, along with the expressions for c and b in Eq. (6.57), are the same as the expressions for a, b, and c in Eq. (6.10) (which came from Eq. (6.8)), with m replaced by k. So we already knew from our original proof in Exercise 6.1 that the $a^2 + b^2 = c^2$ relation holds. But again, the purpose of this exercise was to verify that $a^2 + b^2 = c^2$ holds by doing some algebra.

8. The overall square has side length $a + b$, so its area is $(a + b)^2$. The area of the smaller square is c^2. And the area of each of the four triangles is $ab/2$ (since each one has a base of a and a height of b, or vice versa). So the statement that the overall area equals the sum of the areas of the sub-regions is

$$\begin{aligned}(a+b)^2 &= 4 \cdot \frac{ab}{2} + c^2 \\ \Longrightarrow \quad a^2 + \cancel{2ab} + b^2 &= \cancel{2ab} + c^2 \\ \Longrightarrow \quad a^2 + b^2 &= c^2,\end{aligned} \tag{6.60}$$

as desired. The third line is obtained by canceling the $2ab$ terms on each side of the second line (or more precisely, by subtracting $2ab$ from both sides of the second line, as explained in the second terminology bullet point on page 163).

9. The area of the overall square is c^2, and the area of the smaller square is $(b-a)^2$. The area of each of the four triangles is $ab/2$ (since each one has a base of a and a height of b, or vice versa), So the statement that the overall area equals the sum of the areas of the sub-regions is

$$\begin{aligned}c^2 &= 4 \cdot \frac{ab}{2} + (b-a)^2 \\ &= \cancel{2ab} + (b^2 - \cancel{2ab} + a^2) \\ &= b^2 + a^2,\end{aligned} \tag{6.61}$$

as desired.

10. Fig. 6.24 shows the rearrangement. The white squares that are formed have side lengths a and b, because those are the legs of the four shaded rectangles in the original figure. So the area of the white region is indeed $a^2 + b^2$.

    Note that we don't even need to know that the area of a triangle is $bh/2$ here, as we did in the first two proofs above. All we need to know is that the area of a square is the side length squared.

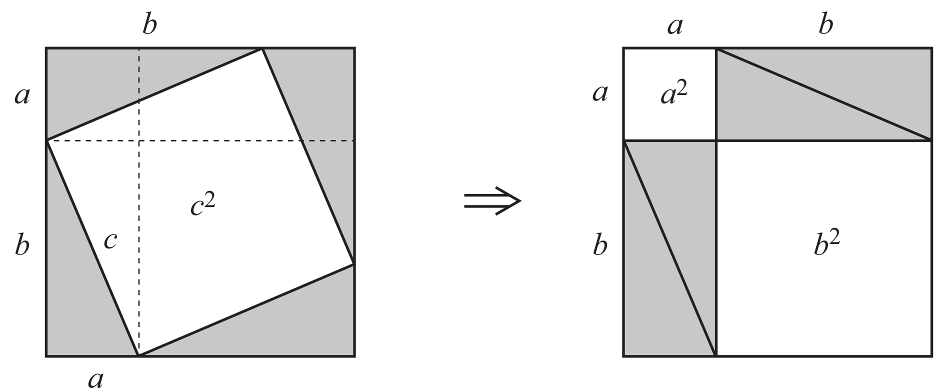

Figure 6.24

11. We just need to move the two dark-shaded triangles in Fig. 6.25 as indicated. (It doesn't matter which one goes where, since they're identical.) The total shaded area (light and dark) is the same in the two figures. And since this area is $a^2 + b^2$ in the left figure and c^2 in the right, it follows that $a^2 + b^2 = c^2$.

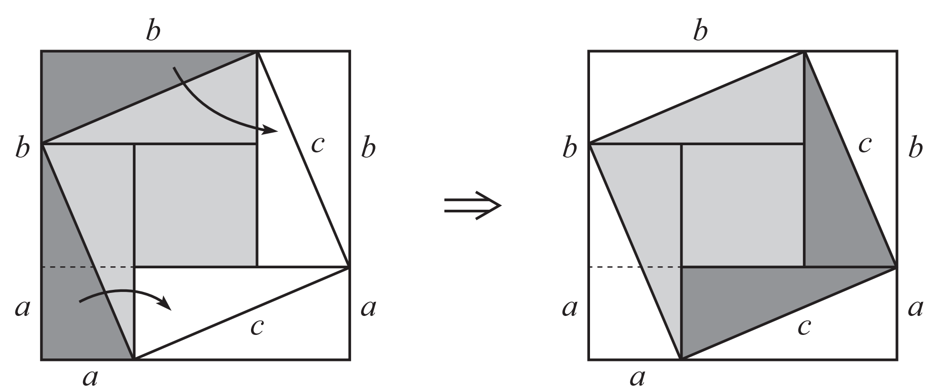

Figure 6.25

This proof and the preceding one show that sometimes a proof doesn't require any words (even though we did use some). A simple picture by itself can do the trick!

The proof she gave somehow succeeded
(a bit odd, given how it proceeded).
But the picture she drew
Soon convinced us it's true
That in some cases words are not needed!

12. For convenience in Fig. 6.26, let $\angle A$ be labeled as α, as shown. Then $\angle B = 90° - \alpha$ because $\angle ACB = 90°$, and the three angles in triangle ABC must add up to 180°.

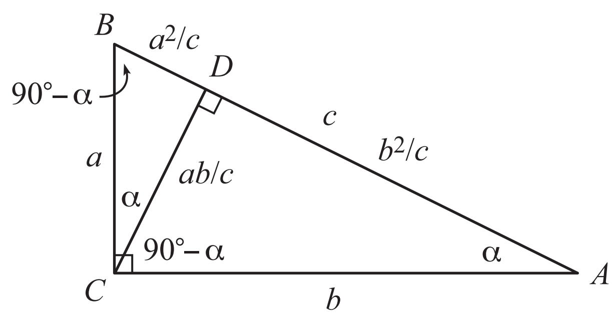

Figure 6.26

Similarly, in right triangle ACD, we have $\angle ACD = 90° - \alpha$ because $\angle ADC = 90°$, and the three angles in triangle ACD must add up to 180°.

Finally, due to the original right angle at C, we have

$$\angle DCB = 90° - \angle DCA = 90° - (90° - \alpha) = \alpha. \tag{6.62}$$

You can also deduce this α angle by demanding that the angles in triangle CBD add up to 180°.

