Chapter 6

Appendices

Special Relativity, For the Enthusiastic Beginner (Draft version, December 2016)

6.1 Appendix A: Qualitative relativity questions

Basic principles

1. QUESTION: You are in a spaceship sailing along in outer space. Is there any way you can measure your speed without looking outside?

   ANSWER: There are two points to be made here. First, the question is meaningless, because absolute speed doesn't exist. The spaceship doesn't have a speed; it only has a speed relative to something else. Second, even if the question asked for the speed with respect to, say, a given piece of stellar dust, the answer would be "no." Uniform speed is not measurable from within the spaceship. Acceleration, on the other hand, is measurable (assuming there is no gravity around to confuse it with).

2. QUESTION: Two people, A and B, are moving with respect to each other in one dimension. Is the speed of B as viewed by A equal to the speed of A as viewed by B?

   ANSWER: Yes. The first postulate of relativity states that all inertial frames are equivalent, which implies that there is no preferred location or direction in space. If the relative speed measured by the left person were larger than the relative speed measured by the right person, then there would be a preferred direction in space. Apparently people on the left always measure a larger speed. This violates the first postulate. Likewise if the left person measures a smaller speed. The two speeds must therefore be equal.

3. QUESTION: Is the second postulate of relativity (that the speed of light is the same in all inertial frames) really necessary? Or is it already implied by the first postulate (that the laws of physics are the same in all inertial frames)?

   ANSWER: It is necessary. The speed-of-light postulate is not implied by the laws-of-physics postulate. The latter doesn't imply that baseballs have the same speed in all inertial frames, so it likewise doesn't imply that light has the same speed.

   It turns out that nearly all the results in special relativity can be deduced by using only the laws-of-physics postulate. What you can find (with some work) is that there is some limiting speed, which may be finite or infinite; see Section 2.7. The speed-of-light postulate fills in the last bit of information by telling us what the limiting speed is.

4. QUESTION: Is the speed of light equal to c, under all circumstances?

   ANSWER: No. In Section 1.2 we stated the speed-of-light postulate as, "The speed of light in vacuum has the same value c in any inertial frame." There are two key words here: "vacuum" and "inertial." If we are dealing with a medium (such as glass or water) instead of vacuum, then the speed of light is smaller than c. And in an accelerating (noninertial) reference frame, the speed of light can be larger or smaller than c.

5. QUESTION: In a given frame, a clock reads noon when a ball hits it. Do observers in all other frames agree that the clock reads noon when the ball hits is?

   ANSWER: Yes. Some statements, such as the one at hand, are frame independent. The relevant point is that everything happens at one location, so we don't have to worry about any Lv/c^2 loss-of-simultaneity effects. The L here is zero. Equivalently, the ball-hits-clock event and the clock-reads-noon event are actually the same event. They are described by the same space and time coordinates. Therefore, since we have only one event, there is nothing for different observers to disagree on.

6. QUESTION: Can information travel faster than the speed of light?

   ANSWER: No. If it could, we would be able to generate not only causality-violating setups, but also genuine contradictions. See the discussions at the ends of Sections 2.3 and 2.4.

7. QUESTION: Is there such a thing as a perfectly rigid object?

   ANSWER: No. From the discussions of causality in Sections 2.3 and 2.4, we know that information can't travel infinitely fast (in fact, it can't exceed the speed of light). So it takes time for the atoms in an object to communicate with each other. If you push on one end of a stick, the other end won't move right away.

8. QUESTION: Can an object (with nonzero mass) move at the speed of light?

   ANSWER: No. Here are four reasons why. (1) Energy: $v = c$ implies $\gamma = \infty$, which implies that $E = \gamma mc^2$ is infinite. The object must therefore have an infinite amount of energy (unless $m = 0$, as for a photon). All the energy in the universe, let alone all the king's horses and all the king's men, can't accelerate something to speed c. (2) Momentum: again, $v = c$ implies $\gamma = \infty$, which implies that $p = \gamma mv$ is infinite. The object must therefore have an infinite amount of momentum (unless $m = 0$, as for a photon). (3) Force: since $F = \gamma^3 ma$, we have $a = F/m\gamma^3$. And since $\gamma \to \infty$ as $v \to c$, the acceleration a becomes smaller and smaller (for a given F) as v approaches c. It can be shown that with the γ^3 factor in the denominator, the acceleration drops off quickly enough so that the speed c is never reached in a finite time. (4) Velocity-addition formula: no matter what speed you give an object with respect to the frame it was just in (that is, no matter how you accelerate it), the velocity-addition formula always yields a speed that is less than c. The only way the resulting speed can equal c is if one of the two speeds in the formula is c.

9. QUESTION: Imagine closing a very large pair of scissors. If arranged properly, it is possible for the point of intersection of the blades to move faster than the speed of light. Does this violate anything in relativity?

   ANSWER: No. If the angle between the blades is small enough, then the tips of the blades (and all the other atoms in the scissors) can move at a speed well below c, while the intersection point moves faster than c. But this doesn't violate anything in relativity. The intersection point isn't an actual object, so there is nothing wrong with it moving faster than c.

You might be worried that this result allows you to send a signal down the scissors at a speed faster than c. However, since there is no such thing as a perfectly rigid object, it is impossible to get the far end of the scissors to move right away, when you apply a force at the handle. The scissors would have to already be moving, in which case the motion is independent (at least for a little while) of any decision you make at the handle to change the motion of the blades.

10. QUESTION: A mirror moves toward you at speed v. You shine a light toward it, and the light beam bounces back at you. What is the speed of the reflected beam?

    ANSWER: The speed is c, as always. You will observe the light (which is a wave) having a higher wave frequency due to the Doppler effect. But the speed is still c.

11. QUESTION: Person A chases person B. As measured in the ground frame, they have speeds v_A and v_B. If they start a distance L apart (as measured in the ground frame), how much time will it take (as measured in the ground frame) for A to catch B?

    ANSWER: In the ground frame, the relative speed is $v_A - v_B$. Person A must close the initial gap of L, so the time it takes is $L/(v_A - v_B)$. There is no need to use any fancy velocity-addition or length-contraction formulas, because all quantities in this problem are measured with respect to the *same* frame. So it quickly reduces to a simple "(rate)(time) = (distance)" problem. Alternatively, the two positions in the ground frame are given by $x_A = v_A t$ and $x_B = L + v_B t$. Setting these positions equal to each other gives $t = L/(v_A - v_B)$.

    Note that no object in this setup moves with speed $v_A - v_B$. This is simply the rate at which the gap between A and B closes, and a gap isn't an actual thing.

12. QUESTION: How do you synchronize two clocks that are at rest with respect to each other?

    ANSWER: One way is to put a light source midway between the two clocks and send out signals, and then set the clocks to a certain value when the signals hit them. Another way is to put a watch right next to one of the clocks and synchronize it with that clock, and then move the watch very slowly over to the other clock and synchronize that clock with it. Any time-dilation effects can be made arbitrarily small by moving the watch sufficiently slowly, because the time-dilation effect is second order in v (and because the travel time is only first order in $1/v$).

The fundamental effects

13. QUESTION: Two clocks at the ends of a train are synchronized with respect to the train. If the train moves past you, which clock shows a higher time?

    ANSWER: The rear clock shows a higher time. It shows Lv/c^2 more than the front clock, where L is the proper length of the train.

14. QUESTION: Does the rear-clock-ahead effect imply that the rear clock runs faster than the front clock?

    ANSWER: No. Both clocks run at the same rate in the ground frame. It's just that the rear clock is always a fixed time of Lv/c^2 ahead of the front clock.

15. QUESTION: Moving clocks run slow. Does this result have anything to do with the time it takes light to travel from the clock to your eye?

    ANSWER: No. When we talk about how fast a clock is running in a given frame, we are referring to what the clock actually reads in that frame. It will certainly take time for the light from the clock to reach your eye, but it is understood that you subtract off this transit time in order to calculate the time (in your frame) at which the clock actually shows a particular reading. Likewise, other relativistic effects, such as length contraction and the loss of simultaneity, have nothing to do with the time it takes light to reach your eye. They deal only with what really *is*, in your frame. One way to avoid the complication of the travel time of light is to use the lattice of clocks and meter sticks described in Section 1.3.4.

16. QUESTION: A clock on a moving train reads T. Does the clock read γT or T/γ in the ground frame?

    ANSWER: Neither. It reads T. Clock *readings* don't get dilated. *Elapsed times* are what get dilated. If the clock advances from T_1 to T_2, then the time between these readings, as measured in the train frame, is just $T_2 - T_1$. But the time between the readings, as measured in the ground frame, is $\gamma(T_2 - T_1)$, due to time dilation.

17. QUESTION: Does time dilation depend on whether a clock is moving across your vision or directly away from you?

    ANSWER: No. A moving clock runs slow, no matter which way it is moving. This is clearer if you think in terms of the lattice of clocks and meter sticks in Section 1.3.4. If you imagine a million people standing at the points of the lattice, then they all observe the clock running slow. Time dilation is an effect that depends on the *frame* and the speed of a clock with respect to it. It doesn't matter where you are in the frame (as long as you're at rest in it), as you look at a moving clock.

18. QUESTION: Does special-relativistic time dilation depend on the acceleration of the moving clock you are looking at?

    ANSWER: No. The time-dilation factor is $\gamma = 1/\sqrt{1 - v^2/c^2}$, which doesn't depend on the acceleration a. The only relevant quantity is the v at a given instant; it doesn't matter if v is changing. As long as you represent an inertial frame, then the clock you are viewing can undergo whatever motion it wants, and you will observe it running slow by the simple factor of γ. See the third remark in the solution to Problem 2.12.

    However, if *you* are accelerating, then you can't naively apply the results of special relativity. To do things correctly, it is easiest to think in terms of general relativity (or at least the Equivalence Principle). This is discussed in Chapter 5.

19. QUESTION: Two twins travel away from each other at relativistic speed. The time-dilation result says that each twin sees the other twin's clock running slow. So each says the other has aged less. How would you reply to someone who asks, "But which twin really *is* younger?"

    ANSWER: It makes no sense to ask which twin really is younger, because the two twins aren't in the same reference frame; they are using different coordinates to measure time. It's as silly as having two people run away from each other into the distance (so that each person sees the other become small), and then asking: Who is really smaller?

20. QUESTION: A train moves at speed v. A ball is thrown from the back to the front. In the train frame, the time of flight is T. Is it correct to use time dilation to say that the time of flight in the ground frame is γT?

    ANSWER: No. The time-dilation result holds only for two events that happen at the *same place* in the relevant reference frame (the train, here). Equivalently, it holds if you are looking at a *single* moving clock. The given information tells us that the reading on the front clock (when the ball arrives) minus the reading on the back clock (when the ball is thrown) is T. It makes no sense to apply time dilation to the difference in these readings, because they come from two different clocks.

Another way of seeing why simple time dilation is incorrect is to use the Lorentz transformation. If the proper length of the train is L, then the correct time on the ground (between the ball-leaving-back and ball-hitting-front events) is given by $\Delta t_g = \gamma(\Delta t_t + v\Delta x_t/c^2) = \gamma T + \gamma vL/c^2$, which isn't equal to γT. Equivalently, if you look at a *single* clock on the train, for example the back clock, then it starts at zero but ends up at $T + Lv/c^2$ due to the rear-clock-ahead effect (because the front clock shows T when the ball arrives). Applying time dilation to this elapsed time on a *single* clock gives the correct time in the ground frame. This is the argument we used when deriving Δt_g in Section 2.1.2.

