Chapter 15

Appendices

Note: This draft version of the appendices is essentially the same as the published version.

15.1 Appendix A: Useful formulas

15.1.1 Taylor series

The general form of a Taylor series is

$$f(x_0 + x) = f(x_0) + f'(x_0)x + \frac{f''(x_0)}{2!}\,x^2 + \frac{f'''(x_0)}{3!}\,x^3 + \cdots, \tag{15.1}$$

which can be verified by taking derivatives and then setting $x = 0$. For example, taking the first derivative and then setting $x = 0$ gives $f'(x_0)$ on the left, and also $f'(x_0)$ on the right, because the first term is a constant and gives zero, the second term gives $f'(x_0)$, and all the rest of the terms give zero once we set $x = 0$ because they all have at least one x left in them. Likewise, if we take the second derivative of each side and then set $x = 0$, we obtain $f''(x_0)$ on both sides. And so on for all derivatives. Therefore, since the two functions on each side of the above equation are equal at $x = 0$ and also have their nth derivatives equal at $x = 0$ for all n, they must in fact be the same function (assuming that they're nicely behaved functions, which we generally assume in physics).

Some specific Taylor series that come up often are listed below. They are all derivable via eq. (15.1), but sometimes there are quicker ways of obtaining them. For example, eq. (15.3) is most easily obtained by taking the derivative of eq. (15.2), which itself is simply the sum of a geometric series.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \tag{15.2}$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots \tag{15.3}$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots \tag{15.4}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{15.5}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \tag{15.6}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \tag{15.7}$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \cdots \tag{15.8}$$

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3x^2}{8} + \cdots \tag{15.9}$$

$$(1+x)^n = 1 + nx + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots \tag{15.10}$$

15.1.2 Nice formulas

The first formula here can be quickly proved by showing that the Taylor series for both sides are equal.

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{15.11}$$

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}), \qquad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \tag{15.12}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}, \qquad \sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \tag{15.13}$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta} \tag{15.14}$$

$$\sin 2\theta = 2\sin\theta\cos\theta, \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta \tag{15.15}$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{15.16}$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{15.17}$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \tag{15.18}$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x}), \qquad \sinh x = \frac{1}{2}(e^x - e^{-x}) \tag{15.19}$$

$$\cosh^2 x - \sinh^2 x = 1 \tag{15.20}$$

$$\frac{d}{dx}\cosh x = \sinh x, \qquad \frac{d}{dx}\sinh x = \cosh x \tag{15.21}$$

15.1.3 Integrals

$$\int \ln x \, dx = x \ln x - x \tag{15.22}$$

$$\int x \ln x \, dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} \tag{15.23}$$

$$\int x e^x \, dx = e^x (x - 1) \tag{15.24}$$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x \qquad \text{or} \qquad -\cot^{-1} x \tag{15.25}$$

$$\int \frac{dx}{x(1+x^2)} = \frac{1}{2} \ln \left(\frac{x^2}{1+x^2} \right) \tag{15.26}$$

$$\int \frac{dx}{1-x^2} = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) \qquad \text{or} \qquad \tanh^{-1} x \qquad (x^2 < 1) \tag{15.27}$$

$$\int \frac{dx}{1-x^2} = \frac{1}{2} \ln \left(\frac{x+1}{x-1} \right) \qquad \text{or} \qquad \coth^{-1} x \qquad (x^2 > 1) \tag{15.28}$$

$$\int \sqrt{1+x^2} \, dx = \frac{1}{2} \left(x\sqrt{1+x^2} + \ln \left(x + \sqrt{1+x^2} \right) \right) \tag{15.29}$$

$$\int \frac{1+x}{\sqrt{1-x}} \, dx = -\frac{2}{3}(5+x)\sqrt{1-x} \tag{15.30}$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \qquad \text{or} \qquad -\cos^{-1} x \tag{15.31}$$