We therefore see that all three of the triangles ABC, ACD, and CBD have angles of 90°, α, and $90° - \alpha$. Hence they are all similar, as we wanted to show.

We can now use the similarity of the triangles to write down some useful ratios. In the overall triangle ABC, the short leg a is a/c times the hypotenuse c, and the long leg b is b/c times the hypotenuse c. Since the other two smaller right triangles are similar to the overall one, they must have these same ratios. That is, in each triangle, the short leg is a/c times the hypotenuse, and the long leg is b/c times the hypotenuse.

So in the smallest triangle, the short leg BD is a/c times the hypotenuse, which is $CB = a$. So $BD = (a/c)a = a^2/c$, as shown.

And in the medium triangle, the long leg AD is b/c times the hypotenuse, which is $AC = b$. So $AD = (b/c)b = b^2/c$, as shown.

We can now use the fact that $BD + AD$ equals the hypotenuse c of the overall triangle. This gives

$$BD + AD = c \quad \Longrightarrow \quad \frac{a^2}{c} + \frac{b^2}{c} = c \quad \Longrightarrow \quad a^2 + b^2 = c^2, \qquad (6.63)$$

where this last equation is obtained from the middle one by multiplying both sides by c.

Remark: We can also find the length of the altitude, CD, in various different ways. In the smallest triangle, the long leg CD is b/c times the hypotenuse, which is $CB = a$. So $CD = (b/c)a = ab/c$, as shown.

Alternatively, in the medium triangle, the short leg CD is a/c times the hypotenuse, which is $AC = b$. So $CD = (a/c)b = ab/c$.

Alternatively again, we can find CD by writing down two valid $bh/2$ expressions for the area of the overall triangle. We can consider b to be the base and a to be the height. Or we can consider c to be the base and CD to be the height. Equating the two resulting expressions for the area gives

$$\frac{ba}{2} = \frac{c(CD)}{2} \quad \Longrightarrow \quad \frac{ab}{c} = CD\,, \qquad (6.64)$$

where the second equation is obtained from the first by multiplying both sides by $2/c$. ♣

13. Let the three triangles (overall, medium, and smallest) be labeled T_1, T_2, and T_3, respectively. The hypotenuse of T_1 is c, and the hypotenuse of T_2 is b. So T_2 is obtained by scaling down T_1 by the factor $f = b/c$. The results from Section 5.5 then tell us that the areas are related by $A_2 = f^2 A_1 \Longrightarrow A_2 = (b/c)^2 A_1$.

    Similarly, the hypotenuse of T_3 is a, so T_3 is obtained by scaling down T_1 by the factor $f = a/c$. The areas are therefore related by $A_3 = f^2 A_1 \Longrightarrow A_3 = (a/c)^2 A_1$.

    The $A_2 + A_3 = A_1$ relation for areas then becomes

$$\frac{b^2}{c^2}A_1 + \frac{a^2}{c^2}A_1 = A_1 \quad \Longrightarrow \quad \frac{b^2}{c^2} + \frac{a^2}{c^2} = 1 \quad \Longrightarrow \quad b^2 + a^2 = c^2, \qquad (6.65)$$

    as desired. The last equation is obtained by dividing both sides of the first equation by A_1, and then multiplying both sides of the second equation by c^2. (Or you can just multiply by c^2/A_1 in one step.)

14. The dark-shaded triangles A_1BA and CBC_1 in Fig. 6.7 both have sides with lengths a and c. So if we can show that the angles $\angle A_1BA$ and $\angle CBC_1$ between these common sides are equal, then by the SAS postulate we'll know that the two triangles are congruent. These angles are indeed equal because

$$\begin{aligned} \angle A_1BA &= \angle A_1BC + \angle CBA = 90^\circ + \angle CBA, \\ \text{and} \quad \angle CBC_1 &= \angle ABC_1 + \angle CBA = 90^\circ + \angle CBA. \end{aligned} \tag{6.66}$$

So both angles are 90° plus $\angle CBA$, and hence are equal. Triangles A_1BA and CBC_1 are therefore congruent by the SAS postulate. So they have the same area.

Note: Another (quick) way of seeing why triangles A_1BA and CBC_1 are congruent is to imagine rotating triangle CBC_1 clockwise by 90° around point B. It will turn into triangle A_1BA.

All of the above reasoning holds with the light-shaded triangles too, so they are also congruent and hence have the same area.

We now claim that the area of triangle A_1BA is half the area of the square with side a. This is true because in the standard $(1/2)bh$ expression for the area of a triangle, we can pick the base to be the $A_1B = a$ side, in which case the altitude from A to the (extension of the) A_1B side has length a, as shown in Fig. 6.27(a). So the area of triangle A_1BA is $(1/2)bh = (1/2)a \cdot a = a^2/2$, which is indeed half the area of the square with side a.

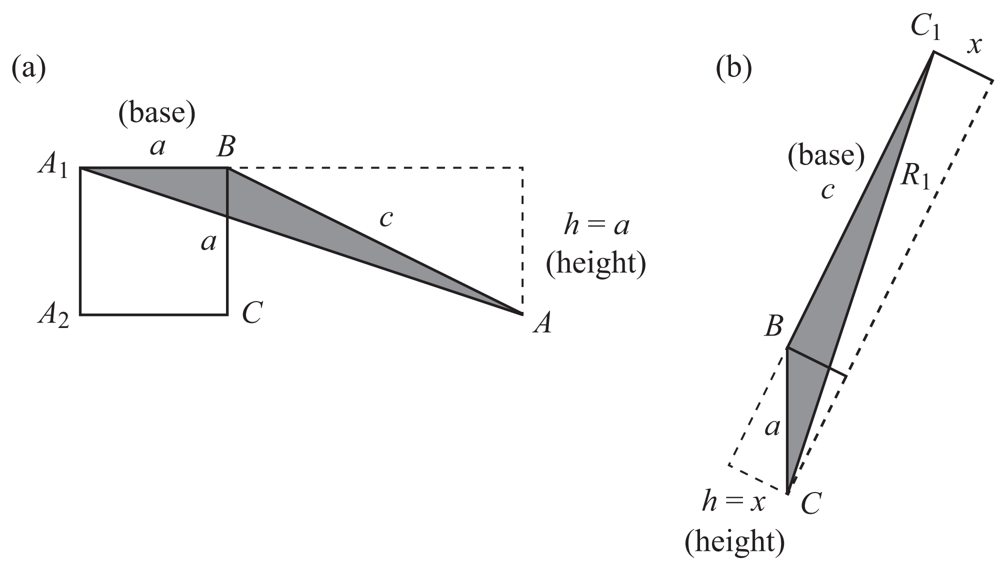

Figure 6.27

We also claim that the area of triangle CBC_1 is half the area of the R_1 rectangular part of the square with side c that is above/left of the dashed line. This is true because in the standard $(1/2)bh$ expression for the area of a triangle, we can pick the base to be the $BC_1 = c$ side, in which case the altitude from C to the (extension of the) BC_1 side has the length x shown in Fig. 6.27(b). So the area of triangle CBC_1 is $(1/2)bh = (1/2)c \cdot x = cx/2$, which is indeed half the area of the R_1 rectangle with sides c and x.