21. QUESTION: Someone says, "A stick that is length-contracted isn't *really* shorter, it just *looks* shorter." How do you respond?

    ANSWER: The stick really *is* shorter in your frame. Length contraction has nothing to do with how the stick looks, because light takes time to travel to your eye. It has to do with where the ends of the stick are at simultaneous times in your frame. This is, after all, how you measure the length of something. At a given instant in your frame, the distance between the ends of the stick is genuinely less than the proper length of the stick. If a green sheet of paper slides with a relativistic speed v over a purple sheet (of the same proper size), and if you take a photo when the centers coincide, then the photo will show some of the purple sheet (or essentially all of it in the $v \to c$ limit).

22. QUESTION: Consider a stick that moves in the direction in which it points. Does its length contraction depend on whether this direction is across your vision or directly away from you?

    ANSWER: No. The stick is length-contracted in both cases. Of course, if you look at the stick in the latter case, then all you see is the end, which is just a dot. But the stick is indeed shorter in your reference frame. As in Question 17 above concerning time dilation, length contraction depends on the frame, not where you are in it.

23. QUESTION: If you move at the speed of light, what shape does the universe take in your frame?

    ANSWER: The question is meaningless, because it's impossible for you to move at the speed of light. A meaningful question to ask is: What shape does the universe take if you move at a speed very close to c (with respect to, say, the average velocity of all the stars)? The answer is that in your frame everything will be squashed along the direction of your motion, due to length contraction. Any given region of the universe will be squashed down to a pancake.

24. QUESTION: Eq. (1.14) says that the time in the observer's frame is *longer* than the proper time, while Eq. (1.20) says that the length in the observer's frame is *shorter* than the proper length. Why does this asymmetry exist?

    ANSWER: The asymmetry arises from the different assumptions that lead to time dilation and length contraction. If a clock and a stick are at rest on a train moving with speed v relative to you, then time dilation is based on the assumption that $\Delta x_{\text{train}} = 0$ (this holds for two ticks on a train clock), while length contraction is based on the assumption that $\Delta t_{\text{you}} = 0$ (you measure a length by observing where the ends are at simultaneous times in your frame). These conditions deal with *different frames*, and this causes the asymmetry. Mathematically, time dilation follows from the second equation in Eq. (2.2):

$$\Delta t_{\text{you}} = \gamma(\Delta t_{\text{train}} + (v/c^2)\Delta x_{\text{train}}) \implies \Delta t_{\text{you}} = \gamma \Delta t_{\text{train}} \quad (\text{if } \Delta x_{\text{train}} = 0). \quad (6.1)$$

Length contraction follows from the first equation in Eq. (2.4):

$$\Delta x_{\text{train}} = \gamma(\Delta x_{\text{you}} - v\Delta t_{\text{you}}) \implies \Delta x_{\text{train}} = \gamma \Delta x_{\text{you}} \quad (\text{if } \Delta t_{\text{you}} = 0). \quad (6.2)$$

These equations *are* symmetric with respect to the γ *factors*. The γ goes on the side of the equation associated with the frame in which the given space or time interval is zero. However, the equations *aren't* symmetric with respect to *you*. You appear on different sides of the equations because for length contraction your frame is the one where the given interval is zero, whereas for time dilation it isn't. Since the end goal is to solve for the "you" quantity, we must divide Eq. (6.2) by γ to isolate Δx_{you}. This is why Eqs. (1.14) and (1.20) aren't symmetric with respect to the "observer" (that is, "you") label.

25. Question: When relating times via time dilation, or lengths via length contraction, how do you know where to put the γ factor?

    Answer: There are various answers to this, but probably the safest method is to (1) remember that moving clocks run slow and moving sticks are short, then (2) identify which times or lengths are larger or smaller, and then (3) put the γ factor where it needs to be so that the relative size of the times or lengths is correct.

Other kinematics topics

26. Question: Two objects fly toward you, one from the east with speed u, and the other from the west with speed v. Is it correct that their relative speed, as measured by you, is $u + v$? Or should you use the velocity-addition formula, $V = (u + v)/(1 + uv/c^2)$? Is it possible for their relative speed, as measured by you, to exceed c?

    Answer: Yes, no, yes, to the three questions. It is legal to simply add the two speeds to obtain $u + v$. There is no need to use the velocity-addition formula, because both speeds are measured with respect to the *same thing* (namely you), and because we are asking for the relative speed as measured by that thing. It is perfectly legal for the result to be greater than c, but it must be less than (or equal to, for photons) $2c$.

27. Question: In what situations is the velocity-addition formula relevant?

    Answer: The formula is relevant in the two scenarios in Fig. 1.41, assuming that the goal is to find the speed of A as viewed by C (or vice versa). The second scenario is the same as the first scenario, but from B's point of view. In the first scenario, the two speeds are measured with respect to different frames, so it isn't legal to simply add them. In the second scenario, although the speeds are measured with respect to the same frame (B's frame), the goal is to find the relative speed as viewed by someone else (A or C). So the simple sum $v_1 + v_2$ isn't relevant (as it was in Question 26).

28. Question: Two objects fly toward you, one from the east with speed u, and the other from the west with speed v. What is their relative speed?

    Answer: The question isn't answerable. It needs to be finished with, ". . . as viewed by so-and-so." The relative speed as viewed by you is $u + v$, and the relative speed as viewed by either object is $(u + v)/(1 + uv/c^2)$.

    However, if someone says, "A and B move with relative speed v," and if there is no mention of a third entity, then it is understood that v is the relative speed as viewed by either A or B.

29. Question: A particular event has coordinates (x, t) in one frame. How do you use a Lorentz transformation (L.T.) to find the coordinates of this event in another frame?

    Answer: You don't. L.T.'s have nothing to do with single events. They deal only with *pairs* of events and the *separation* between them. As far as a single event goes, its coordinates in another frame can be anything you want, simply by defining your origin to be wherever and whenever you please. But for pairs of events, the separation is a well-defined quantity, independent of your choice of origin. It is therefore a meaningful question to ask how the separations in two different frames are related, and the L.T.'s answer this question.

    This question is similar to Question 16, where we noted that clock *readings* (that is, time *coordinates*) don't get dilated. Rather, *elapsed times* (the *separations* between time coordinates) are what get dilated.

30. Question: When using the L.T.'s, how do you tell which frame is the moving "primed" frame?

    Answer: You don't. There is no preferred frame, so it doesn't make sense to ask which frame is moving. We used the primed/unprimed notation in the derivation in Section 2.1.1 for ease of notation, but don't take this to imply that there is a fundamental frame S and a less fundamental frame S'. In general, a better strategy is to use subscripts that describe the two frames, such as "g" for ground and "t" for train, as we did in Section 2.1.2. For example, if you know the values of Δt_{t} and Δx_{t} on a train (which we'll assume is moving in the positive x direction with respect to the ground), and if you want to find the values of Δt_{g} and Δx_{g} on the ground, then you can write down:

$$\begin{aligned} \Delta x_{\text{g}} &= \gamma(\Delta x_{\text{t}} + v\,\Delta t_{\text{t}}), \\ \Delta t_{\text{g}} &= \gamma(\Delta t_{\text{t}} + v\,\Delta x_{\text{t}}/c^2). \end{aligned} \tag{6.3}$$

    If instead you know the intervals on the ground and you want to find them on the train, then you just need to switch the subscripts "g" and "t" and change both signs to "−" (see the following question).

31. Question: How do you determine the sign in the L.T.'s in Eq. (6.3)?

    Answer: The sign is a "+" if the frame associated with the left side of the equation (the ground, in Eq. (6.3)) sees the frame associated with the right side (the train) moving in the positive direction. The sign is a "−" if the motion is in the negative direction. This rule follows from looking at the motion of a specific point in the train frame. Since $\Delta x_{\text{t}} = 0$ for two events located at a specific point in the train, the L.T. for x becomes $\Delta x_{\text{g}} = \pm\gamma v\,\Delta t_{\text{t}}$. So if the point moves in the positive (or negative) direction in the ground frame, then the sign must be "+" (or "−") so that Δx_{g} is positive (or negative).

32. Question: In relativity, the temporal order of two events in one frame may be reversed in another frame. Does this imply that there exists a frame in which I get off a bus before I get on it?

    Answer: No. The order of two events can be reversed in another frame only if the events are spacelike separated, that is, if $\Delta x > c\,\Delta t$ (which means that the events are too far apart for even light to go from one to the other). The two relevant events here (getting on the bus, and getting off the bus) are not spacelike separated, because the bus travels at a speed less than c, of course. They are timelike separated. Therefore, in all frames it is the case that I get off the bus after I get on it.

There would be causality problems if there existed a frame in which I got off the bus before I got on it. If I break my ankle getting off a bus, then I wouldn't be able to make the mad dash that I made to catch the bus in the first place, in which case I wouldn't have the opportunity to break my ankle getting off the bus, in which case I could have made the mad dash to catch the bus and get on, and, well, you get the idea.

33. QUESTION: Does the longitudinal Doppler effect depend on whether the source or the observer is the one that is moving in a given frame?

    ANSWER: No. Since there is no preferred reference frame, only the relative motion matters. If light needed an "ether" to propagate in, then there would be a preferred frame. But there is no ether. This should be contrasted with the everyday Doppler effect for sound. Sounds needs air (or some other medium) to propagate in. So in this case there *is* a preferred frame – the rest frame of the air.

Dynamics

34. QUESTION: How can you prove that $E = \gamma mc^2$ and $p = \gamma mv$ are conserved?

    ANSWER: You can't. Although there are strong theoretical reasons why the E and p given by these expressions should be conserved, in the end it comes down to experiment. And every experiment that has been done so far is consistent with these E and p being conserved. But this is no proof, of course. As is invariably the case, these expressions are undoubtedly just the limiting expressions of a more correct theory.

35. QUESTION: The energy of an object with mass m and speed v is $E = \gamma mc^2$. Is the statement, "A photon has zero mass, so it must have zero energy," correct or incorrect?

    ANSWER: It is incorrect. Although m is zero, the γ factor is infinite because $v = c$ for a photon. And infinity times zero is undefined. A photon does indeed have energy, and it happens to equal hf, where h is Planck's constant and f is the frequency of the light.

36. QUESTION: A particle has mass m. Is its relativistic mass equal to γm?

    ANSWER: Maybe. Or more precisely: if you want it to be. You can define the quantity γm to be whatever you want. But calling it "relativistic mass" isn't the most productive definition, because γm already goes by another name. It's just the energy, up to factors of c. The use of the word "mass" for this quantity, although quite permissible, is certainly not needed. See the discussion on page 129.

37. QUESTION: When using conservation of energy in a relativistic collision, do you need to worry about possible heat generated, as you do for nonrelativistic collisions?

    ANSWER: No. The energy γmc^2 is conserved in relativistic collisions, period. Any heat that is generated in a particle shows up as an increase in mass. Of course, energy is also conserved in nonrelativistic collisions, period. But if heat is generated, then the energy isn't all in the form of $mv^2/2$ kinetic energies of macroscopic particles.

38. QUESTION: How does the relativistic energy γmc^2 reduce to the nonrelativistic kinetic energy $mv^2/2$?

    ANSWER: The Taylor approximation $1/\sqrt{1 - v^2/c^2} \approx 1 + v^2/2c^2$ turns γmc^2 into $mc^2 + mv^2/2$. The first term is the rest energy. If we assume that a collision is elastic, which means that the masses don't change, then conservation of γmc^2 reduces to conservation of $mv^2/2$.