$$\int \frac{dx}{\sqrt{x^2+1}} = \ln \left(x + \sqrt{x^2+1} \right) \qquad \text{or} \qquad \sinh^{-1} x \tag{15.32}$$

$$\int \frac{dx}{\sqrt{x^2-1}} = \ln \left(x + \sqrt{x^2-1} \right) \qquad \text{or} \qquad \cosh^{-1} x \tag{15.33}$$

$$\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x \qquad \text{or} \qquad -\csc^{-1} x \tag{15.34}$$

$$\int \frac{dx}{x\sqrt{1+x^2}} = -\ln \left(\frac{1+\sqrt{1+x^2}}{x} \right) \qquad \text{or} \qquad -\mathrm{csch}^{-1} x \tag{15.35}$$

$$\int \frac{dx}{x\sqrt{1-x^2}} = -\ln \left(\frac{1+\sqrt{1-x^2}}{x} \right) \qquad \text{or} \qquad -\mathrm{sech}^{-1} x \tag{15.36}$$

$$\int \frac{dx}{\cos x} = \ln \left(\frac{1+\sin x}{\cos x} \right) \tag{15.37}$$

$$\int \frac{dx}{\sin x} = \ln \left(\frac{1-\cos x}{\sin x} \right) \tag{15.38}$$

15.2 Appendix B: Multivariable, vector calculus

This appendix gives a brief review of multivariable calculus, often known as vector calculus. The first three topics below (dot product, cross product, partial derivatives) are used often in this book, so if you haven't seen them before, you should read these parts carefully. But the last three topics (gradient, divergence, curl) are used only occasionally, so it isn't crucial that you master these (for this book, at least). For all of the topics, it's possible to go much deeper into them, but I'll present just the basics here. If you want further material, any book on multivariable calculus should do the trick.

15.2.1 Dot product

The *dot product*, or *scalar product*, between two vectors is defined to be

$$\mathbf{a} \cdot \mathbf{b} \equiv a_x b_x + a_y b_y + a_z b_z. \tag{15.39}$$

The dot product takes two vectors and produces a scalar, which is just a number. You can quickly use eq. (15.39) to show that the dot product is commutative and distributive. That is, $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$, and $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$. Note that the dot product of a vector with itself is $\mathbf{a} \cdot \mathbf{a} = a_x^2 + a_y^2 + a_z^2$, which is just its length squared, $|\mathbf{a}|^2 \equiv a^2$.

Taking the sum of the products of the corresponding components of two vectors, as we did in eq. (15.39), might seem like a silly and arbitrary thing to do. Why don't we instead look at the sum of the cubes of the products of the corresponding components? The reason is that the dot product as we've defined it has many nice properties, the most useful of which is that it can be written as

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta \equiv ab\cos\theta, \tag{15.40}$$

where θ is the angle between the two vectors. We can demonstrate this as follows. Consider the dot product of the vector $\mathbf{c} \equiv \mathbf{a} + \mathbf{b}$ with itself, which is simply the square of the length of $\mathbf{c}$. Using the commutative and distributive properties, we have

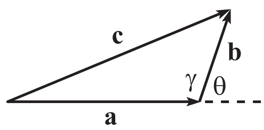

Figure 15.1

$$\begin{aligned} c^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) &= \mathbf{a} \cdot \mathbf{a} + 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} \\ &= a^2 + 2\mathbf{a} \cdot \mathbf{b} + b^2. \end{aligned} \tag{15.41}$$

But from the law of cosines applied to the triangle in Fig. 15.1, we have

$$c^2 = a^2 + b^2 - 2ab\cos\gamma = a^2 + b^2 + 2ab\cos\theta, \tag{15.42}$$

because $\gamma = \pi - \theta$. Comparing this with eq. (15.41) yields $\mathbf{a} \cdot \mathbf{b} = ab\cos\theta$, as desired. The angle between two vectors is therefore given by

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\,. \tag{15.43}$$

A nice corollary of this result is that if the dot product of two vectors is zero, then $\cos\theta = 0$, which means that the vectors are perpendicular. If someone gives you the vectors $(1, -2, 3)$ and $(4, 5, 2)$, it's by no means obvious that they're perpendicular. But the know from eq. (15.43) that they indeed are.