The preceding two paragraphs, combined with the fact that triangles A_1BA and CBC_1 are congruent, tell us that $a^2/2 = cx/2$, which (after multiplying both sides by 2) means that $a^2 = cx$. So the cx area of the R_1 rectangle is simply a^2.

We can now repeat (or rather, just imagine repeating) the entire process above, but with the light-shaded congruent triangles. The result will be that the area of the R_2 rectangular part of the square with side c that is below/right of the dashed line equals b^2.

Finally, since the a^2 and b^2 areas of the R_1 and R_2 rectangles add up to the c^2 area of the square with side c, we conclude that $a^2 + b^2 = c^2$, as desired.

The above proof might seem long, but here's the quick summary: Triangles A_1BA and CBC_1 are congruent because they can be rotated into each other. So they have the same area. Fig. 6.27 then shows that these triangles have (with appropriately chosen bases) the same heights as the a^2 square and the R_1 rectangle. This square and rectangle therefore also have the same area (both twice the common triangle area). So the R_1 rectangle has area a^2. Likewise, the R_2 rectangle has area b^2. Finally, the R_1 and R_2 areas add up to c^2.

15. In terms of R, plugging $h = R/16$ into Eqs. (6.21) and (6.23) gives

$$\begin{aligned} d_{\text{exact}} &= \sqrt{2Rh + h^2} = \sqrt{2R(R/16) + (R/16)^2} \\ &= \sqrt{R^2(1/8 + 1/256)} = R\sqrt{0.1289} = (0.3590)R, \\ d_{\text{approx}} &= \sqrt{2Rh} = \sqrt{2R(R/16)} \\ &= \sqrt{R^2(1/8)} = R\sqrt{0.125} = (0.3536)R. \end{aligned} \tag{6.67}$$

With $R = 6{,}400\,\text{km}$, these results become

$$\begin{aligned} d_{\text{exact}} &= (0.3590)(6{,}400\,\text{km}) = 2298\,\text{km}, \\ d_{\text{approx}} &= (0.3536)(6{,}400\,\text{km}) = 2263\,\text{km}. \end{aligned} \tag{6.68}$$

The difference in these answers is only 35 km, which is about 0.015 (equivalently, 1.5%) of the exact 2298 km distance. So the approximation in Eq. (6.23) is still very good, even for the large (but still small compared with R) h value of the Space Station.

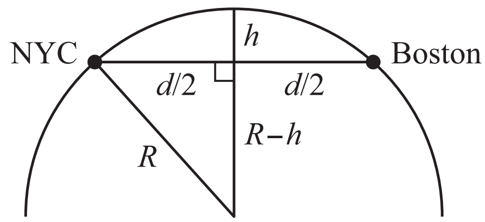

Figure 6.28

16. The setup is shown in Fig. 6.28. The desired maximum depth is h, and the total distance from Boston to NYC is $d = 300\,\text{km}$.

    There is technically an ambiguity about whether the 300 km is the straight-line distance or the curved distance along the surface of the earth. But it doesn't matter since these two distances are essentially the same. If the two cities were significantly farther apart, then we would have to worry about this issue. The straight-line distance is the useful one in Fig. 6.28, so if (as would be the case in real life) we're given the curved distance, the first thing we'd have to do is somehow calculate the straight-line distance. But there's no need to do that here, since the two distances are indistinguishable (because 300 km is sufficiently small compared with the 6,400 km radius of the earth). It's possible to show (with trigonometry; see Chapter 16) that the two distances differ by only about 0.01%.

    The right triangle in Fig. 6.28 has legs $d/2$ and $R - h$, and hypotenuse R. So the Pythagorean theorem gives

$$(d/2)^2 + (R-h)^2 = R^2 \implies (d/2)^2 = R^2 - (R-h)^2$$
$$\implies d^2/4 = \cancel{R^2} - (\cancel{R^2} - 2Rh + h^2). \qquad (6.69)$$

    As we did in the distance-to-horizon problem in the text, we can ignore the h^2 term, because it is negligible compared with the $2Rh$ term. We're then left with

$$\frac{d^2}{4} = 2Rh \implies h = \frac{d^2}{8R}, \qquad (6.70)$$

    where we have divided both sides by $2R$ (and then switched sides). Plugging in the Boston-NYC distance of 300 km gives

$$h = \frac{(300\,\text{km})^2}{8(6{,}400\,\text{km})} \approx 1.75\,\text{km} \approx 1.1\text{ miles}. \qquad (6.71)$$

    Is this answer larger or smaller than what you expected? Personally, my first guess was that the tunnel would be deeper than this. But in retrospect, the earth is nearly flat on the scale of 300 km, so this small value of h is quite believable.

Note that our earlier result for the tower height h in Eq. (6.28) is 4 times the result for the depth h in Eq. (6.70). So if you build a tower in Boston tall enough to see NYC (the ground there, not just the skyline), and if you also dig a tunnel between the two cities, then the tower will be 4 times as tall as the tunnel is deep. Since we just found that the tunnel will be 1.75 km deep, the tower will be $4(1.75\,\text{km}) = 7\,\text{km}$ (or 4.3 miles) tall.

17. The setup is shown in Fig. 6.29. In the bottom face of the box, we have a right triangle with legs a and b, so our first application of the Pythagorean theorem gives the hypotenuse d_1 as

$$a^2 + b^2 = d_1^2 \implies d_1 = \sqrt{a^2 + b^2}. \tag{6.72}$$

This is the length of a diagonal of the bottom (or top) *face* of the box, not the whole box itself.

