Chapter 2

Kinematics in 1-D

From *Problems and Solutions in Introductory Mechanics* (Draft version, August 2014)
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As mentioned in the preface, this book should not be thought of as a textbook. The introduction to each chapter is brief and is therefore no substitute for an actual textbook. You will most likely want to have a textbook on hand when reading the introductions.

2.1 Introduction

In this chapter and the next, we won't be concerned with the forces that cause an object to move in the particular way it is moving. We will simply take the motion as given, and our goal will be to relate positions, velocities, and accelerations as functions of time. Our objects can be treated like point particles; we will not be concerned with what they are actually made of. This is the study of *kinematics.* In Chapter 4 we will move on to *dynamics,* where we will deal with mass, force, energy, momentum, etc.

Velocity and acceleration

In one dimension, the *average* velocity and acceleration over a time interval Δt are given by

$$v_{\text{avg}} = \frac{\Delta x}{\Delta t} \qquad \text{and} \qquad a_{\text{avg}} = \frac{\Delta v}{\Delta t}\,. \tag{2.1}$$

The *instantaneous* velocity and acceleration at a particular time t are obtained by letting the interval Δt become infinitesimally small. In this case we write the "Δ" as a "d," and the instantaneous v and a are given by

$$v = \frac{dx}{dt} \qquad \text{and} \qquad a = \frac{dv}{dt}\,. \tag{2.2}$$

In calculus terms, v is the derivative of x, and a is the derivative of v. Equivalently, v is the slope of the x vs. t curve, and a is the slope of the v vs. t curve. In the case of the velocity v, you can see how this slope arises by taking the limit of $v = \Delta x/\Delta t$, as Δt becomes very small; see Fig. 2.1. The smaller Δt is, the better the slope $\Delta x/\Delta t$ approximates the actual slope of the tangent line at the given point P.

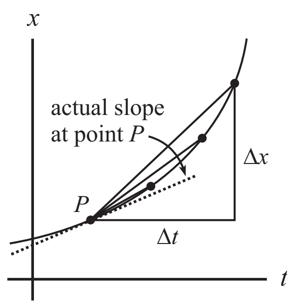


Figure 2.1

In 2-D and 3-D, the velocity and acceleration are vectors. That is, we have a separate pair of equations of the form in Eq. (2.2) for each dimension; the x components are given by $v_x = dx/dt$ and $a_x = dv_x/dt$, and likewise for the y and z components. The velocity and acceleration are also vectors in 1-D, although in 1-D a vector can be viewed simply as a number (which may be positive or negative). In any dimension, the *speed* is the magnitude of the velocity, which means the absolute value of v in 1-D and the length of the vector $\mathbf{v}$ in 2-D and 3-D. So the speed is a positive number by definition. The units of velocity and speed are m/s, and the units of acceleration are m/s^2.

Displacement as an area

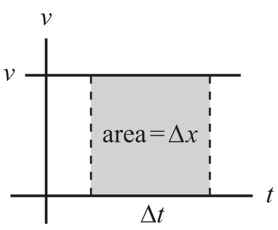


Figure 2.2

If an object moves with constant velocity v, then the displacement Δx during a time Δt is $\Delta x = v\Delta t$. In other words, the displacement is the area of the region (which is just a rectangle) under the v vs. t "curve" in Fig. 2.2. Note that the *displacement* (which is Δx by definition), can be positive or negative. The *distance* traveled, on the other hand, is defined to be a positive number. In the case where the displacement is negative, the v vs. t line in Fig. 2.2 lies below the t axis, so the (signed) area is negative.

If the velocity varies with time, as shown in Fig. 2.3, then we can divide time into a large number of short intervals, with the velocity being essentially constant over each interval. The displacement during each interval is essentially the area of each of the narrow rectangles shown. In the limit of a very large number of very short intervals, adding up the areas of all the thin rectangles gives exactly the total area under the curve; the areas of the tiny triangular regions at the tops of the rectangles become negligible in this limit. So the general result is:

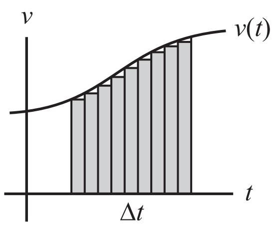
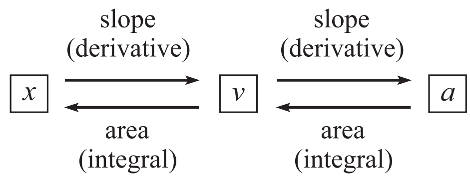


Figure 2.3

- *The displacement (that is, the change in x) equals the area under the v vs. t curve.*

Said in a more mathematical way, the displacement equals the time integral of the velocity. This statement is equivalent (by the fundamental theorem of calculus) to the fact that v is the time derivative of x.

All of the relations that hold between x and v also hold between v and a. In particular, the change in v equals the area under the a vs. t curve. And conversely, a is the time derivative of v. This is summarized in the following diagram:



Motion with constant acceleration

For motion with constant acceleration a, we have

$$
\begin{aligned}
a(t) &= a, \\
v(t) &= v_0 + at, \\
x(t) &= x_0 + v_0 t + \frac{1}{2}at^2, \qquad (2.3)
\end{aligned}
$$

where x_0 and v_0 are the initial position and velocity at $t = 0$. The above expressions for $v(t)$ and $x(t)$ are correct, because $v(t)$ is indeed the derivative of $x(t)$, and $a(t)$ is indeed the derivative of $v(t)$. If you want to derive the expression for $x(t)$ in a graphical manner, see Problem 2.1.

The above expressions are technically all you need for any setup involving constant acceleration, but one additional formula might make things easier now and then. If an object has a displacement d with constant acceleration a, then the initial and final velocities satisfy

$$v_{\mathrm{f}}^2 - v_{\mathrm{i}}^2 = 2ad. \qquad (2.4)$$

See Problem 2.2 for a proof. If you know three out of the four quantities v_{f}, v_{i}, a, and d, then this formula quickly gives the fourth. In the special case where the object starts at rest (so $v_{\mathrm{i}} = 0$), we have the simple result, $v_{\mathrm{f}} = \sqrt{2ad}$.

