Preface

The Green-Eyed Dragons and Other Mathematical Monsters (Draft version, September 2018)
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This book is a collection of 57 *very* difficult math problems I have compiled over the years. The collection started long ago in graduate school. Every now and then during those years, one of my fellow graduate students would come into the office and say, "Hey, I just came across a new problem, have you heard of this one?..." Whenever research was going slowly, it was always comforting to have an interesting problem to puzzle over! Many of those problems eventually found their way to an old "Problem of the Week" webpage of mine (www.physics.harvard.edu/academics/undergrad/problems). After letting the problems sit there for a while, I finally (re)polished up a number of them, added some new ones, and created this book.

The book is written for anyone who (a) enjoys pondering difficult problems for great lengths of time, and (b) can tolerate the frustration of not being able to figure something out. This book isn't for the faint of heart. If you use it properly (that is, without looking at the solutions too soon; see the comments below), you *will* get frustrated at times, and you *will* pull out a few hairs. But just because "No pain, no gain" is a cliche, that doesn't mean it's not true!

Chapter 1 contains the problems (57 in all), Chapter 2 gives some hints, and Chapter 3 presents the solutions. There is also an appendix on Taylor series. The hints in Chapter 2 are fairly minimal, so don't expect a problem to be easy after looking at a hint. For better or for worse, I decided not to rate the problems with a difficulty level.

For most of the problems, algebra is the only formal prerequisite. But a few require calculus, if you want to steer clear of those. They are: Problems 8, 11, 12, 37, 39, 40, 42, 44, 45, 52, and 53. This list doesn't include ones that make use of the results given in Problems 52 and 53 (even though calculus is an ingredient in those). It also doesn't include problems that use (but don't require the derivation of) a Taylor series from the list given in the appendix, because the *use* of a Taylor series involves only algebra; see the appendix for more on this.

The problems are mostly classics that have been around for ages. The solutions (for the most part) are mine, although I'm sure that every one of them has appeared countless times elsewhere. The problems are divided into four categories: General, Geometry, Probability, and Foundational. The probability section is the longest of the four. If you want to review some concepts from probability and combinatorics (binomial coefficients, expectation value, etc.), you may want to take a look at my book *Probability for the Enthusiastic Beginner*. A few problems from that book also appear in this one.

The Foundational problems contain results that are useful in other problems throughout this book (mostly in the probability section). The results from Problems 52 (Stirling's Formula) and 53 (A Handy Formula) are the most useful, so you might want to do those (or at least note their results) before diving into the probability problems. If a given problem requires a result from a foundational one, I will usually make a reference to that in the statement of the problem. I chose to put the foundational problems last in the book instead of first, because they tend to be of the more technical math type, and I didn't want readers to assume that those problems had to be done first. You can think of them sort of as appendices. Overall, there is no preferred order for doing the problems. They are arranged somewhat randomly within each section, so you can jump around and tackle whichever problem looks appealing on a given day.

The solutions often contain extensions/variations of the given problems. You can treat these as additional problems that are waiting to be solved. Just set the book aside and try to derive/prove the result yourself, without looking at how I did it. That way, there are even more than 57 problems in the book!

The most important advice I have for using this book is:

Don't look at the solutions too soon!

The problems are designed to be brooded over for a while. If you look at a solution too soon and thereby remove any chance of solving things yourself, then the problem is gone forever. It's never coming back. There are only so many of these classics in the world, so don't waste them by looking at the solution without thinking about the problem for a *long* time. How long? Well, if you can't solve a problem, wait at least a week before looking at the hint. If that doesn't do the trick, then wait at least a month before looking at the solution. Actually, even a month is probably too short. There's really no hurry. Just move on to another problem; there are lots of them. As long as there are other problems to work on, there's no need to look at any solutions. You can be pondering many at a time.

If you do eventually need to look at a solution (after at least a month), you should read only one line at a time, covering up the page with a piece a paper, so that you don't accidentally see too much. As soon as you read enough to get a hint, set the book aside and try to work things out. That way, you'll still be able to (mostly) solve the problem on your own. Repeat as necessary, with a week between peeks at the solution. You will learn a great deal this way. If you instead head right to the solution and read it straight through, you will learn very little.

A few informational odds and ends: This book contains many supplementary remarks that are separated off from the main text; these end with a shamrock, ♣. The figures were drawn with Adobe Illustrator. The numerical plots were generated with Wolfram Mathematica. I often use an " 's" to indicate the plural of one-letter items (like 6's on dice rolls). I refer to the normal distribution by its other name, the "Gaussian" distribution. I am occasionally sloppy with the distinction between "average value" (dealing with past events) and "expected/expectation value" (dealing

with future events). And in quotients such as $a/(bc)$, I often drop the parentheses and just write a/bc; I do *not* mean $(a/b) \cdot c$ by this.

I am grateful to the many friends and colleagues who have offered valuable input over the years, ranging from ideas for problems to lively discussions of solutions. I would like to thank Jacob Barandes, Joe Blitzstein, Nancy Chen, Carol Davis, Louis Deslauriers, Eric Dunn, Niell Elvin, Dan Eniceicu, Howard Georgi, Theresa Morin Hall, Brian Hall, Lev Kaplan, Alex Johnson, Abijith Krishnan, Matt McIrvin, Lenny Ng, Dave Patterson, Sharad Ramanathan, Mike Robinson, Nate Salwen, Aravi Samuel, Alexia Schulz, Bob Silverman, Steve Simon, Igor Smolyarenko, Joe Swingle, Corri Taylor, Carey Witkov, Eric Zaslow, Tanya Zelevinsky, and Keith Zengel. My memory has certainly faded over the past 20 years, so I have surely left out other people who contributed to the book. Please accept my apologies!

Despite careful editing, there is zero probability that this book is error free. If anything looks amiss, please check for typos, updates, additional material, etc., at the webpage: www.people.fas.harvard.edu/~djmorin/book.html. And please let me know if you discover something that isn't already posted. Suggestions are always welcome.

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