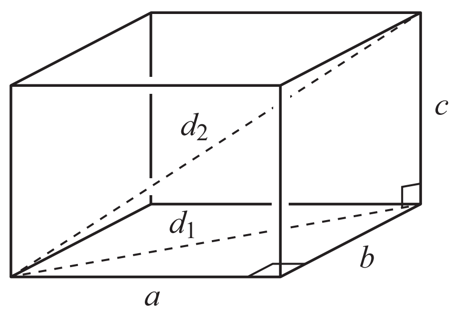

Figure 6.29

We now note that we have another right triangle – the vertical one with legs d_1 and c, and hypotenuse d_2. So our second application of the Pythagorean theorem gives

$$d_1^2 + c^2 = d_2^2. \tag{6.73}$$

Substituting the value of d_1^2 from Eq. (6.72) into this equation gives

$$(a^2 + b^2) + c^2 = d_2^2 \implies d_2 = \sqrt{a^2 + b^2 + c^2}. \tag{6.74}$$

This is the desired length of the diagonal of the box. This expression is symmetric in a, b, and c, because (just as with the Pythagorean theorem) it can't matter which of the three dimensions you arbitrarily choose to label as a, or b, or c. Said in another way, if your first application of the Pythagorean theorem instead involved one of the side faces, you would still end up with the same final $d_2 = \sqrt{a^2 + b^2 + c^2}$ result.

18. The key is to unroll the cylinder into a flat rectangle, as shown in Fig. 6.30. The width of the rectangle is the πd circumference of the cylinder, and the diagonal L is the desired length of the staircase. (Yes, the sides in Fig. 6.30 are in fact in the correct proportion, for the given cylinder shown in Fig. 6.15. The πd circumference is longer than you might think.)

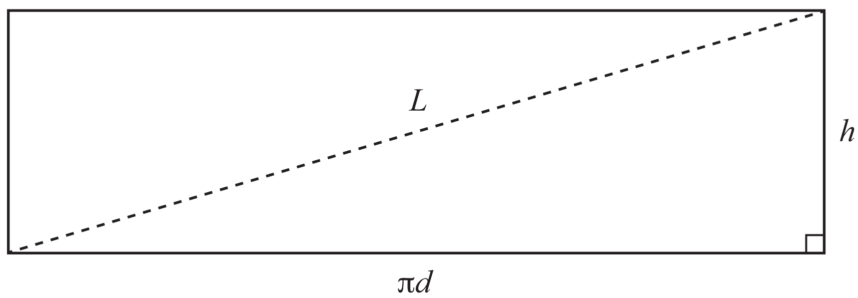

Figure 6.30

Applying the Pythagorean theorem to the right triangle in the figure gives the length L of the staircase as

$$(\pi d)^2 + h^2 = L^2 \implies L = \sqrt{\pi^2 d^2 + h^2}. \tag{6.75}$$

This is the answer in terms of the general lengths d and h. But in the special case where $d = h$, we have

$$L = \sqrt{\pi^2 h^2 + h^2} = \sqrt{(\pi^2 + 1)h^2} = \sqrt{\pi^2 + 1} \cdot h \approx (3.3)h. \tag{6.76}$$

We see that the length L of the staircase is only slightly longer than the circumference πd. This is because the long leg of the right triangle, πh, is significantly longer than the short leg, h.

We can check our general result in Eq. (6.75) for some special cases. If $h = 0$, then the cylinder has zero height, so the "staircase" is just a horizontal circle. And Eq. (6.75) correctly gives the $\sqrt{\pi^2 d^2 + 0^2} = \pi d$ circumference of the circle. In the other extreme, if $d = 0$, the cylinder is just a vertical segment, so the "staircrase" is a vertical ladder. And Eq. (6.75) correctly gives the $\sqrt{\pi^2 \cdot 0^2 + h^2} = h$ height of the segment.

19. If the pendulum is presently y above the lowest point, then it is $R - y$ below the pivot (the center of the circular arc), as shown in Fig. 6.31. So the sides of the right triangle in the figure are x, $R - y$, and R. The Pythagorean theorem then gives

$$\begin{aligned} x^2 + (R-y)^2 = R^2 \quad &\implies \quad x^2 = R^2 - (R-y)^2 \\ &\implies \quad x^2 = \cancel{R^2} - (\cancel{R^2} - 2Ry + y^2) \\ &\implies \quad x^2 = 2Ry - y^2. \end{aligned} \tag{6.77}$$

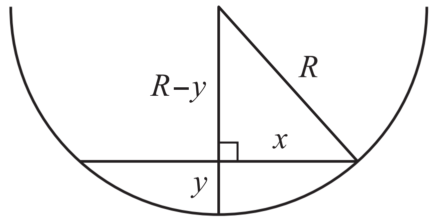

Figure 6.31

We'll now make the approximation where we ignore the y^2 term since it is much smaller than the $2Ry$ term. (y^2 is $y/2R$ times $2Ry$, and this factor of $y/2R$ is small, since we're assuming that y is much smaller than R.) We're therefore left with

$$x^2 \approx 2Ry \quad \Longrightarrow \quad y \approx \frac{x^2}{2R}. \tag{6.78}$$

This problem is just an upside-down version of the tunnel setup in Exercise 6.16, with $d/2$ replaced with x, and h replaced with y. The pendulum moves in the arc of a circle with radius R, just like the surface of the earth took the shape of a circle with radius R in Exercise 6.16.

The x^2 in Eq. (6.78) means that the pendulum's motion takes (approximately, assuming y is much smaller than R) the form of a *parabola*. Parabolas are functions of the form $y = Ax^2$. We'll talk about these, along with other types of functions, in Chapter 8. What we've shown in this exercise is that a circle looks like a parabola, at least near the bottom point. The circle/parabola starts out flat and then gradually gets steeper. If you double the x value from, say, a to $2a$, then the y value quadruples from $a^2/2R$ to $(2a)^2/2R = 4 \cdot a^2/2R$. Similarly, increasing x by a factor of 10 increases y by a factor of $10^2 = 100$ (assuming y is still much smaller than R).

20. Since we've chosen the radii of the large circles be 1, the hypotenuse of the right triangle has length $1 + r$. And the vertical leg has length $1 - r$, because its top and bottom ends are at heights of, respectively, 1 and r above the bottom line. The right triangle therefore has legs 1 and $1 - r$, and hypotenuse $1 + r$.