39. QUESTION: Given the energy E and momentum p of a particle, what is the quickest way to obtain the mass m?

    ANSWER: The quickest way is to use the "Very Important Relation" in Eq. (3.12). Whenever you know two of the three quantities E, p, and m, this equation gives you the third. This isn't the only way to obtain m, of course. For example, you can use $v/c^2 = p/E$ to get v, and then plug the result into $E = \gamma mc^2$. But the nice thing about Eq. (3.12) is that you never have to deal with v.

40. QUESTION: For a collection of particles, why is the value of $E^2_{\text{total}} - p^2_{\text{total}}c^2$ invariant, as Eq. (3.26) states? What is the invariant value?

    ANSWER: $E^2_{\text{total}} - p^2_{\text{total}}c^2$ is invariant because E_{total} and p_{total} transform according to the L.T.'s (see Eq. (3.25)), due to the fact that the single-particle transformations in Eq. (3.20) are *linear*. Given that E_{total} and p_{total} do indeed transform via the L.T.'s, the invariance of $E^2_{\text{total}} - p^2_{\text{total}}c^2$ follows from a calculation similar to the one for $c^2(\Delta t)^2 - (\Delta x)^2$ in Eq. (2.25).

    In the center-of-mass (or really center-of-momentum) frame, the total momentum is zero. So the invariant value of $E^2_{\text{total}} - p^2_{\text{total}}c^2$ equals $(E^{\text{CM}}_{\text{total}})^2$. For a single particle, this is simply $(mc^2)^2 = m^2c^4$, as we know from Eq. (3.12).

41. QUESTION: Why do E and p transform the same way Δt and Δx do, via the L.T.'s?

    ANSWER: In Section 3.2 we used the velocity-addition formula to show that E and p transform via the L.T.'s. This derivation, however, doesn't make it intuitively clear why E and p should transform like Δt and Δx. In contrast, the 4-vector approach in Section 4.2 makes it quite clear. To obtain the energy-momentum 4-vector $(E, \mathbf{p})$ from the displacement 4-vector $(dt, d\mathbf{r})$, we simply need to divide by the proper time $d\tau$ (which is an invariant) and then multiply by the mass m (which is again an invariant). The result is therefore still a 4-vector (which is by definition a 4-tuple that transforms according to the L.T.'s).

42. QUESTION: Why do the *differences* in the coordinates, Δx and Δt, transform via the L.T.'s, while it is the actual *values* of E and p that transform via the L.T.'s?

    ANSWER: First, note that it wouldn't make any sense for the x and t coordinates themselves to transform via the L.T.'s, as we saw in Question 29. Second, as we noted in the preceding question, the 4-vector approach in Section 4.2 shows that E and p are proportional to the differences Δt and Δx. So E and p have these differences built into them. (Of course, due to the linearity of the L.T.'s, differences of E's and p's also transform via the L.T.'s.)

43. QUESTION: In a nutshell, why isn't F equal to ma (or even γma) in relativity?

    ANSWER: F equals dp/dt in relativity, as it does in Newtonian physics. But the relativistic momentum is $p = \gamma mv$, and γ changes with time. So $dp/dt = m(\gamma\dot{v} + \dot{\gamma}v) = \gamma ma + \dot{\gamma}mv$. The second term here isn't present in the Newtonian case.

44. QUESTION: In a given frame, does the acceleration vector $\mathbf{a}$ necessarily point along the force vector $\mathbf{F}$?

    ANSWER: No. We showed in Eq. (3.65) that $\mathbf{F} = m(\gamma^3 a_x, \gamma a_y)$. This isn't proportional to (a_x, a_y). The different powers of γ come from the facts that $\mathbf{F} = d\mathbf{p}/dt = d(\gamma m\mathbf{v})/dt$, and that γ has a first-order change if v_x changes, but not if v_y changes, assuming that v_y is initially zero. The particle therefore responds differently to forces in the x and y directions. It is easier to accelerate something in the transverse direction.

General relativity

45. QUESTION: How would the non equality (or non proportionality) of gravitational and inertial mass be inconsistent with the Equivalence Principle?

    ANSWER: In a box floating freely in space, two different masses that start at rest with respect to each other will remain that way. But in a freefalling box near the earth, two different masses that start at rest with respect to each other will remain that way if and only if their accelerations are equal. And since $F = ma \implies m_g g = m_i a \implies a = (m_g/m_i)g$, we see that the m_g/m_i ratios must be equal if the masses are to remain at rest with respect to each other. That is, m_g must be proportional to m_i. If this isn't the case, then the masses will diverge, which means that it is possible to distinguish between the two settings, in contradiction to the Equivalence Principle.

46. QUESTION: You are in either a large box accelerating at g in outer space or a large box on the surface of the earth. Is there any experiment you can do that will tell you which box you are in?

    ANSWER: Yes. The Equivalence Principle involves the word "local," or equivalently the words "small box." Under this assumption, you can't tell which box you are in. However, if the box is large, you can imagine letting go of two balls separated by a nonnegligible "vertical" distance. In the box in outer space, the balls will keep the same distance as they "fall." But near the surface of the earth, the gravitational force decreases with height, due to the $1/r^2$ dependence. The top ball will therefore fall slower, making the balls diverge. Alternatively, you can let go of two balls separated by a nonnegligible "horizontal" distance. In the box on the earth, the balls will head toward each other because the gravitational field lines converge to the center of the earth.

47. QUESTION: In a gravitational field, if a low clock sees a high clock run fast by a factor f_1, and if a high clock sees a low clock run slow by a factor f_2, then $f_1 f_2 = 1$. But in a special-relativistic setup, both clocks see the other clock running slow by the factor $f_1 = f_2 = 1/\gamma$. So we have $f_1 f_2 = 1/\gamma^2 \neq 1$. Why is the product $f_1 f_2$ equal to 1 in the GR case but not in the SR case?

    ANSWER: In the GR case, both clocks are in the same frame. After a long time, the two clocks can be slowly moved together without anything exciting or drastic happening to their readings. And when the clocks are finally sitting next to each other, it is certainly true that if B's clock reads, say, twice what A's reads, then A's must read half of what B's reads.

    In contrast, the clocks in the SR case are *not* in the same frame. If we want to finally compare the clocks by bringing them together, then something drastic *does* happen with the clocks. The necessary acceleration that must take place leads to the accelerating clock seeing the other clock whip ahead; see Exercise 1.30. It is still certainly true (as it was in the GR case) that when the clocks are finally sitting next to each other, if B's clock reads twice what A's reads, then A's reads half of what B's reads. But this fact implies nothing about the product $f_1 f_2$ while the clocks are sailing past each other, because the clock readings (or at least one of them) necessarily change in a drastic manner by the time the clocks end up sitting next to each other.

48. QUESTION: How is the maximal-proper-time principle consistent with the result of the standard twin paradox?

    ANSWER: If twin A floats freely in outer space, and twin B travels to a distant star and back, then a simple time-dilation argument from A's point of view tells us that B is younger when he returns. This is consistent with the maximal-proper-time principle, because A is under the influence of only gravity (zero gravity, in fact), whereas B feels a normal force from the spaceship during the turning-around period. So A ends up older.

6.2 Appendix B: Derivations of the Lv/c^2 result

In the second half of Section 1.3.1, we showed that if a train with proper length L moves at speed v with respect to the ground, then as viewed in the ground frame, the rear clock reads Lv/c^2 more than the front clock, at any given instant (assuming, as usual, that the clocks are synchronized in the train frame). There are various other ways to derive this rear-clock-ahead result, so for the fun of it I've listed here all the derivations I can think of. The explanations are terse, but I refer you to the specific problem or section in the text where things are discussed in more detail. Many of these derivations are slight variations of each other, so perhaps they shouldn't all count as separate ones, but here's my list:

1. **Light source on train**: This is the original derivation in Section 1.3.1. Put a light source on a train, at distances $d_\text{f} = L(c-v)/2c$ from the front and $d_\text{b} = L(c+v)/2c$ from the back. You can show that the photons hit the ends of the train simultaneously in the ground frame. However, they hit the ends at different times in the train frame; the difference in the readings on clocks at the ends when the photons hit them is $(d_\text{b} - d_\text{f})/c = Lv/c^2$. Therefore, at a given instant in the ground frame (for example, the moment when the clocks at the ends are simultaneously illuminated by the photons), a person on the ground sees the rear clock read Lv/c^2 more than the front clock.

2. **Lorentz transformation**: The second of Eqs. (2.2) is $\Delta t_\text{g} = \gamma(\Delta t_\text{t} + v\,\Delta x_\text{t}/c^2)$, where the subscripts refer to the frame (ground or train). If two events (for example, two clocks flashing their times) located at the ends of the train are simultaneous in the ground frame, then we have $\Delta t_\text{g} = 0$. And $\Delta x_\text{t} = L$, of course. The above Lorentz transformation therefore gives $0 = \Delta t_\text{t} + vL/c^2 \implies \Delta t_\text{t} = -Lv/c^2$. The minus sign here means that the event with the larger x_t value has the smaller t_t value. In other words, the front clock reads Lv/c^2 less than the rear clock, at a given instant in the ground frame.

3. **Invariant interval**: This is just a partial derivation, because it determines only the magnitude of the Lv/c^2 result, and not the sign. The invariant interval says that $c^2\Delta t_\text{g}^2 - \Delta x_\text{g}^2 = c^2\Delta t_\text{t}^2 - \Delta x_\text{t}^2$. If two events (for example, two clocks flashing their times) located at the ends of the train are simultaneous in the ground frame, then we have $\Delta t_\text{g} = 0$. And $\Delta x_\text{t} = L$, of course. And we also know from length contraction that $\Delta x_\text{g} = L/\gamma$. The invariant interval then gives $c^2(0)^2 - (L/\gamma)^2 = c^2\Delta t_\text{t}^2 - L^2$, which yields $c^2\Delta t_\text{t}^2 = L^2(1 - 1/\gamma^2) = L^2v^2/c^2 \implies \Delta t_\text{t} = \pm Lv/c^2$. As mentioned above, the sign isn't determined by this method. (The correct sign is "$-$" since the front clock is behind.)

4. **Minkowski diagram**: The task of Exercise 2.27 is to use a Minkowski diagram to derive the Lv/c^2 result. The basic goal is to determine how many ct units fit in the segment BC in Fig. 2.14, and also how many ct' units fit in the segment DF in Fig. 2.15.

5. **Walking slowly on a train**: In Problem 1.20 a person walks very slowly at speed u from the back of a train of proper length L to the front. In the frame of the train, the time-dilation effect is second order in u/c and therefore negligible (because the travel time is only first order in $1/u$). But in the frame of the ground, the time-dilation effect is (as you can show) *first* order in u/c. So there is a nonzero effect of the speed u; the γ factor for the person is different from the γ factor for a clock that is fixed on the train. An observer on the ground therefore sees the person's clock advance less than a clock that is fixed on the train.

   Now, the person's clock agrees with clocks at the rear and front at the start and finish, because of the negligible time dilation in the train frame. Therefore, since less time elapses on the person's clock than on the front clock (as measured in the ground frame), the person's clock must have started out reading more time than the front clock (in the ground frame). This then implies that the rear clock must show more time than the front clock, at any given instant in the ground frame. A quantitative analysis shows that this excess time is Lv/c^2.