Geometrically, the dot product $\mathbf{a} \cdot \mathbf{b} = ab\cos\theta$ equals the length of $\mathbf{a}$ times the component of $\mathbf{b}$ along $\mathbf{a}$. Or vice versa, depending on which length you want to group the $\cos\theta$ factor with. If we rotate our coordinate system, the dot product of two vectors remains the same, because it depends only on their lengths and the angle between them, and these are unaffected by the rotation. In other words, the dot product is a scalar. This certainly isn't obvious from looking at the original definition in eq. (15.39), because the coordinates get all messed up during the rotation.

Example (Distance on the earth): Given the longitude angle ϕ and the polar angle θ (measured down from the north pole, so θ is 90° minus the latitude angle) for two points on the earth, what is the distance between them, as measured along the earth?

Solution: Our goal is to find the angle, β, between the radii vectors to the two points, because the desired distance is then $R\beta$. This would be a tricky problem if we didn't have the dot product at our disposal, but things are easy if we make use of eq. (15.43) to say that $\cos\beta = \mathbf{r}_1 \cdot \mathbf{r}_2/R^2$. The problem then reduces to finding $\mathbf{r}_1 \cdot \mathbf{r}_2$. The Cartesian components of these vectors are

$$\begin{aligned} \mathbf{r}_1 &= R(\sin\theta_1\cos\phi_1,\ \sin\theta_1\sin\phi_1,\ \cos\theta_1), \\ \mathbf{r}_2 &= R(\sin\theta_2\cos\phi_2,\ \sin\theta_2\sin\phi_2,\ \cos\theta_2). \end{aligned} \tag{15.44}$$

The desired distance is then $R\beta = R\cos^{-1}(\mathbf{r}_1 \cdot \mathbf{r}_2/R^2)$, where

$$\begin{aligned} \mathbf{r}_1 \cdot \mathbf{r}_2/R^2 &= \sin\theta_1\sin\theta_2(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2) + \cos\theta_1\cos\theta_2 \\ &= \sin\theta_1\sin\theta_2\cos(\phi_2 - \phi_1) + \cos\theta_1\cos\theta_2. \end{aligned} \tag{15.45}$$

We can check some limits: If $\phi_1 = \phi_2$, then this gives $\beta = \theta_2 - \theta_1$ (or $\theta_1 - \theta_2$, depending on which is larger), as expected. And if $\theta_1 = \theta_2 = 90°$, then it gives $\beta = \phi_2 - \phi_1$ (or $\phi_2 - \phi_1$), as expected.

15.2.2 Cross product

The *cross product*, or *vector product*, between two vectors is defined via a determinant to be

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &\equiv \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \\ &= \hat{\mathbf{x}}(a_y b_z - a_z b_y) + \hat{\mathbf{y}}(a_z b_x - a_x b_z) + \hat{\mathbf{z}}(a_x b_y - a_y b_x). \end{aligned} \tag{15.46}$$

The cross product takes two vectors and produces another vector. As with the dot product, you can show that the cross product is distributive. However, it is *anti*-commutative (that is, $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$), which is evident from eq. (15.46). So the cross product of any vector with itself is zero.