Since the side lengths of $1 - r$, 1, and $1 + r$ are equally spaced (with the common spacing being r), we can simply invoke the result from Example 6.3, which tells us that the sides are in the ratio of 3 : 4 : 5, as desired. So we're done. But let's work it out again anyway. The Pythagorean theorem applied to the right triangle in Fig. 6.17 gives

$$\begin{aligned}1 + (1-r)^2 = (1+r)^2 \quad &\Longrightarrow \quad 1 = (1+r)^2 - (1-r)^2 \\ &\Longrightarrow \quad 1 = (\cancel{1} + 2r + \cancel{r^2}) - (\cancel{1} - 2r + \cancel{r^2}) \\ &\Longrightarrow \quad 1 = 4r \quad \Longrightarrow \quad r = 1/4. \end{aligned} \tag{6.79}$$

The legs of the right triangle are then $1 - r = 3/4$, and 1. And the hypotenuse is $1 + r = 5/4$. Scaling all of these up by a factor of 4 gives sides of 3, 4, and 5, as desired.

If you instead start with a general radius R for the large circles, instead of 1, you will (as you can verify) end up with sides of $3R/4$, R, and $5R/4$, which are again in the ratio of $3:4:5$. The value of r is different ($R/4$ instead of $1/4$), but the $3:4:5$ ratio isn't affected.

21. This exercise is very similar to Example 6.2. Fig. 6.32 shows a 30° pie piece (just the triangle, without the rounded end). Letting the radius be 1 as usual, the 30-60-90 triangle in the left part of the pie piece has legs with lengths $1/2$ and $\sqrt{3}/2$ (from Section 5.4), as shown.

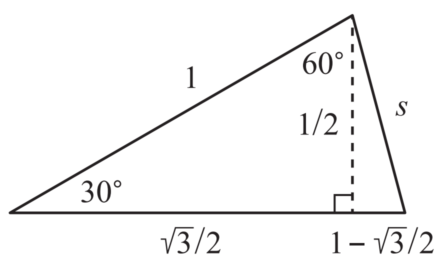

Figure 6.32

Since the bottom side of the pie piece has length 1 (because it's also a radius), a length $1 - \sqrt{3}/2$ is left for the short segment on the right side, as shown. The Pythagorean theorem applied to the right triangle in the right part of the pie piece then gives the dodecagon's side length s as

$$s^2 = (1/2)^2 + \left(1 - \sqrt{3}/2\right)^2 = 1/4 + \left(1 - \sqrt{3} + 3/4\right) = 2 - \sqrt{3}. \tag{6.80}$$

So $s = \sqrt{2 - \sqrt{3}} \approx 0.518$. Multiplying this by 12 to find the perimeter of the dodecagon gives $P_{\text{dodec}} \approx 6.21$. The $C_{\text{circ}} > P_{\text{dodec}}$ statement that the circumference of the circle is greater than the perimeter of the dodecagon is then $2\pi > 6.21$, or equivalently $\pi > 3.1$, after dividing by 2. This value is about 99% of the true $\pi \approx 3.14$ value, so the approximation is a very good one.

22. (a) The upper-left right triangle in Fig. 6.18 has hypotenuse 1 and vertical leg $a_n/2$. So the Pythagorean theorem gives the horizontal leg x as

$$x^2 + (a_n/2)^2 = 1^2 \implies x^2 = 1 - a_n^2/4 \implies x = \sqrt{1 - a_n^2/4}. \quad (6.81)$$

Since $x + y$ equals the radius 1, we have $y = 1 - x = 1 - \sqrt{1 - a_n^2/4}$. We can now use the upper-right right triangle in Fig. 6.18 to solve for the hypotenuse a_{2n}. The legs are $a_n/2$ and y (which we just found), so the Pythagorean theorem gives

$$\begin{aligned} a_{2n}^2 &= (a_n/2)^2 + y^2 \\ &= (a_n/2)^2 + \left(1 - \sqrt{1 - a_n^2/4}\right)^2 \\ &= \cancel{a_n^2/4} + \left(1 - 2\sqrt{1 - a_n^2/4} + (1 - \cancel{a_n^2/4})\right) \\ &= 2 - 2\sqrt{1 - a_n^2/4} \\ \implies a_{2n} &= \sqrt{2 - 2\sqrt{1 - a_n^2/4}}, \end{aligned} \quad (6.82)$$

as desired. This isn't the cleanest answer, but as least it's an answer. If we know a_n, we just need to plug it into this expression, and a_{2n} pops out.

(b) Table 6.3 shows the a_n values for various n's (powers of 2), along with the resulting estimate of π (a lower bound), and also the ratio of this estimate to the true value of π. The perimeter of the n-gon is $n \cdot a_n$, so the $C_{\text{circ}} > P_{n\text{-gon}}$ statement is $2\pi \cdot 1 > na_n \implies \pi > na_n/2$. This is the estimate of π in the third column. Each a_n in the table is obtained from the preceding one by plugging that one into Eq. (6.82). Note that the $a_8 = \sqrt{2 - \sqrt{2}}$ value correctly agrees with the result in Example 6.2.

n	a_n	$na_n/2$	$(na_n/2)/\pi$
2	2	2	0.64
4	$\sqrt{2}$	$2\sqrt{2} = 2.83$	0.90
8	$\sqrt{2 - \sqrt{2}}$	$4\sqrt{2 - \sqrt{2}} = 3.06$	0.974
16	0.39018064	3.1214	0.9936
32	0.19603428	3.13655	0.9984
64	0.09813535	3.14033	0.99960
128	0.04908246	3.141277	0.99990
256	0.02454308	3.141514	0.999975

Table 6.3: Estimates of π using n-gons, where n is a power of 2

It is in fact legal to start the table with the $n = 2$ case, as we have done. Of course, a 2-sided polygon isn't much of a polygon. It's the back-and-forth diameter we encountered in Exercise 5.23, so it has "sides" of $a_2 = 2$. Plugging this into Eq. (6.82) correctly gives the $a_4 = \sqrt{2}$ side of a square. However, if you're uncomfortable using the $n = 2$ case, you can simply start the table with the $n = 4$ case.

REMARKS: It's possible to write the a_n lengths in terms of square roots for $n = 16$ and higher, but the expressions become long and tedious. So we opted to use the (approximate) decimal forms in the table. In most cases, we didn't actually need to keep as many digits as we did, as far as a specific a_n is concerned. However, the accuracy of any given a_n affects the accuracy of higher a_n's. And the higher the n, the more digits we need to know. For example, for $n = 16$ our $na_n/2$ estimate for π differs from the actual value of π (≈ 3.14159) in the second digit after the decimal point, whereas for $n = 256$ it differs in the fifth digit.