6. **Consistency arguments**: There are many setups (for example, see the four problems in Section 1.4) where the Lv/c^2 result is an ingredient in explaining a result. Without it, you would encounter a contradiction, such as two different frames giving two different answers to a frame-independent question. So if you wanted to, you could work backwards (under the assumption that everything is consistent in relativity) and let the rear-clock-ahead effect be some unknown time T (which might be zero, for all you know), and then solve for the T that makes everything consistent. You would arrive at $T = Lv/c^2$.

7. **Gravitational time dilation**: This derivation holds only for small v. The task of Exercise 5.18 is to derive (for small v) the Lv/c^2 result by making use of the fact that Lv/c^2 looks a lot like the gh/c^2 term in the GR time-dilation result. If you stand on the ground near the front of a train of length L and then accelerate toward the back with acceleration g, you will see a clock at the back running faster by a factor $(1 + gL/c^2)$, which will cause it to read $(gL/c^2)t = Lv/c^2$ more than a clock at the front, after a time t. (Assume that you accelerate for a short period of time, so that $v \approx gt$, and so that the distance in the gL/c^2 term remains essentially L.)

8. **Accelerating rocket**: The task of Problem 5.9 is to show that if a rocket with proper length L accelerates at g and reaches a speed v, then in the ground frame the readings on the front and rear clocks are related by $t_\text{f} = t_\text{b}(1 + gL/c^2) - Lv/c^2$. In other words, the front clock reads $t_\text{b}(1 + gL/c^2) - Lv/c^2$ simultaneously with the rear clock reading t_b, in the ground frame. But in the rocket frame, gravitational time dilation tells us that the front clock reads $t_\text{b}(1 + gL/c^2)$ simultaneously with the rear clock reading t_b. The difference in clock readings (front minus rear) is therefore smaller in the ground frame than in the rocket frame, by an amount equal to Lv/c^2. This is the desired result. (Of course, the 4-star Problem 5.9 certainly isn't the quickest way to derive it!)

If the above derivations aren't sufficient to make you remember the rear-clock-ahead result, here's a little story that should do the trick:

Once upon a time there was a family with the surname of Rhee. This family was very wealthy, although it so happened that they had only one son. This son was destined to inherit the entire Rhee fortune, but despite the promise of future riches, his two prized possessions as a child were very modest ones. The first was a teddy bear he named Elvie, and the second was a clock in the shape of a large head. The face of the head served as the face of the clock, with the mouth at 6 o'clock and the eyes at 1:15 and 10:30. It was said to have been built in the likeness of a great-great-grandfather in the Rhee line, who was moderately disfigured.

When not playing with Elvie or watching the hands of the clock go round and round, the son spent a great deal of time making letters out of pieces of wire. His parents' view was that this was obviously the best way for him to learn the alphabet. As the saying goes, you never forget what you've built out of pieces of wire. However, the h's and y's caused great consternation, not to mention the f's, k's, t's and x's. (It is amusing to note that his parents created a personalized 24-letter alphabet for him, due to his repeated convulsions at the sight of i's and j's.)

His favorite letter was c, being the simplest letter, except perhaps for the l (which, however, requires at least an honest attempt at a straight line). He therefore made lots of c's. In fact, he made so many that he eventually got bored with them. So he started making variations. His favorite variation was bending his old c's into square ones with two corners, like block letters. That was fun. Lots of fun. He liked squaring the c's.

One day he was playing with his friend Albert in his room, tossing Elvie back and forth. An errant throw landed Elvie on top of a big c that he had squared, which happened to be right by the large clock head. Both children sat in silence for a few moments, contemplating the implications of what they were seeing. Albert smacked his hand on his forehead, wondering how he had never realized before what now seemed so obvious. "Of course!" he exclaimed, "The Rhee heir clock is a head by Elvie over c squared!"

6.3 Appendix C: Resolutions to the twin paradox

The twin paradox appeared in Chapters 1, 2, and 5, both in the text and in various problems. To summarize, the twin paradox deals with twin A who stays on the earth[1] and twin B who travels quickly to a distant star and back. When they meet up again, they discover that B is younger. This is true because A can use the standard special-relativistic time-dilation result to say that B's clock runs slow by a factor γ.

The "paradox" arises from the fact that the situation seems symmetrical. That is, it seems as though each twin should be able to consider herself to be at rest, so that she sees the other twin's clock running slow. So why does B turn out to be younger? The resolution to the paradox is that the setup is in fact *not* symmetrical, because B must turn around and thus undergo acceleration. She is therefore not always in an inertial frame, so she cannot always apply the simple special-relativistic time-dilation result.

While the above reasoning is sufficient to get rid of the paradox, it isn't quite complete, because (a) it doesn't explain how the result from B's point of view quantitatively

[1] We should actually have A floating in space, to avoid any general-relativistic time-dilation effects from the earth's gravity. But if B travels quickly enough, the special-relativistic effects will dominate the gravitational ones.

agrees with the result from A's point of view, and (b) the paradox can be formulated without any mention of acceleration, in which case a slightly different line of reasoning applies.

Below is a list of all the complete resolutions (explaining things from B's point of view) that I can think of. The descriptions are terse, but I refer you to the specific problem or section in the text where things are discussed in more detail. As with the Lv/c^2 derivations in Appendix B, many of these resolutions are slight variations of each other, so perhaps they shouldn't all count as separate ones, but here's my list:

1. **Rear-clock-ahead effect**: Let the distant star be labeled as C. Then on the outward part of the journey, B sees C's clock ahead of A's by Lv/c^2, because C is the rear clock in the universe as the universe flies by B. But after B turns around, A becomes the rear clock and is therefore now Lv/c^2 ahead of C. Since nothing unusual happens with C's clock, which is right next to B during the turnaround, A's clock must therefore jump forward very quickly (by an amount $2Lv/c^2$), from B's point of view. See Exercise 1.30.

2. **Looking out the portholes**: Imagine many clocks lined up between the earth and the star, all synchronized in the earth-star frame. And imagine looking out the portholes of the spaceship and making a movie of the clocks as you fly past them (or rather as the clocks fly past you, as viewed from the spaceship frame). Although you see each individual clock running slow, you see the "effective" clock in the movie (which is really many successive clocks) running fast. This is true because each successive clock is a "rear" clock relative to the previous one, which means that it is a little bit ahead of the previous one. This effect is just a series of small applications of the calculation in the first example in Section 1.4. The same reasoning applies during the return trip.

3. **Minkowski diagram**: Draw a Minkowski diagram with the axes in A's frame perpendicular. Then the lines of simultaneity (that is, the successive x axes) in B's frame are titled in different directions for the outward and return parts of the journey. The change in the tilt at the turnaround causes a large amount of time to advance on A's clock, as measured in B's frame. See Fig. 2.41.

4. **General-relativistic turnaround effect**: The acceleration that B feels when she turns around may equivalently be thought of as a gravitational field. Twin A on the earth is high up in the gravitational field, so B sees A's clock run very fast during the turnaround. This causes A's clock to show more time in the end. See Problem 5.12.

5. **Doppler effect**: By equating the total number of signals one twin sends out with the total number of signals the other twin receives, we can relate the total times on their clocks. See Exercise 2.32.

6.4 Appendix D: Lorentz transformations

In this Appendix, we will present an alternative derivation of the Lorentz transformations given in Eq. (2.2). The goal here will be to derive them from scratch, using only the two postulates of relativity. We will *not* use any of the fundamental effects from Section 1.3. Our strategy will be to use the relativity postulate ("all inertial frames are equivalent") to figure out as much as we can, and to then invoke the speed-of-light postulate at the end. The main reason for doing things in this order is that it will allow us to derive a very interesting result in Section 2.7.

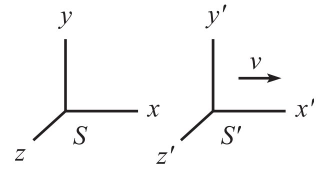

Figure 6.1

As in Section 2.1, consider a reference frame S' moving relative to another frame S; see Fig. 6.1. Let the constant relative speed between the frames be v. Let the corresponding axes of S and S' point in the same direction, and let the origin of S' move along the x axis of S. As in Section 2.1, we want to find the constants, A, B, C, and D, in the relations,

$$\begin{aligned} \Delta x &= A\,\Delta x' + B\,\Delta t', \\ \Delta t &= C\,\Delta t' + D\,\Delta x'. \end{aligned} \tag{6.4}$$

The four constants will end up depending on v (which is constant, given the two inertial frames). Since we have four unknowns, we need four facts. The facts we have at our disposal (using only the two postulates of relativity) are the following.

1. The physical setup: S' moves with velocity v with respect to S.

2. The principle of relativity: S should see things in S' in exactly the same way as S' sees things in S (except perhaps for a minus sign in some relative positions, but this just depends on our arbitrary choice of directional signs for the axes).

3. The speed-of-light postulate: A light pulse with speed c in S' also has speed c in S.

The second statement here contains two independent bits of information. (It contains at least two, because we will indeed be able to solve for our four unknowns. And it contains no more than two, because otherwise our four unknowns would be over-constrained.) The two bits that are used depend on personal preference. Three that are commonly used are: (a) the relative speed looks the same from either frame, (b) time dilation (if any) looks the same from either frame, and (c) length contraction (if any) looks the same from either frame. It is also common to recast the second statement in the form: The Lorentz transformations are the same as their inverse transformations (up to a minus sign). We'll choose to work with (a) and (b). Our four independent facts are then:

1. S' moves with velocity v with respect to S.

2. S moves with velocity $-v$ with respect to S'. The minus sign here is due to the fact that we picked the positive x axes of the two frames to point in the same direction.

3. Time dilation (if any) looks the same from either frame.

4. A light pulse with speed c in S' also has speed c in S.

Let's see what these imply, in the above order. In what follows, we could obtain the final result a little quicker if we invoked the speed-of-light fact prior the time-dilation one. But we'll do things in the above order so that we can easily carry over the results of this appendix to the discussion in Section 2.7.

- Fact 1 says that a given point in S' moves with velocity v with respect to S. Letting $x' = 0$ (which is understood to be $\Delta x' = 0$, but we'll drop the Δ's from here on) in Eqs. (6.4) and dividing them gives $x/t = B/C$. This must equal v. Therefore, $B = vC$, and the transformations become

$$\begin{aligned} x &= Ax' + vCt', \\ t &= Ct' + Dx'. \end{aligned} \tag{6.5}$$

- Fact 2 says that a given point in S moves with velocity $-v$ with respect to S'. Letting $x = 0$ in the first of Eqs. (6.5) gives $x'/t' = -vC/A$. This must equal $-v$. Therefore, $C = A$, and the transformations become

$$\begin{aligned} x &= Ax' + vAt', \\ t &= At' + Dx'. \end{aligned} \tag{6.6}$$

Note that these are consistent with the Galilean transformations, which have $A = 1$ and $D = 0$.

- Fact 3 can be used in the following way. How fast does a person in S see a clock in S' tick? (The clock is assumed to be at rest with respect to S'.) Let our two events be two successive ticks of the clock. Then $x' = 0$, and the second of Eqs. (6.6) gives

$$t = At'. \tag{6.7}$$

In other words, one second on S''s clock takes a time of A seconds in S's frame.