As with the dot product, the reason why we study this particular combination of components is that it has many nice properties, the most useful of which are that its direction is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ (in the orientation determined by the right-hand rule; see below), and its magnitude is

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta \equiv ab\sin\theta. \tag{15.47}$$

Let's first show that $\mathbf{a} \times \mathbf{b}$ is indeed perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. We'll do this by making use of the above handy fact that if the dot product of two vectors is zero, then the vectors are perpendicular. We have

$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = a_x(a_y b_z - a_z b_y) + a_y(a_z b_x - a_x b_z) + a_z(a_x b_y - a_y b_x) = 0, \tag{15.48}$$

as desired. Likewise for $\mathbf{b}$. There is still an ambiguity, however, because although we know that $\mathbf{a}\times\mathbf{b}$ points along the direction perpendicular to the plane spanned by $\mathbf{a}$ and $\mathbf{b}$, there are two possible directions along this line. Assuming that our coordinate system has been chosen to be "right-handed" (that is, if you point the fingers of your right hand in the direction of $\hat{\mathbf{x}}$ and then swing them to $\hat{\mathbf{y}}$, your thumb points along $\hat{\mathbf{z}}$), then the direction of $\mathbf{a}\times\mathbf{b}$ is determined by the right-hand rule. That is, if you point the fingers of your right hand in the direction of $\mathbf{a}$ and then swing them to $\mathbf{b}$ (through the angle that is less than 180°), your thumb points along $\mathbf{a}\times\mathbf{b}$. This is consistent with the fact that eq. (15.46) gives $(1,0,0)\times(0,1,0)=(0,0,1)$, or $\hat{\mathbf{x}}\times\hat{\mathbf{y}}=\hat{\mathbf{z}}$.

Let's now demonstrate the $|\mathbf{a}\times\mathbf{b}| = ab\sin\theta$ result, which is equivalent to $|\mathbf{a}\times\mathbf{b}|^2 = a^2b^2(1-\cos^2\theta)$, which is equivalent to $|\mathbf{a}\times\mathbf{b}|^2 = a^2b^2-(\mathbf{a}\cdot\mathbf{b})^2$. Written in terms of the components, this last equation is

$$\begin{aligned}(a_yb_z-a_zb_y)^2+(a_zb_x-a_xb_z)^2+(a_xb_y-a_yb_x)^2 \;&=\; (a_x^2+a_y^2+a_z^2)(b_x^2+b_y^2+b_z^2)\\ &\quad -(a_xb_x+a_yb_y+a_zb_z)^2. \qquad (15.49)\end{aligned}$$

If you stare at this long enough, you'll see that it's true. The three different types of terms agree on both sides. For example, both sides have an $a_y^2b_z^2$ term, a $-2a_yb_ya_zb_z$ term, and no $a_x^2b_x^2$ term.

15.2.3 Partial derivatives

When dealing with a function of only one variable, there is no ambiguity when taking a derivative. However, with a function of many variables, we have to specify which one of the variables we're differentiating with respect to. If we have function of, say, two variables, $f(x,y)$, and if we want to take the derivative with respect to x, then we use the terminology "*partial derivative* with respect to x," with the notation $\partial f/\partial x$. To evaluate this partial derivative, we don't have to do anything fancy. We just take a regular derivative with respect to x, while assuming that y is constant. For example, if $f(x,y)=x-2y+x^2y^3$, then $\partial f/\partial x = 1+2xy^3$, and $\partial f/\partial y=-2+3x^2y^2$. If we plot the value of f as the height above the x-y plane, then when we take the partial derivative with respect to x, we're simply finding the slope of the curve formed by the intersection of the function's surface with the vertical plane parallel to the x axis and passing through the point in question. Similarly for y.

If we want to maximize or minimize a function of more than one variable, we need to set all the partial derivatives equal to zero. This is true because if the partial derivative with respect to a certain variable isn't zero, then the slope of the function in that direction is nonzero, which means that the point can't be a local maximum or minimum. This argument is the same as in the single-variable case. It's just that now we can make the argument for each of the variables independently.

Demanding that all the partial derivatives equal zero doesn't actually guarantee having a local maximum or minimum. The point in question might be a *saddle point,* which means that the function is a local maximum in some directions and a local minimum in others (so in two dimensions the function looks like a saddle; hence the name). For example, consider the function of two variables, $f(x,y)=3x^2-y^2$. Then the point $(0,0)$ is a local minimum in the x direction and a local maximum in the y direction.