Falling bodies

Perhaps the most common example of constant acceleration is an object falling under the influence of only gravity (that is, we'll ignore air resistance) near the surface of the earth. The constant nature of the gravitational acceleration was famously demonstrated by Galileo. (He mainly rolled balls down ramps instead of dropping them, but it's the same idea.) If we take the positive y axis to point upward, then the acceleration due to gravity is $-g$, where $g = 9.8\,\text{m/s}^2$. After every second, the velocity becomes more negative by 9.8 m/s; that is, the downward speed increases by 9.8 m/s. If we substitute $-g$ for a in Eq. (2.3) and replace x with y, the expressions become

$$\begin{aligned} a(t) &= -g, \\ v(t) &= v_0 - gt, \\ y(t) &= y_0 + v_0 t - \frac{1}{2}gt^2, \end{aligned} \tag{2.5}$$

For an object dropped from rest at a point we choose to label as $y = 0$, Eq. (2.5) gives $y(t) = -gt^2/2$.

In some cases it is advantageous to choose the positive y axis to point downward, in which case the acceleration due to gravity is g (with no minus sign). In any case, it is always a good idea to take g to be the *positive* quantity $9.8\,\text{m/s}^2$, and then throw in a minus sign by hand if needed, because working with quantities with minus signs embedded in them can lead to confusion.

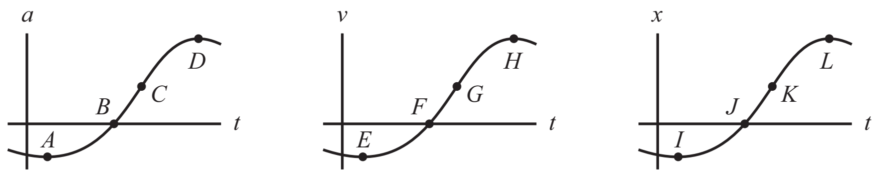
The expressions in Eq. (2.5) hold only in the approximation where we neglect air resistance. This is generally a good approximation, as long as the falling object isn't too light or moving too quickly. Throughout this book, we will ignore air resistance unless stated otherwise.

2.2 Multiple-choice questions

2.1. If an object has negative velocity and negative acceleration, is it slowing down or speeding up?

(a) slowing down
(b) speeding up

2.2. The first figure below shows the a vs. t plot for a certain setup. The second figure shows the v vs. t plot for a different setup. The third figure shows the x vs. t plot for a yet another setup. Which of the twelve labeled points correspond(s) to zero acceleration? Circle all that apply. (To repeat, the three setups have nothing to do with each other. That is, the v plot is *not* the velocity curve associated with the position in the x plot. etc.)



2.3. If the acceleration as a function of time is given by $a(t) = At$, and if $x = v = 0$ at $t = 0$, what is $x(t)$?

(a) $\dfrac{At^2}{2}$ (b) $\dfrac{At^2}{6}$ (c) At^3 (d) $\dfrac{At^3}{2}$ (e) $\dfrac{At^3}{6}$

2.4. Under what condition is the average velocity (which is defined to be the total displacement divided by the time) equal to the average of the initial and final velocities, $(v_\text{i} + v_\text{f})/2$?

(a) The acceleration must be constant.
(b) It is true for other motions besides constant acceleration, but not for all possible motions.
(c) It is true for all possible motions.

2.5. Two cars, with initial speeds of $2v$ and v, lock their brakes and skid to a stop. Assume that the deceleration while skidding is independent of the speed. The ratio of the distances traveled is

(a) 1 (b) 2 (c) 4 (d) 8 (e) 16

2.6. You start from rest and accelerate with a given constant acceleration for a given distance. If you repeat the process with twice the acceleration, then the time required to travel the same distance

(a) remains the same
(b) is doubled
(c) is halved
(d) increases by a factor of $\sqrt{2}$
(e) decreases by a factor of $\sqrt{2}$

2.7. A car travels with constant speed v_0 on a highway. At the instant it passes a stationary police motorcycle, the motorcycle accelerates with constant acceleration and gives chase. What is the speed of the motorcycle when it catches up to the car (in an adjacent lane on the highway)? *Hint*: Draw the v vs. t plots on top of each other.

(a) v_0 (b) $3v_0/2$ (c) $2v_0$ (d) $3v_0$ (e) $4v_0$

2.8. You start from rest and accelerate to a given final speed v_0 after a time T. Your acceleration need not be constant, but assume that it is always positive or zero. If d is the total distance you travel, then the range of possible d values is

(a) $d = v_0T/2$
(b) $0 < d < v_0T/2$
(c) $v_0T/2 < d < v_0T$
(d) $0 < d < v_0T$
(e) $0 < d < \infty$

2.9. You are driving a car that has a maximum acceleration of a. The magnitude of the maximum deceleration is also a. What is the maximum distance that you can travel in time T, assuming that you begin and end at rest?

(a) $2aT^2$ (b) aT^2 (c) $aT^2/2$ (d) $aT^2/4$ (e) $aT^2/8$

2.10. A golf club strikes a ball and sends it sailing through the air. Which of the following choices best describes the sizes of the position, speed, and acceleration of the ball at a moment in the middle of the strike? ("Medium" means a non-tiny and non-huge quantity, on an everyday scale.)

(a) x is tiny, v is medium, a is medium
(b) x is tiny, v is medium, a is huge
(c) x is tiny, v is huge, a is huge
(d) x is medium, v is medium, a is medium
(e) x is medium, v is medium, a is huge

2.11. Which of the following answers is the best estimate for the time it takes an object dropped from rest to fall a vertical mile (about 1600 m)? Ignore air resistance, as usual.

(a) 5 s (b) 10 s (c) 20 s (d) 1 min (e) 5 min

2.12. You throw a ball upward. After half of the time to the highest point, the ball has covered

(a) half the distance to the top
(b) more than half the distance
(c) less than half the distance
(d) It depends on how fast you throw the ball.

2.13. A ball is dropped, and then another ball is dropped from the same spot one second later. As time goes on while the balls are falling, the distance between them (ignoring air resistance, as usual)

(a) decreases
(b) remains the same
(c) increases and approaches a limiting value
(d) increases steadily

2.14. You throw a ball straight upward with initial speed v_0. How long does it take to return to your hand?

(a) $v_0^2/2g$ (b) v_0^2/g (c) $v_0/2g$ (d) v_0/g (e) $2v_0/g$

2.15. Ball 1 has mass m and is fired directly upward with speed v. Ball 2 has mass $2m$ and is fired directly upward with speed $2v$. The ratio of the maximum height of Ball 2 to the maximum height of Ball 1 is

(a) 1 (b) $\sqrt{2}$ (c) 2 (d) 4 (e) 8

2.3 Problems

The first three problems are foundational problems.

2.1. **Area under the curve**

At $t = 0$ an object starts with position x_0 and velocity v_0 and moves with constant acceleration a. Derive the $x(t) = x_0 + v_0 t + at^2/2$ result by finding the area under the v vs. t curve (without using calculus).