Consider the analogous situation from S''s point of view. How fast does a person in S' see a clock in S tick? (The clock is now assumed to be at rest with respect to S, in order to create the analogous setup. This is important.) If we invert Eqs. (6.6) to solve for x' and t' in terms of x and t, we find

$$\begin{aligned} x' &= \frac{x - vt}{A - Dv}, \\ t' &= \frac{At - Dx}{A(A - Dv)}. \end{aligned} \tag{6.8}$$

Two successive ticks of the clock in S satisfy $x = 0$, so the second of Eqs. (6.8) gives

$$t' = \frac{t}{A - Dv}. \tag{6.9}$$

In other words, one second on S's clock takes a time of $1/(A - Dv)$ seconds in S''s frame.

Eqs. (6.7) and (6.9) apply to the same situation (someone looking at a clock flying by). Therefore, the factors on the righthand sides must be equal, that is,

$$A = \frac{1}{A - Dv} \qquad \Longrightarrow \qquad D = \frac{1}{v}\left(A - \frac{1}{A}\right). \tag{6.10}$$

Our transformations in Eq. (6.6) therefore take the form

$$\begin{aligned} x &= A(x' + vt'), \\ t &= A\left(t' + \frac{1}{v}\left(1 - \frac{1}{A^2}\right)x'\right). \end{aligned} \tag{6.11}$$

These are consistent with the Galilean transformations, which have $A = 1$.

- Fact 4 may now be used to say that if $x' = ct'$, then $x = ct$. In other words, if $x' = ct'$, then

$$c = \frac{x}{t} = \frac{A\big((ct') + vt'\big)}{A\left(t' + \frac{1}{v}\left(1 - \frac{1}{A^2}\right)(ct')\right)} = \frac{c + v}{1 + \frac{c}{v}\left(1 - \frac{1}{A^2}\right)}. \tag{6.12}$$

Solving for A gives

$$A = \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{6.13}$$

We have chosen the positive square root so that the coefficient of t' is positive; if t' increases, then t should also increase. (The forward direction of time should be the same in the two frames.) The transformations are now no longer consistent with the Galilean transformations, because c is not infinite, which means that A is not equal to 1.

The constant A is commonly denoted by γ, so we may finally write our Lorentz transformations, Eqs. (6.11), in the form (using $1 - 1/\gamma^2 = v^2/c^2$),

$$\begin{aligned} x &= \gamma(x' + vt'), \\ t &= \gamma(t' + vx'/c^2), \end{aligned} \tag{6.14}$$

where

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}, \tag{6.15}$$

in agreement with Eq. (2.2).

6.5 Appendix E: Nonrelativistic dynamics

Chapter 3 covers relativistic dynamics, that is, relativistic momentum, energy, force, etc. If you haven't taken a standard mechanics course, the present appendix will get you up to speed on Newtonian (nonrelativistic) dynamics. The discussion here will necessarily be brief. The goal is to cover the basics of the relevant topics, as opposed to presenting a comprehensive introduction to dynamics.

Newton's laws

The subject of dynamics is governed by Newton's three laws. The *first law* states that an object continues to move with constant velocity (which may be zero) unless acted on by a force. Of course, we haven't defined what a force is yet, so this law might seem a little circular. But what the first law does is define an *inertial frame*, which is simply a frame in which the first law holds. If an object isn't interacting with any other objects and is moving with constant velocity in a given frame, then that frame is an inertial one. If, on the other hand, the object's velocity is changing (still assuming no interaction with any other objects), then the frame isn't inertial. For example, if an object is floating freely in space, and if it is enclosed in a box and the box is accelerating, then with respect to the box, the object's velocity changes. The box therefore represents a *noninertial* reference frame. In Newtonian physics, inertial frames are important because they are the frames in which the second law (discussed below) holds. In relativistic physics, inertial frames are important because they are the frames in which Einstein's two postulates hold.

Newton's *second law* is often written as

$$\mathbf{F} = m\mathbf{a}, \tag{6.16}$$

where m is the mass of an object and $\mathbf{a}$ is its acceleration. ($\mathbf{F} = m\mathbf{a}$ is a vector equation, so it is really shorthand for the three equations, $F_x = ma_x$, $F_y = ma_y$, and $F_z = ma_z$.) However, what the law actually states is

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \qquad \text{(Newton's second law)} \tag{6.17}$$

where the momentum $\mathbf{p}$ of an object is defined as

$$\mathbf{p} = m\mathbf{v}, \tag{6.18}$$

with $\mathbf{v}$ being the object's velocity. In the common case where the mass m is constant, the time derivative in Eq. (6.17) acts only on the $\mathbf{v}$, so the second law simplifies to the form in Eq. (6.16):

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d(m\mathbf{v})}{dt} = m\frac{d\mathbf{v}}{dt} \implies \mathbf{F} = m\mathbf{a}. \tag{6.19}$$

Since most mechanics problems involve objects with constant mass, the $\mathbf{F} = m\mathbf{a}$ form of the law usually suffices.

Newton's *third law* states that the forces that two objects exert on each other are equal and opposite. If we use the notation $\mathbf{F}_{ij}$ to denote the force on object i due to object j, then we have

$$\mathbf{F}_{12} = -\mathbf{F}_{21} \qquad \text{(Newton's third law)} \tag{6.20}$$

Conservation of momentum

In an isolated system of particles, the combination of the second and third laws implies that the total momentum is conserved (constant in time). Consider first the simple case of two isolated particles. Since the second law tells us that $\mathbf{F} = d\mathbf{p}/dt$, we have $\mathbf{F}_{12} = d\mathbf{p}_1/dt$ and $\mathbf{F}_{21} = d\mathbf{p}_2/dt$. The third law, $\mathbf{F}_{12} = -\mathbf{F}_{21}$, then becomes

$$\frac{d\mathbf{p}_1}{dt} = -\frac{d\mathbf{p}_2}{dt} \implies \frac{d(\mathbf{p}_1 + \mathbf{p}_2)}{dt} = 0 \implies \mathbf{p}_1 + \mathbf{p}_2 = \text{constant}. \tag{6.21}$$

In other words, the total momentum is conserved. Whatever momentum particle 2 gives to particle 1, particle 1 gives an equal and opposite momentum to particle 2. We can alternatively show that the total momentum is conserved by writing the second law in integral form:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \implies \int \mathbf{F}\,dt = \int d\mathbf{p} \implies \int \mathbf{F}\,dt = \Delta\mathbf{p}. \tag{6.22}$$

This implies

$$\Delta\mathbf{p}_{\text{total}} = \Delta\mathbf{p}_1 + \Delta\mathbf{p}_2 = \int \mathbf{F}_{12}\,dt + \int \mathbf{F}_{21}\,dt = 0, \tag{6.23}$$

because $\mathbf{F}_{12} = -\mathbf{F}_{21}$, by the third law.

Consider now the case of three isolated particles. The rate of change of the total momentum of the system equals

$$\begin{aligned}
\frac{d(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3)}{dt} &= \frac{d\mathbf{p}_1}{dt} + \frac{d\mathbf{p}_2}{dt} + \frac{d\mathbf{p}_3}{dt} \\
&= \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 \\
&= (\cancel{\mathbf{F}_{12}} + \cancel{\mathbf{F}_{13}}) + (\cancel{\mathbf{F}_{21}} + \cancel{\mathbf{F}_{23}}) + (\cancel{\mathbf{F}_{31}} + \cancel{\mathbf{F}_{32}}) \\
&= 0.
\end{aligned} \tag{6.24}$$

We have used the fact that Newton's third law, $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$, tells us that the forces cancel in pairs, as indicated. This reasoning easily extends to a general number N of particles. For every term $\mathbf{F}_{ij}$ in the generalization of the third line above, there is also a term $\mathbf{F}_{ji}$, and the sum of these two terms is zero by Newton's third law.

Work

The (nonrelativistic) kinetic energy of an object is defined as

$$K = \frac{mv^2}{2}. \tag{6.25}$$

The reason why this quantity is important (and why it is conserved) is that it appears in the *work-energy theorem*, which we will prove below. In 1-D, the *work* done by a constant force acting on an object (directed parallel to the line of motion of the object) is defined to be the force times the displacement:

$$W = F\,\Delta x. \tag{6.26}$$

Work is a signed quantity; if the force points opposite to the direction of motion (so that F and Δx have opposite signs), then W is negative. For a force that isn't constant, the work is defined to be the integral of the force:

$$W = \int F\,dx. \tag{6.27}$$

The *work-energy theorem* states that the work done on a particle equals the change in kinetic energy of the particle. That is,

$$W = \Delta K \qquad \text{(work-energy theorem)} \tag{6.28}$$

This theorem is consistent with our intuition. From $F = ma$, we know that if the force points in the same direction as the velocity of a particle, then the speed increases, so the kinetic energy increases. That is, ΔK is positive. This is consistent with the fact that the work W is positive if the force points in the same direction as the velocity. Conversely, if the force points opposite to the velocity of the particle, then the speed decreases, so the kinetic energy decreases. This is consistent with the fact that the work is negative.

To prove the work-energy theorem, we will need to use the fact that the acceleration a of an object can be written as $a = v\,dv/dx$. This is true because if we write a as dv/dt and then multiply by 1 in the form of dx/dx, we obtain

$$a = \frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx}, \tag{6.29}$$

as desired. Plugging $a = v\,dv/dx$ into $F = ma$ and then multiplying by dx and integrating from x_1 (and the corresponding v_1) to x_2 (and the corresponding v_2) gives

$$F = ma \implies F = mv\frac{dv}{dx} \implies \int_{x_1}^{x_2} F\,dx = \int_{v_1}^{v_2} mv\,dv. \tag{6.30}$$

The integral on the lefthand side is the work done, by definition. The integral of v on the righthand side is $v^2/2$, so we obtain

$$W = \frac{mv^2}{2}\bigg|_{v_1}^{v_2} = \frac{mv_2^2}{2} - \frac{mv_1^2}{2} = \Delta K, \tag{6.31}$$

as desired.

Conservation of energy

Consider the simple case of an isolated system of two particles (with no internal structure; see below). If these particles collide and interact only via contact forces (as opposed to, say, the gravitational force, which acts over a distance), then we claim that the sum of the kinetic energies of the particles is conserved. This follows from the work-energy theorem and Newton's third law:

$$\Delta K_{\text{total}} = \Delta K_1 + \Delta K_2 = W_1 + W_2 = \int \mathbf{F}_{12}\,dx + \int \mathbf{F}_{21}\,dx = 0, \tag{6.32}$$

because $\mathbf{F}_{12} = -\mathbf{F}_{21}$, by the third law. Whatever work particle 2 does on particle 1, particle 1 does an equal and opposite amount of work on particle 2. We have used the fact that the dx's in the two integrals in Eq. (6.32) are the same. This is true because the force is a *contact* force, which means that the point of application on particle 1 is the same point in space as the point of application on particle 2. As with conservation of momentum, the above conservation-of-energy result generalizes to more than two particles. The forces (and hence works) cancel in pairs due to Newton's third law, so the total kinetic energy of any isolated system of particles (with no internal structure) is conserved.

Note the parallel between the above derivations of conservation of momentum and conservation of energy. Conservation of momentum follows from integrating $\mathbf{F}$ with respect to *time* (that is, $\int \mathbf{F}\,dt$) and applying Newton's third law; see Eq. (6.23). Similarly, conservation of energy follows from integrating $\mathbf{F}$ with respect to *displacement* (that is, $\int \mathbf{F}\,dx$) and applying Newton's third law; see Eq. (6.32). The only difference is replacing t with x. The critical similarity is that the forces $\mathbf{F}_{ij}$ and $\mathbf{F}_{ji}$ (a third-law pair) between two objects act not only for the same *time*, but also for the same *displacement* (assuming that the force is a contact force).