Fig. 6.33 shows a 32-gon, which is very close to a smooth circle. The 0.9984 (equivalently, 99.84%) ratio in Table 6.3 for $n = 32$ seems quite reasonable, since the perimeter of the 32-gon is essentially equal to the circumference of the circle in which it is inscribed. We haven't drawn the circle, because it would be nearly indistinguishable from the 32-gon.

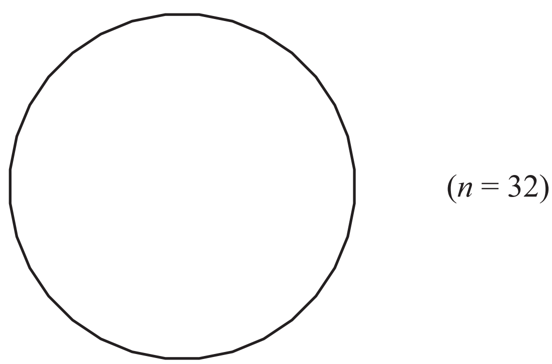

Figure 6.33

An inspection of Table 6.3 shows that the error (the difference from 1) in the $(na_n/2)/\pi$ value in the last column decreases by a factor of 4 from one n to the next (except for the first few small values of n). For example, 0.9984 differs from 1 by 0.0016, and then 0.99960 differs from 1 by 0.0004, and then 0.99990 differs from 1 by 0.0001. Each of these differences is 1/4 of the previous one. To prove that this pattern holds in general, we would need more machinery than we have at our disposal, so we'll just accept it as an interesting fact here.

When producing a π estimation,
Use n-gons without hesitation,
Because doubling your n,
Yet again, and again,
Yields improvement with each iteration!

You can also make another table of doubled n values, but now starting with $n = 3$. The n values will be 3, 6, 12, 24, 48, and so on. From Exercise 5.23, the a_3 side of an equilateral triangle (assuming a circle radius of 1, as usual) is $a_3 = \sqrt{3}$. You can quickly show that plugging this into Eq. (6.82) correctly gives the $a_6 = 1$ side of a hexagon. And then plugging this into Eq. (6.82) reproduces the $a_{12} = \sqrt{2 - \sqrt{3}}$ dodecagon result from Exercise 6.21. ♣

23. (a) The two sub-triangles in Fig. 6.34 are indeed isosceles, because they each have two sides equal to the radius r. Let the angle at A be α. Then because triangle ADC is isosceles, we have the other angle α shown.

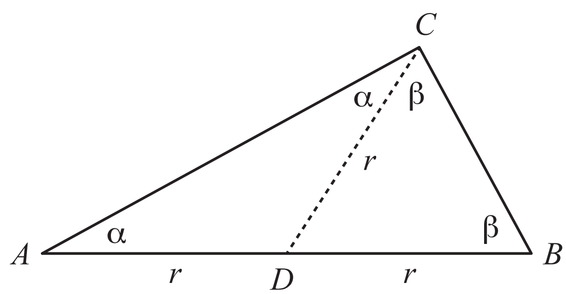

Figure 6.34

Likewise, let the angle at B be β. Then because triangle BDC is isosceles, we have the other angle β shown. The three angles in the overall triangle are then $\angle A = \alpha$, $\angle B = \beta$, and $\angle C = \alpha + \beta$. The sum of these angles must be 180°, so

$$\begin{aligned} \alpha + \beta + (\alpha + \beta) = 180^\circ \quad &\Longrightarrow \quad 2\alpha + 2\beta = 180^\circ \\ &\Longrightarrow \quad \alpha + \beta = 90^\circ. \end{aligned} \tag{6.83}$$

The lefthand side of this last relation is simply $\angle C$, so we have shown that $\angle C = 90^\circ$, as desired. This 90° result for $\angle C$ is a special case of a more general theorem we'll prove in Chapter 11.

Remark: The above $\angle C = 90^\circ$ result provides a quick answer to the question of how to classify all the different possible shapes of rectangles that have a specified length L for their diagonal. We can do this by drawing a circle with diameter L. If we draw the horizontal diameter, along with any other arbitrary diameter (the dashed lines in Fig. 6.35), we'll end up with a rectangle, because the result of this exercise tells us that the horizontal diameter leads to two right angles (the solid little boxes shown), and the titled dashed diameter leads to two others (the dashed little boxes). Depending on how tilted the dashed diameter is, we can produce a thin rectangle like the one on the left, or the "fat" square on the right.

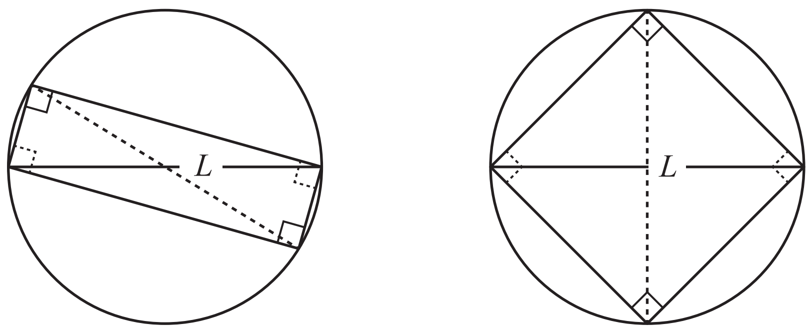

Figure 6.35

Exercise 6.4 dealt with a computer screen with a 13.3 inch (or 17 inch) diagonal. We now see how to generate all possible rectangular screens with a 13.3 inch diagonal. However, probably no one is going to want to use a screen as squat as the left rectangle in Fig. 6.35. In the old days, screens were more square-ish, but they've gotten wider (although not as wide as the left one in Fig. 6.35) as the years have gone by. ♣

(b) In Fig. 6.36, let the sides of right triangle ABC be $2a$, $2b$, and $2c$, for convenience (so that we won't have a bunch a $1/2$ factors floating around). Let D be the midpoint of the hypotenuse, so that $BD = DA = c$. Draw the vertical segment DE. Then triangle ADE is similar to triangle ABC (they both have a right angle along with the common angle at A, which means that their third angles are also the same), and it is half a large (since its $AD = c$ hypotenuse is half the $AB = 2c$ hypotenuse). So $DE = a$ (half of BC), and $AE = b$ (half of AC). This leaves b for CE, as shown. From the Pythagorean theorem applied to triangles ADE and CDE, the lengths of AD and CD are both $\sqrt{a^2 + b^2}$ (which equals c). So they are equal (and hence also equal to DB), as desired. D is therefore the center of the circle passing through A, B, and C. And since any chord of a circle containing the center is a diameter, we see that AB is a diameter, as we wanted to show.