2.2. **A kinematic relation**

Use the relations in Eq. (2.3) to show that if an object moves through a displacement d with constant acceleration a, then the initial and final velocities satisfy $v_\text{f}^2 - v_\text{i}^2 = 2ad$.

2.3. **Maximum height**

If you throw a ball straight upward with initial speed v_0, it reaches a maximum height of $v_0^2/2g$. How many derivations of this result can you think of?

2.4. **Average speeds**

(a) If you ride a bike up a hill at 10 mph, and then down the hill at 20 mph, what is your average speed?
(b) If you go on a bike ride and ride for half the time at 10 mph, and half the time at 20 mph, what is your average speed?

2.5. **Colliding trains**

Two trains, A and B, travel in the same direction on the same set of tracks. A starts at rest at position d, and B starts with velocity v_0 at the origin. A accelerates with acceleration a, and B decelerates with acceleration $-a$. What is the maximum value of v_0 (in terms of d and a) for which the trains don't collide? Make a rough sketch of x vs. t for both trains in the case where they barely collide.

2.6. **Ratio of distances**

Two cars, A and B, start at the same position with the same speed v_0. Car A travels at constant speed, and car B decelerates with constant acceleration $-a$. At the instant when B reaches a speed of zero, what is the ratio of the distances traveled by A and B? Draw a reasonably accurate plot of x vs. t for both cars.

You should find that your answer for the ratio of the distances is a nice simple number, independent of any of the given quantities. Give an argument that explains why this is the case.

2.7. **How far apart?**

An object starts from rest at the origin at time $t = -T$ and accelerates with constant acceleration a. A second object starts from rest at the origin at time $t = 0$ and accelerates with the same a. How far apart are they at time t? Explain the meaning of the two terms in your answer, first in words, and then also with regard to the v vs. t plots.

2.8. **Ratio of odd numbers**

An object is dropped from rest. Show that the distances fallen during the first second, the second second, the third second, etc., are in the ratio of $1 : 3 : 5 : 7 \ldots$.

2.9. **Dropped and thrown balls**

A ball is dropped from rest at height h. Directly below on the ground, a second ball is simultaneously thrown upward with speed v_0. If the two balls collide at the moment the second ball is instantaneously at rest, what is the height of the collision? What is the relative speed of the balls when they collide? Draw the v vs. t plots for both balls.

2.10. **Hitting at the same time**

A ball is dropped from rest at height h. Another ball is simultaneously thrown downward with speed v from height $2h$. What should v be so that the two balls hit the ground at the same time?

2.11. **Two dropped balls**

A ball is dropped from rest at height $4h$. After it has fallen a distance d, a second ball is dropped from rest at height h. What should d be (in terms of h) so that the balls hit the ground at the same time?

2.4 Multiple-choice answers

2.1. $\boxed{\text{b}}$ The object is speeding up. That is, the magnitude of the velocity is increasing. This is true because the negative acceleration means that the change in velocity is negative. And we are told that the velocity is negative to start with. So it might go from, say, −20 m/s to −21 m/s a moment later. It is therefore speeding up.

Remarks: If we had said that the object had negative velocity and positive acceleration, then it would be slowing down. Basically, if the sign of the acceleration is the same as (or the opposite of) the sign of the velocity, then the object is speeding up (or slowing down).

A comment on terminology: The word "decelerate" means to slow down. The word "accelerate" means in a colloquial sense to speed up, but as a physics term it means (in 1-D) to either speed up *or* slow down, because acceleration can be positive or negative. More generally, in 2-D or 3-D it means to change the velocity in any general manner (magnitude and/or direction).

2.2. $\boxed{\text{B,E,H,K}}$ Point B is where a equals zero in the first figure. Points E and H are where the slope (the derivative) of the v vs. t plot is zero; and the slope of v is a. Point K is where the slope of the x vs. t plot is *maximum*. In other words, it is where v is maximum. But the slope of a function is zero at a maximum, so the slope of v (which is a) is zero at K.

Remark: In calculus terms, K is an *inflection point* of the x vs. t curve. It is a point where the slope is maximum. Equivalently, the derivative of the slope is zero. Equivalently again, the second derivative is zero. In the present case, the tangent line goes from lying below the x vs. t curve to lying above it; the slope goes from increasing to decreasing as it passes through its maximum value.

2.3. [e] Since the second derivative of $x(t)$ equals $a(t)$, we must find a function whose second derivative is At. Choice (e) is satisfies this requirement; the first derivative equals $At^2/2$, and then the second derivative equals At, as desired. The standard $At^2/2$ result is valid only for a *constant* acceleration a. Note that all of the choices satisfy $x = v = 0$ at $t = 0$.

Remark: If we add on a constant C to $x(t)$, so that we now have $At^3/6 + C$, then the $x = 0$ initial condition isn't satisfied, even though $a(t)$ is still equal to At. Similarly, if we add on a linear term Bt, then the $v = 0$ initial condition isn't satisfied, even though $a(t)$ is again still equal to At. If we add on quadratic term Dt^2, then although the $x = v = 0$ initial conditions are satisfied, the second derivative is now not equal to At. Likewise for any power of t that is 4 or higher. So not only is the $At^3/6$ choice the only correct answer among the five given choices, it is the only correct answer, period. Formally, the integral of a (which is v) must take the form of $At^2/2 + B$, where B is a constant of integration. And the integral of v (which is x) must then take the form of $At^3/6 + Bt + C$, where C is a constant of integration. The initial conditions $x = v = 0$ then quickly tell us that $C = B = 0$.

2.4. [b] The statement is at least true in the case of constant acceleration, as seen by looking at the v vs. t plot in Fig. 2.4(a). The area under the v vs. t curve is the distance traveled, and the area of the trapezoid (which corresponds to constant acceleration) is the same as the area of the rectangle (which corresponds to constant velocity $(v_i + v_f)/2$). Equivalently, the areas of the triangles above and below the $(v_i + v_f)/2$ line are equal. If you want to work things out algebraically, the displacement is

$$d = v_i t + \frac{1}{2}at^2 = \frac{1}{2}(2v_i + at)t = \frac{1}{2}\Big(v_i + (v_i + at)\Big)t = \frac{1}{2}(v_i + v_f)t. \tag{2.6}$$

The average velocity d/t is therefore equal to $(v_i + v_f)/2$, as desired.

The statement is certainly not true in all cases; a counterexample is shown in Fig. 2.4(b). The distance traveled (the area under the curve) is essentially zero, so the average velocity is essentially zero and hence not equal to $(v_i + v_f)/2$.