In the collisions we are concerned with, we are assuming that each particle is the same before and after the collision. Equivalently, as mentioned above, we are assuming that the particles have no internal structure where other forms of energy might be hiding. The most common form of "hidden" energy is heat. This is simply the kinetic energy (and potential energy, too) of molecules on a microscopic scale, as opposed to the macroscopic kinetic energy associated with the motion of the particle as a whole. If no heat is generated, we call a collision *elastic*; whereas if heat is generated, we call it *inelastic*. So in summary, in an isolated collision,

1. Momentum is always conserved.
2. Energy is also always conserved, although *mechanical energy* (by which we mean the $mv^2/2$ energies of the macroscopic particles involved; that is, excluding heat) is conserved only if the collision is elastic (by definition).

In the more general case where there are non-contact forces (such as gravitational or electric) or where an object deforms (as with a spring), the work done by these types of forces may be (under certain circumstances) associated with a *potential energy*. The general conservation-of-energy statement is then that the total kinetic plus potential energy of a system (including heat) is conserved. But for the collisions we discuss in Chapter 3, we don't need to worry about potential energy.

Collision examples

Figure 6.2

Consider the 1-D setup shown in Fig. 6.2. Masses m_1 and m_2 have initial velocities v_1 and v_2. These are signed quantities, so they may be positive or negative (rightward is positive, leftward is negative). The masses collide elastically in 1-D and acquire final velocities u_1 and u_2. To determine u_1 and u_2, we simply need to write down the conservation of momentum and energy equations and then do some algebra. Conservation of p gives

$$\begin{aligned} p_{\text{before}} &= p_{\text{after}} \\ \Longrightarrow\quad m_1v_1 + m_2v_2 &= m_1u_1 + m_2u_2. \end{aligned} \tag{6.33}$$

And conservation of E gives

$$\begin{aligned} E_{\text{before}} &= E_{\text{after}} \\ \Longrightarrow\quad \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 &= \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2. \end{aligned} \tag{6.34}$$

Eqs. (6.33) and (6.34) are two equations in the two unknowns u_1 and u_2. Solving them involves solving a quadratic equation (although there are some sneakier and less time consuming ways). The result is fairly messy and not terribly enlightening, so we won't bother writing it down. The point we want to make here is simply that to solve for the final motion of the particles, you just need to write down the conservation of p and E equations and then do some algebra.

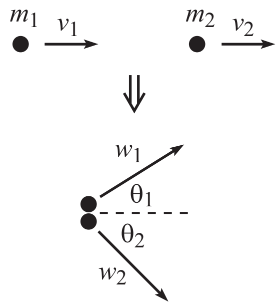

Figure 6.3

Consider now the 2-D setup shown in Fig. 6.3. The initial motion is the same as in Fig. 6.2, but now the masses are free to scatter elastically in two dimensions. The final velocities are described by the final *speeds* w_1 and w_2 (these are positive quantities, by definition) and the angles θ_1 and θ_2. Since momentum is a vector, it is conserved in both the x and y directions. So conservation of **p** gives us *two* equations:

$$\begin{aligned} p_x: &\qquad m_1 v_1 + m_2 v_2 = m_1 w_1 \cos\theta_1 + m_2 w_2 \cos\theta_2, \\ p_y: &\qquad 0 = m_1 w_1 \sin\theta_1 - m_2 w_2 \sin\theta_2. \end{aligned} \tag{6.35}$$

(There is technically also the conservation-of-p_z equation, but that is just the trivial statement $0 = 0$ in the present setup.) The conservation-of-E equation is exactly the same as in Eq. (6.34), because kinetic energy depends only on the speed (the magnitude of the velocity); the direction is irrelevant. So we have

$$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1 w_1^2 + \frac{1}{2}m_2 w_2^2. \tag{6.36}$$

The preceding three equations are three equations in four unknowns (w_1, w_2, θ_1, and θ_2), so it is impossible to solve for all of the unknowns. (The physical reason for this is that we would need to be told exactly how the masses glance off each other, to be able to figure out what the final angles are.) But if someone gives us the additional information of what one of the four unknowns is, then we can solve for the other three (with some tedious algebra).

The nice thing about conservation laws is that we don't need to worry about the messy/intractible specifics of what goes on during a collision. We simply have to write down the initial and final expressions for **p** and E and then solve for whatever variable(s) we're trying to determine.

6.6 Appendix F: Problem-solving strategies

Given the large number of problems that appear in this book, it's a good idea to provide you with some strategies for solving them. The strategies discussed below are certainly not specific to relativity; they can be applied in any subject in physics (along with many other sciences). The examples we'll use are taken from basic mechanics. If you haven't studied mechanics yet, rest assured, you'll still be able to understand the problem-solving strategy being discussed, even without knowing the relevant physics. The strategies we'll cover here are (1) solving problems symbolically (with letters instead of numbers), (2) checking units, and (3) checking limiting cases.

6.6.1 Solving problems symbolically

If you are solving a problem where the given quantities are specified numerically, it is highly advantageous to immediately change the numbers to letters and then solve the problem in terms of the letters. After you obtain a symbolic answer in terms of these letters, you can plug in the actual numerical values to obtain a numerical answer. There are many advantages to using letters:

- It is quicker. It's much easier to multiply a g by an ℓ by writing them down on a piece of paper next to each other, than it is to multiply their numerical values on a calculator. If solving a problem involves five or ten such operations, the time would add up if you performed all the operations on a calculator.

- You are less likely to make a mistake. It's very easy to mistype an 8 for a 9 in a calculator, but you're probably not going to miswrite a q for an a on a piece of paper. But even if you do, you'll quickly realize that it should be an a. You certainly won't just give up on the problem and deem it unsolvable because no one gave you the value of q!

- You can do the problem once and for all. If someone comes along and says, oops, the value of ℓ is actually 2.4 m instead of 2.3 m, then you won't have to do the whole problem again. You can simply plug the new value of ℓ into your symbolic answer.

- You can see the general dependence of your answer on the various given quantities. For example, you can see that it grows with quantities a and b, decreases with c, and doesn't depend on d. There is *much* more information contained in a symbolic answer than in a numerical one. And besides, symbolic answers nearly always look nice and pretty.

- You can check units and special cases. These checks go hand-in-hand with the preceding "general dependence" advantage. We'll discuss these very important checks below.

Two caveats to all this: First, occasionally there are times when things get messy when working with letters. For example, solving a system of three equations in three unknowns might be rather cumbersome unless you plug in the actual numbers. But in the vast majority of problems, it is advantageous to work entirely with letters. Second, if you solve a problem that was posed with letters instead of numbers, it's always a good idea to pick some values for the various parameters to see what kinds of numbers pop out, just to get a general sense of the size of things.

6.6.2 Checking units/dimensions

The words *dimensions* and *units* are often used interchangeably, but there is technically a difference: dimensions refer to the general qualities of mass, length, time, etc., whereas units refer to the specific way we quantify these qualities. For example, in the standard meters-kilogram-second (mks) system of units we use in this book, the meter is the unit associated with the dimension of length, the joule is the unit associated with the dimension of energy, and so on. However, we'll often be sloppy and ignore the difference between units and dimensions.

The consideration of units offers two main benefits when solving problems:

- Consider units at the start. Considering the units of the relevant quantities before you start solving a problem can tell you roughly what the answer has to look like, up to numerical factors. This practice is called *dimensional analysis*.

- Check units at the end. Checking units at the end of a calculation (which is something you should *always* do) can tell you if your answer has a chance at being correct. It won't tell you that your answer is definitely correct, but it might tell you that your answer is definitely *incorrect*. For example, if your goal in a problem is to find a length, and if you end up with a mass, then you know that it's time to look back over your work.

In the mks system of units, the three fundamental mechanical units are the meter (m), kilogram (kg), and second (s). All other units in mechanics, for example the joule (J) or the newton (N), can be built up from these fundamental three. If you want to work with dimensions instead of units, then you can write everything in terms of length (L), mass (M), and time (T). The difference is only cosmetic.

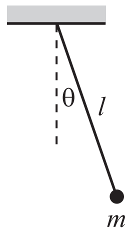

Figure 6.4

As an example of the above two benefits of considering units, consider a pendulum consisting of a mass m hanging from a massless string with length ℓ; see Fig. 6.4. Assume that the pendulum swings back and forth with an angular amplitude θ_0 that is small; that is, the string doesn't deviate far from vertical. What is the period, call it T_0, of the oscillatory motion? (The period is the time of a full back-and-forth cycle.)

With regard to the first of the above benefits, what can we say about the period T_0 (which has units of seconds), by looking only at units and not doing any calculations? Well, we must first make a list of all the quantities the period can possibly depend on. The given quantities are the mass m (with units of kg), the length ℓ (with units of m), and the angular amplitude θ_0 (which is unitless). Additionally, there might be dependence on g (the acceleration due to gravity, with units of m/s^2), If you think for a little while, you'll come to the conclusion that there isn't anything else the period can depend on (assuming that we ignore air resistance).

So the question becomes: How does T_0 depend on m, ℓ, θ_0, and g? Or equivalently: How can we produce a quantity with units of seconds from four quantities with units of kg, m, 1, and m/s^2? (The 1 signifies no units.) We quickly see that the answer can't involve the mass m, because there would be no way to get rid of the units of kg. We then see that if we want to end up with units of seconds, the answer must be proportional to $\sqrt{\ell/g}$, because this gets rid of the meters and leaves one power of seconds in the numerator. Therefore, by looking only at the units involved, we have shown that $T_0 \propto \sqrt{\ell/g}$.[2]

This is all we can say by considering units. For all we know, there might be a numerical factor out front, and also an arbitrary function of θ_0 (which won't mess up the units, because θ_0 is unitless). The correct answer happens to be $T_0 = 2\pi\sqrt{\ell/g}$, but there is no way to know this without solving the problem for real.[3] However, even though we haven't produced an exact result, there is still a great deal of information contained in our $T_0 \propto \sqrt{\ell/g}$ statement. For example, we see that the period is independent of m; a small mass and a large mass swing back and forth at the same frequency. Similarly, the period is independent of θ_0 (as long as θ_0 is small). We also see that if we quadruple the length of the string, then the period gets doubled. And if we place the same pendulum on the moon, where the g factor is $1/6$ of that on the earth, then the period increases by a factor of $\sqrt{6} \approx 2.4$; the pendulum swings back and forth more slowly. Not bad for only considering units!

While this is all quite interesting, the second of the above two benefits (checking the units of an answer) is actually the one that you will get the most mileage out of when solving problems, mainly because you should make use of it *every* time you solve a problem. It only takes a second. In the present example with the pendulum, let's say that you solved the problem correctly and ended up with $T_0 = 2\pi\sqrt{\ell/g}$. You should immediately check the units, which do indeed correctly come out to be seconds. If you

[2]In this setup it was easy to determine the correct combination of the given parameters. But in more complicated setups, you might find it simpler to write down a general product of the given dimensionful quantities raised to arbitrary powers, and then solve a system of equations to determine the correct powers that yield the desired units.