To succinctly summarize the results in parts (a) and (b) of this exercise: If a triangle is inscribed in a circle, then (a) If one side is a diameter, then the triangle is right, and (b) If the triangle is right, then one side

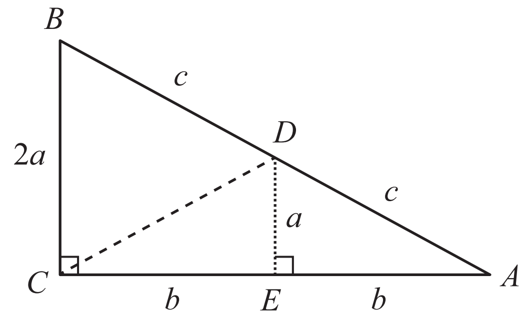

Figure 6.36

(the hypotenuse) is a diameter. These two statements are the *converses* (reverses) of each other.

REMARK: We just showed in part (b) that for a given hypotenuse length, the midpoint of the hypotenuse is always the same distance (half the length of the hypotenuse) from the right angle, no matter what the lengths of the legs are. It doesn't matter if the right triangle is a "fat" 45-45-90 one, or a thin 1-89-90 one. As long as the hypotenuse has a fixed length, the distance from the midpoint to the right angle is always the same (half the hypotenuse).

This fact is relevant to the famous "sliding ladder" problem. A ladder slides down a wall, as shown in Fig. 6.37, always maintaining contact with the wall and the floor. What is the path taken by the midpoint? Since the midpoint of the ladder (the midpoint of the hypotenuse; the dots shown) is always the same distance from the corner, we see that the midpoint traces out a quarter circle, as shown. ♣

24. (a) FIRST PROOF: We'll present two proofs. The first uses a *symmetry* argument, which is a standard (and slick) method of proof. Fig. 6.38(a) shows a circle sitting on top of a line; the line is tangent to the circle. This setup has left/right symmetry, meaning that if we flip it over (so that left and right are reversed), it looks the same. Equivalently, the mirror image looks the same. Said in yet another way, it looks the same if we view it through the back of the paper.

If we flip over the setup in Fig. 6.38(a), it turns into Fig. 6.38(b), which mean that the α and β angles have switched; α is now on the right, and β is on the left. However, the above left/right symmetry property tells us that Fig. 6.38(b) must be *exactly the same setup* as Fig. 6.38(a), which means that the α angle must *still be on the left*, and likewise the β angle must still be on the right. Putting the preceding two sentences together, we see that

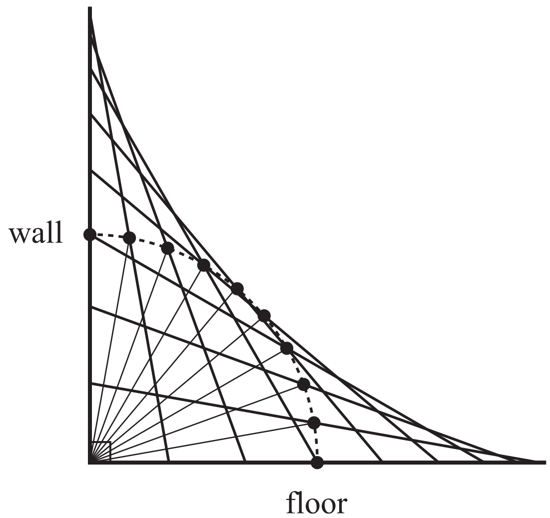

Figure 6.37

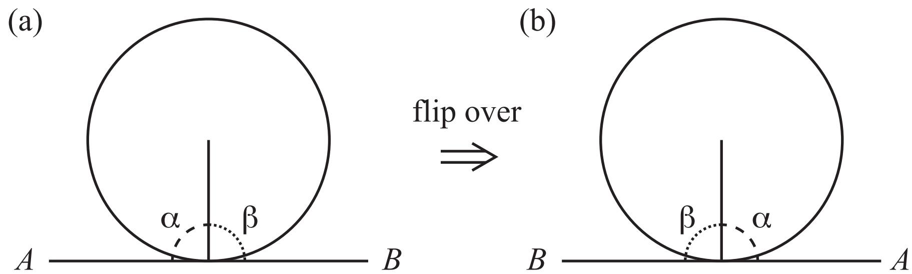

Figure 6.38

the left angle in Fig. 6.38(b) must be equal to both α (from the preceding sentence) and also β (from two sentences ago). We therefore conclude that $\alpha = \beta$. And since these two equal angles add up to 180° (since together they form a straight line), they must each be 90°, as we wanted to show.

SECOND PROOF: For this second proof, we'll use the Pythagorean theorem in what is called a *proof by contradiction.* In a proof by contradiction, if we're trying to prove a particular statement (let's label the statement as "*P*"), the strategy is to make the assumption that *P* is *not* true (which is the opposite of what we're actually trying to show). The goal is to then show that this assumption leads to a false statement. It then logically follows that our not-*P* assumption must have been incorrect, because true things can't logically lead to false things. So our original *P* statement must be correct (because if not-*P* is false, then *P* must be true), as we wanted to show.

For example, consider the statement, "There are no integers (positive or negative) a and b for which $2a + 6b = 5$." Now, there's no chance of testing all of the infinite number of possible a and b values and showing that none of them work. So we need a different method of proof, and a proof by contradiction works well here. In search of a contradiction, let's assume that there *do* exist integers a and b for which $2a + 6b = 5$. It then follows (by dividing both sides by 2) that there exist integers a and b for which $a + 3b = 5/2$. But this is a false statement, because the lefthand side is an integer (if a and b are integers), whereas the righthand side is not. Our initial assumption (that there *do* exist integers. . .) must therefore have been incorrect, which means that we have successfully proved that there are *no* integers a and b for which $2a + 6b = 5$. We'll discuss proofs by contradiction in more detail in Chapter 12.

In our present tangent-line problem, our assumption (which we'll end up showing is incorrect) is that the tangent line is *not* perpendicular to the radius at the point of contact. If this assumption is true, then of the two angles the radius makes with the tangent, one must be larger than 90°, and one must be smaller than 90°, as shown in Fig. 6.39. (Don't try to make too much sense of this figure, since we're going to show that it can't actually look this way.)