However, the statement *can* be true for motions without constant acceleration, as long as the area under the v vs. t curve is the same as the area of the rectangle associated with velocity $(v_i + v_f)/2$, as shown in Fig. 2.4(c). For the curve shown, this requirement is the same as saying that the areas of the two shaded regions are equal.

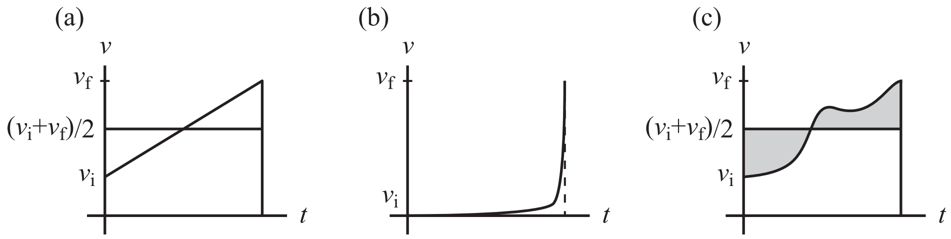


Figure 2.4

2.5. [c] The final speed is zero in each case, so the $v_f^2 - v_i^2 = 2ad$ relation in Eq. (2.4) gives $0 - v_i^2 = 2(-a)d$, where a is the magnitude of the (negative) acceleration. So $d = v_i^2/2a$. Since this is proportional to v_i^2, the car with twice the initial speed has four times the stopping distance.

Alternatively, the distance traveled is $d = v_i t - at^2/2$, where again a is the magnitude of the acceleration. Since the car ends up at rest, the $v(t) = v_i - at$ expression for the velocity tells us that $v = 0$ when $t = v_i/a$. So

$$d = v_i\left(\frac{v_i}{a}\right) - \frac{1}{2}a\left(\frac{v_i}{a}\right)^2 = \frac{v_i^2}{2a}, \tag{2.7}$$

in agreement with the relation obtained via Eq. (2.4).

Alternatively again, we could imagine reversing time and accelerating the cars from rest. Using the fact that one time is twice the other (since $t = v_i/a$), the relation $d = at^2/2$ immediately tells us that twice the time implies four times the distance.

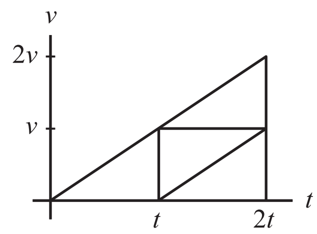


Figure 2.5

Alternatively yet again, the factor of 4 quickly follows from the v vs. t plot shown in Fig. 2.5. The area under the diagonal line is the distance traveled, and the area of the large triangle is four times the area of the small lower-left triangle, because all four of the small triangles have the same area.

REMARK: When traveling in a car, the safe distance (according to many sources) to keep between your car and the car in front of you is dictated by the "three-second rule" (in good weather). That is, your car should pass, say, a given tree at least three seconds after the car in front of you passes it. This rule involves *time*, but it immediately implies that the minimum following *distance* is proportional to your speed. It therefore can't strictly be correct, because we found above that the stopping distance is proportional to the *square* of your speed. This square behavior means that the three-second rule is inadequate for sufficiently high speeds. There are of course many other factors involved (reaction time, the nature of the road hazard, the friction between the tires and the ground, etc.), so the exact formula is probably too complicated to be of much use. But if you take a few minutes to observe some cars and make some rough estimates of how drivers out there are behaving, you'll find that many of them are following at astonishingly unsafe distances, by any measure.

2.6. [e] The distance traveled is given by $d = at^2/2$, so $t = \sqrt{2d/a}$. Therefore, if a is doubled then t decreases by a factor of $\sqrt{2}$.

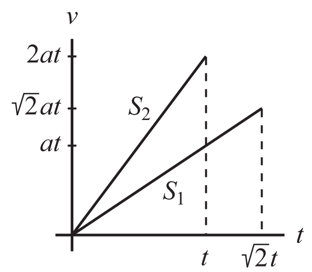


Figure 2.6

REMARK: Since $v = at$ for constant acceleration, the speeds in the two given scenarios (label them S_1 and S_2) differ by a factor of 2 at any given time. So if at all times the speed in S_2 is twice the speed in S_1, shouldn't the time simply be halved, instead of decreased by the factor of $\sqrt{2}$ that we just found? No, because although the S_1 distance is only $d/2$ when the S_2 distance reaches the final value of d, it takes S_1 less time to travel the remaining $d/2$ distance, because its speed increases as time goes on. This is shown in Fig. 2.6. The area under each v vs. t curve equals the distance traveled. Compared with S_1, S_2's final speed is $\sqrt{2}$ times larger, but its time is $1/\sqrt{2}$ times smaller. So the areas of the two triangles are the same.

2.7. [c] The area under a v vs. t curve is the distance traveled. The car's curve is the horizontal line shown in Fig. 2.7, and the motorcycle's curve is the tilted line. The two vehicles will have traveled the same distance when the area of the car's rectangle equals the area of the motorcycle's triangle. This occurs when the triangle has twice the height of the rectangle, as shown. (The area of a triangle is half the base times the height.) So the final speed of the motorcycle is $2v_0$. Note that this result is independent of the motorcycle's (constant) acceleration. If the acceleration is small, then the process will take a long time, but the speed of the motorcycle when it catches up to the car will still be $2v_0$.

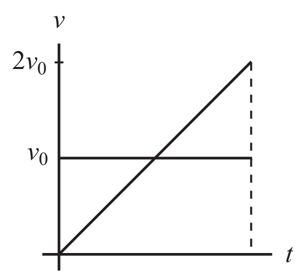


Figure 2.7

Alternatively, the position of the car at time t is $v_0 t$, and the position of the motorcycle is $at^2/2$. These two positions are equal when $v_0 t = at^2/2 \implies at = 2v_0$. But the motorcycle's speed is at, which therefore equals $2v_0$ when the motorcycle catches up to the car.

2.8. [d] A distance of essentially zero can be obtained by sitting at rest for nearly all of the time T, and then suddenly accelerating with a huge acceleration to speed v_0. Approximately zero distance is traveled during this acceleration phase. This is true because Eq. (2.4) gives $d = v_0^2/2a$, where v_0 is a given quantity and a is huge.

Conversely, a distance of essentially $v_0 T$ can be obtained by suddenly accelerating with a huge acceleration to speed v_0, and then coasting along at speed v_0 for nearly all of the time T.