[3]This $T_0 = 2\pi\sqrt{\ell/g}$ result, which is independent of θ_0, holds in the approximation where the amplitude θ_0 is small. For a general value of θ_0, the period actually *does* involve a function of θ_0. This function can't be written in closed form, but it starts off as $1 + \theta_0^2/16 + \cdots$. It takes a lot of work to show this, though. See Exercise 4.23 in Morin (2008).

made a mistake in your solution, such as flipping the square root upside down (so that you instead had $\sqrt{g/\ell}$), then your units check would yield the incorrect units of s^{-1}. You would then know to go back and check over your work. Throughout this book, we often won't bother to explicitly write down the units check if the check is a simple one (as with the above pendulum). But you should of course always do the check in your head.

Having said all this, we should note that it is common practice in relativity to drop the c's in calculations. On one hand, this goes against everything we've just said, because dropping the c's (which have units) means that the units of your answer will in general be incorrect. (You can put the c's back in at the end, of course. But you've lost the ability to catch a mistake by checking units.) On the other hand, the calculations are often *much* simpler without the c's, and this simplicity decreases the probability of making an algebraic mistake. In the end, it comes down to which scenario you prefer – messier calculations with a units check at the end, or simpler calculations with no units check. My view is that in nontrivial calculations, the simplicity usually wins.

6.6.3 Checking limiting/special cases

As with units, the consideration of limiting cases (or perhaps we should say more generally special cases) offers two main benefits. First, it can help you get started on a problem. If you are having trouble figuring out how a given system behaves, you can imagine making, for example, a certain length become very large or very small, and then you can see what happens to the behavior. Having convinced yourself that the length actually affects the system in extreme cases (or perhaps you will discover that the length doesn't affect things at all), it will then be easier to understand how it affects the system in general. This will in turn make it easier to write down the relevant quantitative equations (conservation laws, $F = ma$ equations, etc.), which will allow you to fully solve the problem. In short, modifying the various parameters and seeing the effects on the system can lead to an enormous amount of information.

Second, as with checking units, checking limiting cases (or special cases) is something you should *always* do at the end of a calculation. As with units, checking limiting cases won't tell you that your answer is definitely correct, but it might tell you that your answer is definitely incorrect. Your intuition about limiting cases is invariably *much* better than your intuition about generic values of the parameters. You should use this to your advantage.

As an example, consider the trigonometric formula for $\tan(\theta/2)$. The formula can be written in various ways. Let's say that you're trying to derive it, but you keep making mistakes and getting different answers. However, let's assume that you're fairly sure it takes the form of $\tan(\theta/2) = A(1 \pm \cos\theta)/\sin\theta$, where A is a numerical coefficient. Can you determine the correct form of the answer by checking special cases? Indeed you can, because you know what $\tan(\theta/2)$ equals for a few special values of θ:

- $\theta = 0$: We know that $\tan(0/2) = 0$, so this immediately rules out the $A(1 + \cos\theta)/\sin\theta$ form, because this isn't zero when $\theta = 0$; it actually goes to infinity at $\theta = 0$. The answer must therefore take the $A(1 - \cos\theta)/\sin\theta$ form. This appears to be $0/0$ when $\theta = 0$, but it does indeed go to zero, as you can check by using the Taylor series for $\sin\theta$ and $\cos\theta$; see the subsection on Taylor series below.

- $\theta = 90°$: We know that $\tan(90°/2) = 1$, which quickly gives $A = 1$. So the correct answer must be $\tan(\theta/2) = (1 - \cos\theta)/\sin\theta$.

- $\theta = 180°$: If you want to feel better about this $(1 - \cos\theta)/\sin\theta$ result, you can note that it gives the correct answer for another special value of θ; it correctly goes to infinity when $\theta = 180°$.

Of course, none of what we've done here demonstrates that $(1 - \cos\theta)/\sin\theta$ actually *is* the right answer. But checking the above special cases does two things: it rules out some incorrect answers, and it makes us feel better about the correct answer.

A type of approximation that frequently comes up involves expressions of the form $ab/(a+b)$, that is, a product over a sum. For example, let's say we're trying to determine the value of a mass, and it comes out to be

$$M = \frac{m_1 m_2}{m_1 + m_2}\,. \tag{6.37}$$

What does M look like in the limit where m_1 is much smaller than m_2? In this limit we can ignore the m_1 in the denominator, but we *can't* ignore it in the numerator. So we obtain $M \approx m_1 m_2/(0 + m_2) = m_1$. Why can we can ignore one of the m_1's but not the other? We can ignore the m_1 in the denominator because it appears there as an *additive* term. If m_1 is small, then erasing it essentially doesn't change the value of the denominator. However, in the numerator m_1 appears as a *multiplicative* term. Even if m_1 is small, its value certainly affects the value of the numerator. Decreasing m_1 by a factor of 10 decreases the numerator by the same factor of 10. So we certainly can't just erase it. (That would change the units of M, anyway.)

Alternatively, you can obtain the $M \approx m_1$ result in the limit of small m_1 by applying a Taylor series (discussed below) to M. But this would be overkill. It's much easier to just erase the m_1 in the denominator. In any case, if you're ever unsure about which terms you should keep and which terms you can ignore, just plug some very small numbers (or very large numbers, depending on what limit you're dealing with) into a calculator to see how the expression depends on the various parameters.

It should be noted that there is no need to wait until the end of a solution to check limiting cases (or units, too). Whenever you arrive at an intermediate result that lends itself to checking limiting cases, you should check them. If you find that something is amiss, this will prevent you from wasting time carrying onward with incorrect results.

6.6.4 Taylor series

A tool that is often useful when checking limiting cases is the Taylor series. A Taylor series expresses a function $f(x)$ as a series expansion in x (that is, a sum of terms involving different powers of x). Perhaps the most well-known Taylor series is the one for the function $f(x) = e^x$:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\,. \tag{6.38}$$

A number of other Taylor series are listed near the beginning of Appendix G. The rest of Appendix G contains a discussion of Taylor series and various issues that arise when using them. In the present section, we'll just take the above expression for e^x as given and see where it leads us.[4]

The classic example of the usefulness of a Taylor series in relativity is the demonstration of how the relativistic energy γmc^2 leads to the nonrelativistic energy $mv^2/2$; see Eq. (3.10). However, for the present purposes of illustrating the utility of Taylor

[4]Calculus is required if you want to *derive* a Taylor series. However, if you just want to *use* a Taylor series (which is what we do in this book), then algebra is all you need. So although some Taylor-series manipulations might look a bit scary, there's nothing more than algebra involved.

series, let's consider a mechanics setup in which a beach ball is dropped from rest. It can be shown that if air drag is taken into account, and if the drag force is proportional to the velocity (so that it takes the form $F_{\mathrm{d}} = -bv$, where b is the drag coefficient), then the ball's velocity (with upward taken as positive) as a function of time equals

$$v(t) = -\frac{mg}{b}\left(1 - e^{-bt/m}\right). \tag{6.39}$$

This is a somewhat complicated expression, so you might be a little doubtful of its validity. Let's therefore look at some limiting cases. If these limiting cases yield expected results, then we can feel more comfortable that the expression is actually correct.

If t is very small (more precisely, if $bt/m \ll 1$; see the discussion in Section 6.7.3), then we can use the Taylor series in Eq. (6.38) to make an approximation to $v(t)$, to leading order in t. (The leading-order term is the smallest power of t with a nonzero coefficient.) To first order in x, Eq. (6.38) gives $e^x \approx 1 + x$. If we let x be $-bt/m$, then we see that Eq. (6.39) can be written as

$$\begin{aligned} v(t) &\approx -\frac{mg}{b}\left(1 - \left(1 - \frac{bt}{m}\right)\right) \\ &\approx -gt. \end{aligned} \tag{6.40}$$

This answer makes sense, because the drag force is negligible at the start (because v, and hence bv, is very small), so we essentially have a freely falling body with acceleration g downward. And $v(t) = -gt$ is the correct expression in that case (a familiar result from introductory mechanics). This successful check of a limiting case makes us have a little more faith that Eq. (6.39) is actually correct.

If we mistakenly had, say, $-2mg/b$ as the coefficient in Eq. (6.39), then we would have obtained $v(t) \approx -2gt$ in the small-t limit, which is incorrect. So we would know that we needed to go back and check over our work. Although it isn't obvious that an extra factor of 2 in Eq. (6.39) is incorrect, an extra 2 *is* obviously incorrect in the limiting $v(t) \approx -2gt$ result. As mentioned above, your intuition about limiting cases is generally much better than your intuition about generic values of the parameters.

We can also consider the limit of large t (or rather, large bt/m). In this limit, $e^{-bt/m}$ is essentially zero, so the $v(t)$ in Eq. (6.39) becomes (there's no need for a Taylor series in this case)

$$v(t) \approx -\frac{mg}{b}. \tag{6.41}$$

This is the "terminal velocity" that the ball approaches as time goes on. Its value makes sense, because it is the velocity for which the total force (gravitational plus air drag), $-mg - bv$, equals zero. And zero force means constant velocity. Mathematically, the velocity never quite reaches the value of $-mg/b$, but it gets extremely close as t becomes large.

Whenever you derive approximate answers as we just did, you gain something and you lose something. You lose some truth, of course, because your new answer is an approximation and therefore technically not correct (although the error becomes arbitrarily small in the appropriate limit). But you gain some aesthetics. Your new answer is invariably much cleaner (often involving only one term), and that makes it a lot easier to see what's going on.

In the above beach-ball example, we checked limiting cases of an answer that was correct. This whole process is more useful (and a bit more fun) when you check limiting cases of an answer that is *incorrect* (as in the case of the erroneous coefficient of $-2mg/b$ we mentioned above). When this happens, you gain the unequivocal information that your answer is wrong (assuming that your incorrect answer doesn't just happen to give the correct result in a certain limit, by pure luck). However, rather than leading you into despair, this information is something you should be quite happy about, considering that the alternative is to carry on in a state of blissful ignorance. Once you know that your answer is wrong, you can go back through your work and figure out where the error is (perhaps by checking limiting cases at various intermediate stages to narrow down where the error could be). Personally, if there's any way I'd like to discover that my answer is garbage, this is it. So you shouldn't check limiting cases (and units) because you're being told to, but rather because you *want* to.

6.7 Appendix G: Taylor series

6.7.1 Basics

We saw in Section 6.6.4 that Taylor series can be extremely useful for checking limiting cases, in particular in situations where a given parameter is small. In this appendix we'll discuss Taylor series in more detail.

A Taylor series expresses a given function of x as a series expansion in powers of x. The general form of a Taylor series is (the primes here denote differentiation)

$$f(x_0 + x) = f(x_0) + f'(x_0)x + \frac{f''(x_0)}{2!}x^2 + \frac{f'''(x_0)}{3!}x^3 + \cdots . \tag{6.42}$$

This equality can be verified by taking successive derivatives of both sides of the equation and then setting $x = 0$. For example, taking the first derivative and then setting $x = 0$ yields $f'(x_0)$ on the left. And this operation also yields $f'(x_0)$ on the right, because the first term is a constant and gives zero when differentiated, the second term gives $f'(x_0)$, and all of the rest of the terms give zero once we set $x = 0$, because they all contain at least one power of x. Likewise, if we take the second derivative of each side and then set $x = 0$, we obtain $f''(x_0)$ on both sides. And so on for all derivatives. Therefore, since the two functions on each side of Eq. (6.42) are equal at $x = 0$ and also have their nth derivatives equal at $x = 0$ for all n, they must in fact be the same function (assuming that they're nicely behaved functions, as we generally assume in physics).