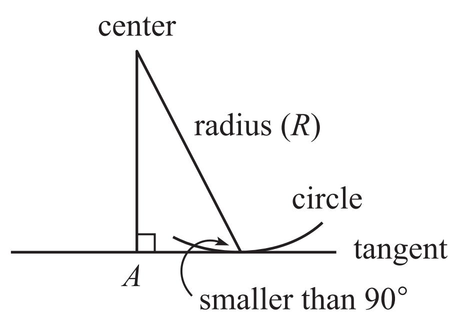

Figure 6.39

Consider the angle that is smaller than 90°, and draw a right triangle containing that angle, as shown. Since the leg of a right triangle is (due to the Pythagorean theorem) always shorter than the hypotenuse, which is R here, the vertical leg of our right triangle is smaller than R. This implies that point A must be *inside* the circle (because its distance from the center is less than the radius R). This contradicts the fact that every point on a tangent line lies *outside* the circle (except for the single point that lies on the circle). Therefore, since our assumption of non-perpendicularity leads to a false statement (that A is inside the circle), we conclude that the assumption must have been incorrect. The radius and tangent line must therefore in fact be perpendicular.

(b) FIRST PROOF: We'll again present two proofs. As in part (a), the first one uses a symmetry argument, and the second one uses the Pythagorean theorem (but isn't a proof by contradiction). In Fig. 6.40(a), the two tangents are drawn from a point directly above the center of the circle. As with the setup in part (a), this system has left/right symmetry. So if we flip it over (or look at it in a mirror, or through the back of the paper), we obtain the identical system in Fig. 6.40(b). The length d_1 is now on the right. But since it's the same setup, the length d_2 must also be on the right. Hence $d_1 = d_2$, as desired. This is exactly the same reasoning that led to the $\alpha = \beta$ conclusion in part (a).

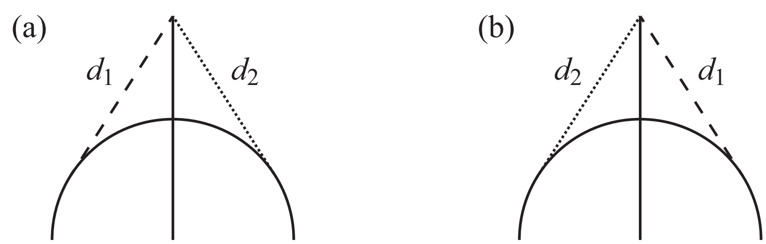

Figure 6.40

SECOND PROOF: From part (a), we know that the tangents are perpendicular to the radii where they touch, as shown in Fig. 6.41. The two right triangles shown have a common hypotenuse L, along with a common leg (the radius R). So the Pythagorean theorem tells us that the other legs have the same length (both equal to $\sqrt{L^2 - R^2}$), as desired.

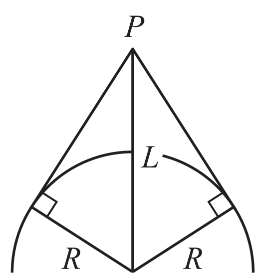

Figure 6.41

25. Eq. (6.35) in Example 6.4 tells us that $BA = 2\sqrt{ab}$. And since Eq. (6.35) is valid for any two circles that touch tangentially, we can also apply it to the left two circles in Fig. 6.21, with radii b and r. B and C are now the relevant points of contact on the tangent line, so Eq. (6.35) tells us that $BC = 2\sqrt{rb}$. Likewise, applying Eq. (6.35) to the right two circles in Fig. 6.21, with radii r and a, gives $CA = 2\sqrt{ra}$.

We can now use the fact that $BC + CA = BA$, which gives

$$\begin{aligned} 2\sqrt{rb} + 2\sqrt{ra} = 2\sqrt{ab} \quad &\Longrightarrow \quad \sqrt{r}\big(\sqrt{b} + \sqrt{a}\big) = \sqrt{ab} \\ &\Longrightarrow \quad \sqrt{r} = \frac{\sqrt{ab}}{\sqrt{a} + \sqrt{b}}\,, \end{aligned} \tag{6.84}$$

where we have divided both sides by 2, factored out the $\sqrt{r}$ on the lefthand side, and then divided both sides by $\sqrt{a} + \sqrt{b}$. Squaring both sides then gives

$$r = \left(\frac{\sqrt{ab}}{\sqrt{a} + \sqrt{b}}\right)^2 = \frac{ab}{(\sqrt{a} + \sqrt{b})^2}\,. \tag{6.85}$$

Note that in the special case where $b = a$, we have (replacing every b in Eq. (6.85) with a)

$$r = \frac{a \cdot a}{(2\sqrt{a})^2} = \frac{a^2}{4a} = \frac{a}{4}\,. \tag{6.86}$$

So the small circle has $1/4$ the radius of the large circles. This agrees with the result in Exercise 6.20, where we solved the $a = b$ case directly (with $a = b = 1$).

26. As in Example 6.4, the short leg of the shaded triangle is the $a - b$ difference of the radii. But the hypotenuse here isn't $a + b$, as it was in Example 6.4. Instead, it equals $\sqrt{a^2 + b^2}$ because it is the hypotenuse of right triangle ABC. This is indeed a right triangle, because the dashed tangent is perpendicular to the radius of the left circle at C, from Exercise 6.24(a).

    The long leg of the shaded triangle is the desired distance d, so applying the Pythagorean theorem to the shaded triangle gives

$$\begin{aligned} d^2 + (a-b)^2 &= \big(\sqrt{a^2 + b^2}\big)^2 \\ \Longrightarrow \quad d^2 &= \big(\sqrt{a^2 + b^2}\big)^2 - (a-b)^2 \\ &= (\cancel{a^2} + \cancel{b^2}) - (\cancel{a^2} - 2ab + \cancel{b^2}) \\ &= 2ab \\ \Longrightarrow \quad d &= \sqrt{2ab} = \sqrt{2}\sqrt{ab} = (1.41)\sqrt{ab}. \end{aligned} \tag{6.87}$$

    This $\sqrt{2}\sqrt{ab} = (1.41)\sqrt{ab}$ answer is smaller than the $2\sqrt{ab}$ answer in Example 6.4. This makes sense, because the centers of the circles are closer together in this exercise (the circles partially overlap here).

    In the special case where $a = b$, we obtain $d = \sqrt{2}\sqrt{a^2} = \sqrt{2}a$. In this case, the BC and AC segments are two sides of a square, with side length a. The distance d is the same as the diagonal BA of this square (the distance between the centers of the circles), as you can verify by drawing a picture. And the diagonal of a square with side a correctly has length $\sqrt{2}a$.