These two cases are shown in the v vs. t plots in Fig. 2.8. The area under the curve (which is the distance traveled) for the left curve is approximately zero, and the area under the right curve is approximately the area of the whole rectangle, which is v_0T. This is the maximum possible distance, because an area larger than the v_0T rectangle would require that the v vs. t plot extend higher than v_0, which would then require a negative acceleration (contrary to the stated assumption) to bring the final speed back down to v_0.

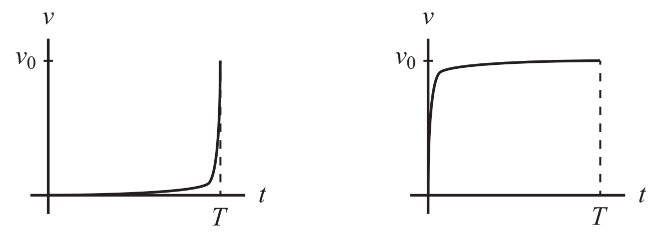


Figure 2.8

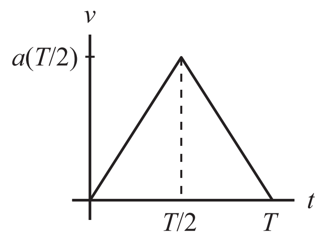


Figure 2.9

2.9. [d] The maximum distance is obtained by having acceleration a for a time $T/2$ and then deceleration $-a$ for a time $T/2$. The v vs. t plot is shown in Fig. 2.9. The distance traveled during the first $T/2$ is $a(T/2)^2/2 = aT^2/8$. Likewise for the second $T/2$, because the two triangles have the same area, and the area under a v vs. t curve is the distance traveled. So the total distance is $aT^2/4$.

Alternatively, we see from the triangular plot that the average speed is half of the maximum v, which gives $v_{avg} = (aT/2)/2 = aT/4$. So the total distance traveled is $v_{avg}T = (aT/4)T = aT^2/4$.

Note that the triangle in Fig. 2.9 does indeed yield the maximum area under the curve (that is, the maximum distance traveled) subject to the given conditions, because (1) the triangle is indeed a possible v vs. t plot, and (2) velocities above the triangle aren't allowed, because the given maximum a implies that it would either be impossible to accelerate from zero initial speed to such a v, or impossible to decelerate to zero final speed from such a v.

2.10. [b] The distance x is certainly tiny, because the ball is still in contact with the club during the (quick) strike. The speed v is medium, because it is somewhere between the initial speed of zero and the final speed (on the order of 100 mph); it would be exactly half the final speed if the acceleration during the strike were constant. The acceleration is huge, because (assuming constant acceleration to get a rough idea) it is given by v/t, where v is medium and t is tiny (the strike is very quick).

REMARK: In short, the ball experiences a very large a for a very small t. The largeness and smallness of these quantities cancel each other and yield a medium result for the velocity $v = at$ (again, assuming constant a). But in the position $x = at^2/2$, the two factors of t win out over the one factor of a, and the result is tiny. These results (tiny x, medium v, and huge a) are consistent with Eq. (2.4), which for the present scenario says that $v^2 = 2ax$.

2.11. [c] From $d = at^2/2$ we obtain (using $g = 10\,\text{m/s}^2$)

$$1600\,\text{m} = \frac{1}{2}(10\,\text{m/s}^2)t^2 \implies t^2 = 320\,\text{s}^2 \implies t \approx 18\,\text{s}. \tag{2.8}$$

So 20 s is the best answer. The speed at this time is $gt \approx 10 \cdot 20 = 200\,\text{m/s}$, which is about 450 mph (see Multiple-Choice Question 1.4). In reality, air resistance is important, and a terminal velocity is reached. For a skydiver in a spread-eagle position, the terminal velocity is around 50 m/s.

2.12. [b] Let the time to the top be t. Since the ball decelerates on its way up, it moves faster in the first $t/2$ time span than in the second $t/2$. So it covers more than half the distance in the first $t/2$.

REMARK: If you want to find the exact ratio of the distances traveled in the two $t/2$ time spans, it is easiest to imagine dropping the ball instead of firing it upward; the answer is the same. In the upper $t/2$ of the motion, the ball falls $g(t/2)^2/2$, whereas in the total time t the ball falls $gt^2/2$. The ratio of these distances is 1 to 4, so the distance in the upper $t/2$ is 1/4 of the total, which means that the distance in the lower $t/2$ is 3/4 of the total. The ratio of the distances traveled in the two $t/2$ time spans is therefore 3 to 1. This also quickly follows from drawing a v vs. t plot like the one in Fig. 2.5.

2.13. [d] If T is the time between the dropping of each ball (which is one second here), then the first ball has a speed of gT when the second ball is dropped. At a time t later than this, the speeds of the two balls are $g(t+T)$ and gt. So the difference in speeds is always gT. That is, the second ball always sees the first ball pulling away with a relative speed of gT. The separation therefore increases steadily at a rate gT.

This result ignores air resistance. In reality, the objects will reach the same terminal velocity (barring any influence of the first ball on the second), so the distance between them will approach a constant value. The real-life answer is therefore choice (c).

2.14. [e] The velocity as a function of time is given by $v(t) = v_0 - gt$. Since the velocity is instantaneously zero at the highest point, the time to reach the top is $t = v_0/g$. The downward motion takes the same time as the upward motion (although it wouldn't if we included air resistance), so the total time is $2v_0/g$. Note that choices (a) and (b) don't have the correct units; choice (a) is the maximum height.

2.15. [d] From general kinematics (see Problem 2.3), or from conservation of energy (the subject of Chapter 5), or from dimensional analysis, the maximum height is proportional to v^2/g (it equals $v^2/2g$). The v^2 dependence implies that the desired ratio is $2^2 = 4$. The difference in the masses is irrelevant.

2.5 Problem solutions

Although this was mentioned many times in the preface and in Chapter 1, it is worth belaboring the point: Don't look at the solution to a problem (or a multiple-choice question) too soon. If you do need to look at it, read it line by line until you get a hint to get going again. If you read through a solution without first solving the problem, you will gain essentially nothing from it!

2.1. **Area under the curve**

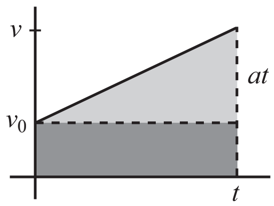


Figure 2.10

The v vs. t curve, which is simply a tilted line in the case of constant acceleration, is shown in Fig. 2.10. The slope of the line equals the acceleration a, which implies that the height of the triangular region is at, as shown. The area under the v vs. t curve is the distance traveled. This area consists of the rectangle with area $t \cdot v_0$ and the triangle with area $(1/2) \cdot t \cdot at$. So the total area is $v_0 t + at^2/2$. To find the present position $x(t)$, we must add the initial position, x_0, to the distance traveled. The present position is therefore $x(t) = x_0 + v_0 t + at^2/2$, as desired.