Some specific Taylor series that often come up are listed below. They are all expanded around $x = 0$. That is, $x_0 = 0$ in Eq. (6.42). They are all derivable via Eq. (6.42), but sometimes there are quicker ways of obtaining them. For example, Eq. (6.44) is most easily obtained by taking the derivative of Eq. (6.43), which itself is just the sum of a geometric series.

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \tag{6.43}$$

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots \tag{6.44}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \tag{6.45}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{6.46}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \tag{6.47}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \tag{6.48}$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \cdots \tag{6.49}$$

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3x^2}{8} + \cdots \tag{6.50}$$

$$(1+x)^n = 1 + nx + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots \tag{6.51}$$

Each of these series has a range of validity, that is, a "radius of convergence." For example, the series for e^x is valid for all x, while the series for $1/(1+x)$ is valid for $|x| < 1$. The various ranges won't particularly concern us, because whenever we use a Taylor series, we will assume that x is small (much smaller than 1). In this case, all of the series are valid.

The above series might look a little scary, but in most situations there is no need to include terms beyond the first-order term in x. For example, $\sqrt{1+x} \approx 1 + x/2$ is usually a good enough approximation. The smaller x is, the better the approximation is, because any term in the expansion is smaller than the preceding term by a factor of order x. Note that you can quickly verify that the $\sqrt{1+x} \approx 1 + x/2$ expression is valid to first order in x, by squaring both sides to obtain $1 + x \approx 1 + x + x^2/4$. Similar reasoning at second order shows that $-x^2/8$ is correctly the next term in the expansion.

As mentioned in Footnote 4 in Appendix F, we won't worry about taking derivatives to rigorously derive the above Taylor series. We'll just take them as given, which means that if you haven't studied calculus yet, that's no excuse for not using Taylor series! Instead of deriving them, let's just check that they're believable. This can easily be done with a calculator. For example, consider what e^x looks like if x is a very small number, say, $x = 0.0001$. Your calculator (or a computer, if your want more digits) will tell you that

$$e^{0.0001} = 1.000\,100\,005\,000\,166\,6\ldots \tag{6.52}$$

This can be written more informatively as

$$\begin{aligned} e^{0.0001} = \;& 1.0 \\ &+ 0.000\,1 \\ &+ 0.000\,000\,005 \\ &+ 0.000\,000\,000\,000\,166\,6\ldots \\ = \;& 1 + (0.0001) + \frac{(0.0001)^2}{2!} + \frac{(0.0001)^3}{3!} + \cdots . \end{aligned} \tag{6.53}$$

This last line agrees with the form of the Taylor series for e^x in Eq. (6.46). If you made x smaller (say, 0.000001), then the same pattern would appear, but just with more zeros between the numbers than in Eq. (6.52). If you kept more digits in Eq. (6.52), you could verify the $x^4/4!$ and $x^5/5!$, etc., terms in the e^x Taylor series. But things aren't quite as obvious for these terms, because we don't have all the nice zeros as we do in the first 12 digits in Eq. (6.52).

Note that the lefthand sides of all of the Taylor series listed above involve 1's and x's. So how do we make an approximation to an expression of the form, say, $\sqrt{N+x}$, where x is small? We could of course use the general Taylor-series expression in Eq. (6.42) to generate the series from scratch by taking derivatives. But we can save ourselves some time by making use of the similar-looking $\sqrt{1+x}$ series in Eq. (6.49). We can turn the N into a 1 by factoring out an N from the square root, which gives $\sqrt{N}\sqrt{1+x/N}$. Having generated a 1, we can now apply Eq. (6.49), with the only modification being that the small quantity x that appears in that equation is replaced by the small quantity

x/N. This gives (to first order in x)

$$\sqrt{N+x} = \sqrt{N}\sqrt{1+\frac{x}{N}} \approx \sqrt{N}\left(1+\frac{1}{2}\frac{x}{N}\right) = \sqrt{N}+\frac{x}{2\sqrt{N}}\,. \tag{6.54}$$

You can quickly verify that this expression is valid to first order in x by squaring both sides. As a numerical example, if $N = 100$ and $x = 1$, then this approximation gives $\sqrt{101} \approx 10 + 1/20 = 10.05$, which is very close to the actual value of $\sqrt{101} \approx 10.0499$.

Example (Calculating a square root): Use the Taylor series $\sqrt{1+x} \approx 1 + x/2 - x^2/8$ to produce an approximate value of $\sqrt{5}$. How much does your answer differ from the actual value?

Solution: We'll first write 5 as 4 + 1, because we know what the square root of 4 is. However, we can't immediately apply the given Taylor series with $x = 4$, because we need x to be small. We must first factor out a 4 from the square root, so that we have an expression of the form $\sqrt{1+x}$, where x is small. Using $\sqrt{1+x} \approx 1 + x/2 - x^2/8$ with $x = 1/4$ (not 4!), we obtain

$$\begin{aligned}\sqrt{5} = \sqrt{4+1} = 2\sqrt{1+1/4} &\approx 2\left(1+\frac{1/4}{2}-\frac{(1/4)^2}{8}\right)\\ &= 2\left(1+\frac{1}{8}-\frac{1}{128}\right) \approx 2.2344.\end{aligned} \tag{6.55}$$

The actual value of $\sqrt{5}$ is about 2.2361. The approximate result is only 0.0017 less than this, so the approximation is quite good (the percentage difference is only 0.08%). Equivalently, the square of the approximate value is 4.9924, which is very close to 5. If you include the next term in the series, which happens to be $+x^3/16$, the result is $\sqrt{5} \approx 2.2363$, with an error of only 0.01%. By keeping a sufficient number of terms, you can produce any desired accuracy.

When trying to determine the square root of a number that isn't a perfect square, you could of course just guess and check, improving your guess on each iteration. But a Taylor series (calculated relative to the closest perfect square) provides a systematic method that doesn't involve guessing.

6.7.2 How many terms to keep?

When making a Taylor-series approximation, how do you know how many terms in the series to keep? For example, if the exact answer to a given problem takes the form of $e^x - 1$, then the Taylor series $e^x \approx 1 + x$ tells us that our answer is approximately equal to x. You can check this by picking a small value for x (say, 0.01) and plugging it in your calculator. This approximate form makes the dependence on x (for small x) much more transparent than the original expression $e^x - 1$ does.

But what if our exact answer had instead been $e^x - 1 - x$? The Taylor series $e^x \approx 1 + x$ would then yield an approximate answer of zero. And indeed, the answer *is* approximately zero. However, when making approximations, it is generally understood that we are looking for the *leading-order* term in the answer (that is, the smallest power of x with a nonzero coefficient). If our approximate answer comes out to be zero, then that means we need to go (at least) one term further in the Taylor series, which means $e^x \approx 1 + x + x^2/2$ in the present case. Our approximate answer is then $x^2/2$. (You should check this by letting $x = 0.01$.) Similarly, if the exact answer had instead been

$e^x - 1 - x - x^2/2$, then we would need to go out to the $x^3/6$ term in the Taylor series for e^x.

You should be careful to be consistent in the powers of x you deal with. If the exact answer is, say, $e^x - 1 - x - x^2/3$, and if you use the Taylor series $e^x \approx 1 + x$, then you will obtain an approximate answer of $-x^2/3$. This is incorrect, because it is inconsistent to pay attention to the $-x^2/3$ term in the exact answer while ignoring the corresponding $x^2/2$ term in the Taylor series for e^x. Including both terms gives the correct approximate answer as $x^2/6$.

So what is the answer to the above question: How do you know how many terms in the series to keep? Well, the answer is that before you do the calculation, there's really no way of knowing how many terms to keep. The optimal strategy is probably to just hope for the best and start by keeping only the term of order x. This will often be sufficient. But if you end up with a result of zero, then you can go to order x^2, and so on. Of course, you could play it safe and always keep terms up to, say, fourth order. But that is invariably a poor strategy, because you will probably never need to go out that far in a series.

6.7.3 Dimensionless quantities

Whenever you use a Taylor series from the above list to make an approximation in a physics problem, the parameter x must be *dimensionless*. If it weren't dimensionless, then the terms with the various powers of x in the series would all have different units, and it makes no sense to add terms with different units.

As an example of an expansion involving a properly dimensionless quantity, consider the approximation we made in going from Eq. (6.39) to Eq. (6.40) in the beach-ball example in Appendix F. In this setup, the small dimensionless quantity x is the bt/m term that appears in the exponent in Eq. (6.39). This quantity is indeed dimensionless, because from the original expression for the drag force, $F_d = -bv$, we see that b has units of N/(m/s), or equivalently kg/s. Hence bt/m is dimensionless.

We can restate the above dimensionless requirement in a more physical way. Consider the question, "What is the velocity $v(t)$ in Eq. (6.39), in the limit of small t?" This question is meaningless, because t has dimensions. Is a year a large or a small time? How about a hundredth of a second? There is no way to answer this without knowing what situation we're dealing with. A year is short on the time scale of galactic evolution, but a hundredth of a second is long on the time scale of an elementary-particle process. It makes sense only to look at the limit of a large or small *dimensionless* quantity x. And by "large or small," we mean compared with the number 1.

Equivalently, in the beach-ball example the quantity m/b has dimensions of time, so the value of m/b is a time that is inherent to the system. It therefore *does* make sense to look at the limit where $t \ll m/b$ (that is, $bt/m \ll 1$), because we are now comparing two things, namely t and m/b, that have the same dimensions. We will sometimes be sloppy and say things like, "In the limit of small t." But you know that we really mean, "In the limit of a small dimensionless quantity that has a t in the numerator (like bt/m)," or, "In the limit where t is much smaller than a certain quantity that has dimensions of time (like m/b)." Similarly, throughout this book the phrase "small v" is always understood to mean "v much smaller than c."

After you make an approximation, how do you know if it is a "good" one? Well, just as it makes no sense to ask if a dimensionful quantity is large or small without comparing it to another quantity with the same dimensions, it makes no sense to ask if an approximation is "good" or "bad" without stating what accuracy you want. In the beach-ball example, let's say that we're looking at a value of t for which $bt/m = 1/100$.

In Eq. (6.40) we kept the bt/m term in the Taylor series for $e^{-bt/m}$, and this directly led to our answer of $-gt$. We ignored the $(bt/m)^2/2$ term in the series. This is smaller than the bt/m term that we kept, by a factor of $(bt/m)/2 = 1/200$. So the error is roughly half a percent. (The corrections from the higher-order terms are even smaller.) If this is enough accuracy for whatever purpose you have in mind, then the approximation is a good one. If not, then it's a bad one, and you need to add more terms in the series until you get your desired accuracy.

6.8 Appendix H: Useful formulas

The first formula here can be quickly proved by showing that the Taylor series for both sides are equal.

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{6.56}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{6.57}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}} \qquad \sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \tag{6.58}$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta} \tag{6.59}$$

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta \tag{6.60}$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{6.61}$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{6.62}$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \tag{6.63}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2} \tag{6.64}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \tag{6.65}$$

$$\cosh^2 x - \sinh^2 x = 1 \tag{6.66}$$

$$\frac{d}{dx}\cosh x = \sinh x \qquad \frac{d}{dx}\sinh x = \cosh x \tag{6.67}$$

$$\sinh(x+y) = \sinh x\cosh y + \cosh x\sinh y \tag{6.68}$$

$$\cosh(x+y) = \cosh x\cosh y + \sinh x\sinh y \tag{6.69}$$

$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x\tanh y} \tag{6.70}$$