2.2. **A kinematic relation**

FIRST SOLUTION: Our strategy will be to eliminate t from the equations in Eq. (2.3) by solving for t in the second equation and plugging the result into the third. This gives

$$\begin{aligned} x &= x_0 + v_0\left(\frac{v-v_0}{a}\right) + \frac{1}{2}a\left(\frac{v-v_0}{a}\right)^2 \\ &= x_0 + \frac{1}{a}(v_0 v - v_0^2) + \frac{1}{a}\left(\frac{v^2}{2} - v v_0 + \frac{v_0^2}{2}\right) \\ &= x_0 + \frac{1}{2a}(v^2 - v_0^2). \end{aligned} \quad (2.9)$$

Hence $2a(x - x_0) = v^2 - v_0^2$. But $x - x_0$ is the displacement d. Changing the notation, $v \to v_f$ and $v_0 \to v_i$, gives the desired result, $2ad = v_f^2 - v_i^2$. A quick corollary is that if d and a have the same (or opposite) sign, then v_f is larger (or smaller) than v_i. You should convince yourself that this makes sense intuitively.

Second solution: A quicker derivation is the following. The displacement equals the average velocity times the time, by definition. The time is $t = (v - v_0)/a$, and the average velocity is $v_{avg} = (v + v_0)/2$, where this second expression relies on the fact that the acceleration is constant. (The first expression does too, because otherwise we wouldn't have a unique a in the denominator.) So we have

$$d = v_{avg}t = \left(\frac{v + v_0}{2}\right)\left(\frac{v - v_0}{a}\right) = \frac{v^2 - v_0^2}{2a}. \tag{2.10}$$

Multiplying by $2a$ gives the desired result.

2.3. **Maximum height**

The solutions I can think of are listed below. Most of them use the fact that the time to reach the maximum height is $t = v_0/g$, which follows from the velocity $v(t) = v_0 - gt$ being zero at the top of the motion. The fact that the acceleration is constant also plays a critical role in all of the solutions.

1. Since the acceleration is constant, the average speed during the upward motion equals the average of the initial and final speeds. So $v_{avg} = (v_0 + 0)/2 = v_0/2$. The distance equals the average speed times the time, so $d = v_{avg}t = (v_0/2)(v_0/g) = v_0^2/2g$.
2. Using the standard expression for the distance traveled, $d = v_0 t - gt^2/2$, we have

$$d = v_0\left(\frac{v_0}{g}\right) - \frac{g}{2}\left(\frac{v_0}{g}\right)^2 = \frac{v_0^2}{2g}. \tag{2.11}$$

3. If we imagine reversing time (or just looking at the downward motion, which takes the same time), then the ball starts at rest and accelerates downward at g. So we can use the simpler expression $d = gt^2/2$, which quickly gives $d = g(v_0/g)^2/2 = v_0^2/2g$.
4. The kinematic relation $v_f^2 - v_i^2 = 2ad$ from Eq. (2.4) gives $0^2 - v_0^2 = 2(-g)d \Longrightarrow d = v_0^2/2g$. We have been careful with the signs here; if we define positive d as upward, then the acceleration is negative.
5. The first three of the above solutions have graphical interpretations (although perhaps these shouldn't count as separate solutions). The v vs. t plots associated with these three solutions are shown in Fig. 2.11. The area under each curve, which is the distance traveled, equals $v_0^2/2g$.

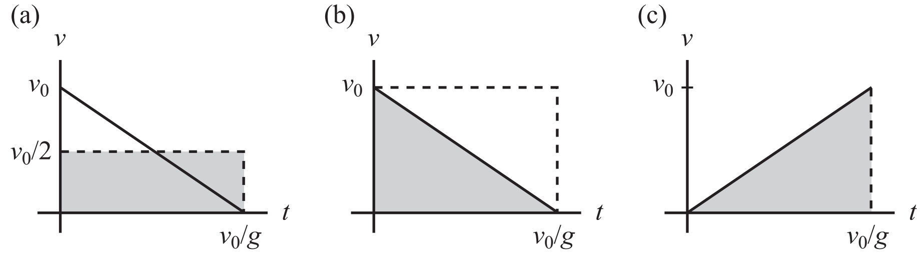


Figure 2.11

6. We can also use conservation of energy to solve this problem. Even though we won't discuss energy until Chapter 5, the solution is quick enough to state here. The

initial kinetic energy $mv_0^2/2$ gets completely converted into the gravitational potential energy mgd at the top of the motion (because the ball is instantaneously at rest at the top). So $mv_0^2/2 = mgd \Longrightarrow d = v_0^2/2g$.

2.4. **Average speeds**

(a) Let the length of the hill be ℓ, and define $v \equiv 10$ mph. Then the time up the hill is ℓ/v, and the time down is $\ell/2v$. Your average speed is therefore

$$v_{\text{avg}} = \frac{d_{\text{total}}}{t_{\text{total}}} = \frac{2\ell}{\ell/v + \ell/2v} = \frac{2}{3/2v} = \frac{4v}{3} = 13.3 \text{ mph}. \tag{2.12}$$

(b) Let $2t$ be the total time of the ride, and again define $v \equiv 10$ mph. Then during the first half of the ride, you travel a distance vt. And during the second half, you travel a distance $(2v)t$. Your average speed is therefore

$$v_{\text{avg}} = \frac{d_{\text{total}}}{t_{\text{total}}} = \frac{vt + 2vt}{2t} = \frac{3v}{2} = 15 \text{ mph}. \tag{2.13}$$

Remark: This result of 15 mph is simply the average of the two speeds, because you spend the *same amount of time* traveling at each speed. This is *not* the case in the scenario in part (a), because you spend longer (twice as long) traveling uphill at the slower speed. So that speed matters more when taking the average. In the extreme case where the two speeds differ greatly (in a multiplicative sense), the average speed in the scenario in part (a) is very close to twice the smaller speed (because the downhill time can be approximated as zero), whereas the average speed in the scenario in part (b) always equals the average of the two speeds. For example, if the two speeds are 1 and 100 (ignoring the units), then the answers to parts (a) and (b) are, respectively,

$$\begin{aligned} v_{\text{avg}}^{(a)} &= \frac{2\ell}{\ell/1 + \ell/100} = \frac{200}{101} = 1.98, \\ v_{\text{avg}}^{(b)} &= \frac{1 \cdot t + 100 \cdot t}{2t} = \frac{101}{2} = 50.5. \end{aligned} \tag{2.14}$$

2.5. **Colliding trains**

The positions of the two trains are given by

$$x_A = d + \frac{1}{2}at^2 \qquad \text{and} \qquad x_B = v_0 t - \frac{1}{2}at^2. \tag{2.15}$$

These are equal when

$$\begin{aligned} d + \frac{1}{2}at^2 = v_0 t - \frac{1}{2}at^2 \quad &\Longrightarrow \quad at^2 - v_0 t + d = 0 \\ &\Longrightarrow \quad t = \frac{v_0 \pm \sqrt{v_0^2 - 4ad}}{2a}. \end{aligned} \tag{2.16}$$

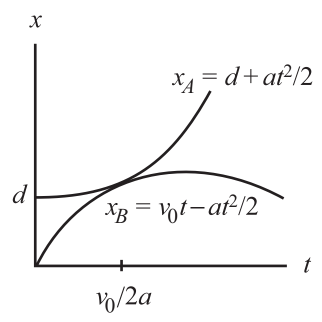


Figure 2.12

The trains *do* collide if there is a real solution for t, that is, if $v_0^2 > 4ad \Longrightarrow v_0 > 2\sqrt{ad}$. The relevant solution is the "–" root. The "+" root corresponds to the case where the trains "pass through" each other and then meet up again a second time.

The trains *don't* collide if the roots are imaginary, that is, if $v_0 < 2\sqrt{ad}$. So the maximum value of v_0 that avoids a collision is $2\sqrt{ad}$. In the cutoff case where $v_0 = 2\sqrt{ad}$, the trains barely touch, so it's semantics as to whether you call that a "collision."

Note that $\sqrt{ad}$ correctly has the units of velocity. And in the limit of large a or d, the cutoff speed $2\sqrt{ad}$ is large, which makes intuitive sense.

A sketch of the x vs. t curves for the $v_0 = 2\sqrt{ad}$ case is shown in Fig. 2.12. If v_0 is smaller than $2\sqrt{ad}$, then the bottom curve stays lower (because its initial slope at the origin is v_0), so the curves don't intersect. If v_0 is larger than $2\sqrt{ad}$, then the bottom curve extends higher, so the curves intersect twice.

REMARKS: As an exercise, you can show that the location where the trains barely collide in the $v_0 = 2\sqrt{ad}$ case is $x = 3d/2$. And the maximum value of x_B is $2d$. If B has normal friction brakes, then it will of course simply stop at this maximum value and not move backward as shown in the figure. But in the hypothetical case of a jet engine with reverse thrust, B would head backward as the curve indicates.

In the $a \to 0$ limit, B moves with essentially constant speed v_0 toward A, which is essentially at rest, initially a distance d away. So the time is simply $t = d/v_0$. As an exercise, you can apply a Taylor series to Eq. (2.16) to produce this $t = d/v_0$ result. A Taylor series is required because if you simply set $a = 0$ in Eq. (2.16), you will obtain the unhelpful result of $t = 0/0$.

2.6. **Ratio of distances**

The positions of the two cars are given by

$$x_A = v_0 t \qquad \text{and} \qquad x_B = v_0 t - \frac{1}{2}at^2. \tag{2.17}$$

B's velocity is $v_0 - at$, and this equals zero when $t = v_0/a$. The positions at this time are

$$x_A = v_0\left(\frac{v_0}{a}\right) = \frac{v_0^2}{a} \qquad \text{and} \qquad x_B = v_0\left(\frac{v_0}{a}\right) - \frac{1}{2}a\left(\frac{v_0}{a}\right)^2 = \frac{v_0^2}{2a}. \tag{2.18}$$

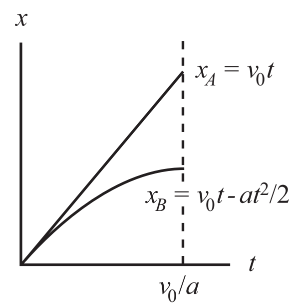


Figure 2.13

The desired ratio is therefore $x_A/x_B = 2$. The plots are shown in Fig. 2.13. Both distances are proportional to v_0^2/a, so large v_0 implies large distances, and large a implies small distances. These make intuitive sense.

The only quantities that the ratio of the distances can depend on are v_0 and a. But the ratio of two distances is a dimensionless quantity, and there is no non-trivial combination of v_0 and a that gives a dimensionless result. Therefore, the ratio must simply be a number, independent of both v_0 and a.

Note that it is easy to see from a v vs. t graph why the ratio is 2. The area under A's velocity curve (the rectangle) in Fig. 2.14 is twice the area under B's velocity curve (the triangle). And these areas are the distances traveled.

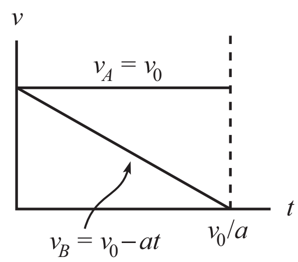


Figure 2.14

2.7. **How far apart?**

At time t, the first object has been moving for a time $t+T$, so its position is $x_1 = a(t+T)^2/2$. The second object has been moving for a time t, so its position is $x_2 = at^2/2$. The difference is

$$x_1 - x_2 = aTt + \frac{1}{2}aT^2. \tag{2.19}$$

The second term here is the distance the first object has already traveled when the second object starts moving. The first term is the relative speed, aT, times the time. The relative speed is always aT because this is the speed the first object has when the second object starts moving. And from that time onward, both speeds increase at the same rate (namely a), so the objects always have the same relative speed. In summary, from the second object's point of view, the first object has a head start of $aT^2/2$ and then steadily pulls away with relative speed aT.

The v vs. t plots are shown in Fig. 2.15. The area under a v vs. t curve is the distance traveled, so the difference in the distances is the area of the shaded region. The triangular region on the left has an area equal to half the base times the height, which gives $T(aT)/2 = aT^2/2$. And the parallelogram region has an area equal to the horizontal width times the height, which gives $t(aT) = aTt$. These terms agree with Eq. (2.19).

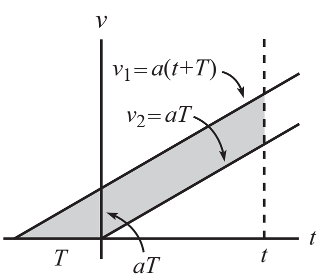


Figure 2.15

2.8. **Ratio of odd numbers**

This general result doesn't depend on the 1-second value of the time interval, so let's replace 1 second with a general time t. The *total* distances fallen after times of 0, t, $2t$, $3t$, $4t$, etc., are

$$0, \quad \frac{1}{2}gt^2, \quad \frac{1}{2}g(2t)^2, \quad \frac{1}{2}g(3t)^2, \quad \frac{1}{2}g(4t)^2, \quad \text{etc.} \tag{2.20}$$

The distances fallen *during* each interval of time t are the differences between the above distances, which yield

$$\frac{1}{2}gt^2, \quad 3\cdot\frac{1}{2}gt^2, \quad 5\cdot\frac{1}{2}gt^2, \quad 7\cdot\frac{1}{2}gt^2, \quad \text{etc.} \tag{2.21}$$

These are in the desired ratio of $1 : 3 : 5 : 7 \ldots$. Algebraically, the difference between $(nt)^2$ and $\big((n+1)t\big)^2$ equals $(2n+1)t^2$, and the $2n+1$ factor here generates the odd numbers.

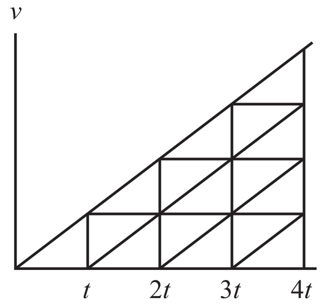


Figure 2.16

Geometrically, the v vs. t plot is shown in Fig. 2.16. The area under the curve (a tilted line in this case) is the distance traveled, and by looking at the number of (identical) triangles in each interval of time t, we quickly see that the ratio of the distances traveled in each interval is $1 : 3 : 5 : 7 \ldots$.

2.9. **Dropped and thrown balls**

The positions of the two balls are given by

$$y_1(t) = h - \frac{1}{2}gt^2 \quad \text{and} \quad y_2(t) = v_0 t - \frac{1}{2}gt^2. \tag{2.22}$$

These are equal (that is, the balls collide) when $h = v_0 t \implies t = h/v_0$. The height of the collision is then found from either of the y expressions to be $y_c = h - gh^2/2v_0^2$. This holds in any case, but we are given the further information that the second ball is instantaneously at rest when the collision occurs. Its speed is $v_0 - gt$, so the collision must occur at $t = v_0/g$. Equating this with the above $t = h/v_0$ result tells us that v_0 must be given by $v_0^2 = gh$. Plugging this into $y_c = h - gh^2/2v_0^2$ gives $y_c = h/2$.

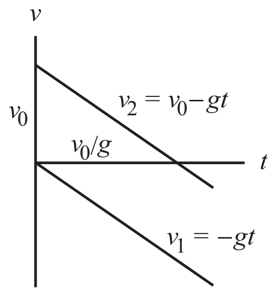


Figure 2.17

The two velocities are given by $v_1(t) = -gt$, and $v_2(t) = v_0 - gt$. The difference of these is v_0. This holds for *all* time, not just at the moment when the balls collide. This is due to the fact that both balls are affected by gravity in exactly the same way, so the initial relative speed (which is v_0) equals the relative speed at any other time. This is evident from the v vs. t plots in Fig. 2.17. The upper line is v_0 above the lower line for all values of t.

2.10. **Hitting at the same time**

The time it takes the first ball to hit the ground is given by

$$\frac{gt_1^2}{2} = h \implies t_1 = \sqrt{\frac{2h}{g}}. \tag{2.23}$$

The time it takes the second ball to hit the ground is given by $vt_2 + gt_2^2/2 = 2h$. We could solve this quadratic equation for t_2 and then set the result equal to t_1. But a much quicker strategy is to note that since we want t_2 to equal t_1, we can just substitute t_1 for t_2 in the quadratic equation. This gives

$$v\sqrt{\frac{2h}{g}} + \frac{g}{2}\left(\frac{2h}{g}\right) = 2h \implies v\sqrt{\frac{2h}{g}} = h \implies v = \sqrt{\frac{gh}{2}}. \tag{2.24}$$

In the limit of small g, the process will take a long time, so it makes sense that v should be small. Note that without doing any calculations, the consideration of units tells us that the answer must be proportional to $\sqrt{gh}$.

REMARK: The intuitive interpretation of the above solution is the following. If the second ball were dropped from rest, it would be at height h when the first ball hits the ground at time $\sqrt{2h/g}$ (after similarly falling a distance h). The second ball therefore needs to be given an initial downward speed v that causes it to travel an extra distance of h during this time. But this is just what the middle equation in Eq. (2.24) says.

2.11. **Two dropped balls**

The total time it takes the first ball to fall a height $4h$ is given by $gt^2/2 = 4h \implies t = 2\sqrt{2h/g}$. This time may be divided into the time it takes to fall a distance d (which is $\sqrt{2d/g}$), plus the remaining time it takes to hit the ground, which we are told is the same as the time it takes the second ball to fall a height h (which is $\sqrt{2h/g}$). Therefore,

$$2\sqrt{\frac{2h}{g}} = \sqrt{\frac{2d}{g}} + \sqrt{\frac{2h}{g}} \implies 2\sqrt{h} = \sqrt{d} + \sqrt{h} \implies d = h. \qquad (2.25)$$

REMARK: Graphically, the process is shown in the v vs. t plot in Fig. 2.18(a). The area of the large triangle is the distance $4h$ the first ball falls. The right small triangle is the distance h the second ball falls, and the left small triangle is the distance d the first ball falls by the time the second ball is released. If, on the other hand, the second ball is released too soon, after the first ball has traveled a distance d that is less than h, then we have the situation shown in Fig. 2.18(b). The second ball travels a distance that is larger than h (assuming it can fall into a hole in the ground) by the time the first ball travels $4h$ and hits the ground. In other words (assuming there is no hole), the second ball hits the ground first. Conversely, if the second ball is released too late, then it travels a distance that is smaller than h by the time the first ball hits the ground. This problem basically boils down to the fact that freefall distances fallen are proportional to t^2 (or equivalently, to the areas of triangles), so twice the time means four times the distance.

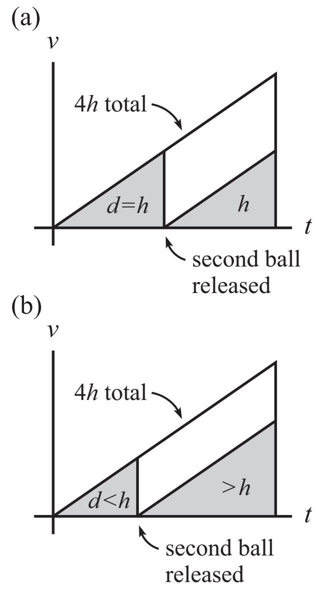


Figure 2.18