For two variables, if the second partial derivatives have opposite signs at a point where the first partial derivatives are zero, then we have a saddle point, because there is an upward parabola in one direction and a downward parabola in the other. However, we might have a saddle point even if the second partial derivatives have the same sign. For example, if we make the change of variables $x\equiv w-z$ and $y\equiv w+z$ in $f(x,y)=3x^2-y^2$, then it becomes

$f(w, z) = 2w^2 + 2z^2 - 8wz$. If we didn't already know from the $f(x, y)$ form that $(0, 0)$ is a saddle point, we could deduce this in the following way. Imagine that z is given, and then solve for the w that makes $f(w, z) = 0$. The result of solving this quadratic equation is that w takes the form of some multiple of z. That is, $w = Az$, where A happens to be $2 \pm \sqrt{3}$ in the present case. Since there are two (real) solutions for A here, there are two lines, namely $w = (2 \pm \sqrt{3})z$, emanating from $(0, 0)$ for which $f(w, z) = 0$. So $(0, 0)$ can't be a local maximum or minimum. It must therefore be a saddle point.[1]

In general, there are two real solutions for A if and only if the discriminant of the quadratic equation is positive. For an arbitrary function of two variables, its shape in the vicinity of a point (which we will take to be $(0, 0)$ after a shift in the coordinates) where both first partial derivatives are zero can be approximated by the Taylor series to second order (which you can verify by taking various derivatives, just as you would do with a function of one variable),

$$f(x, y) = C + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)x^2 + \frac{1}{2}\left(\frac{\partial^2 f}{\partial y^2}\right)y^2 + \left(\frac{\partial^2 f}{\partial x\,\partial y}\right)xy + \cdots, \tag{15.50}$$

where it is understood that the partial derivatives here are evaluated at $(0, 0)$. The condition that the discriminant is positive is therefore

$$\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 - \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) > 0. \tag{15.51}$$

If this is true, then the point is a saddle point. If the left side is less than zero, then the point is a local maximum or minimum, because there are no nearby points for which $f(x, y) = C$; they are all either greater than C or less than C. If the left side equals zero, then the function looks like a trough, at least in the vicinity of the point in question (assuming that there is some quadratic dependence in the function).

15.2.4 Gradient

Given a function $f(x, y, z)$ (we'll work mainly with three variables from now on), we can form the vector whose components are the partial derivatives of f, namely $(\partial f/\partial x,\ \partial f/\partial y,\ \partial f/\partial z)$. This vector is called the *gradient*. If we define the differential vector operator ∇ (usually called "del") to be $\nabla \equiv (\partial/\partial x,\ \partial/\partial y,\ \partial/\partial z)$, then the gradient is simply

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right). \tag{15.52}$$

The gradient takes a function and produces a vector. For example, if $f(x, y, z) = xy^2 - yz^3$, then $\nabla f = (y^2,\ 2xy - z^3,\ -3yz^2)$. We call ∇ an "operator" because it needs to operate on a function to produce the gradient vector.

What is the physical meaning of the gradient? The gradient gives the direction you should march in if you want f to increase at the greatest rate. The reason for this is the following. Consider the value of a function $f(x, y, z)$ at a certain point, and then look at

[1] This is true because along these two lines, the first partial derivatives aren't zero, which means that the plane of the function's surface is tilted there. So the function is positive on one side of each line and negative on the other, which is exactly what happens with a saddle. There is, however, the special case where the discriminant of the quadratic equation is zero (as with, for example, $f(x, y) = (x - y)^2$), in which case there is only one solution for A and thus only one line for which the function is zero. In this case, the function looks like a (possibly upside down) trough. It is zero (or some given constant) along the the line, at least to second order. And it curves up (or down) quadratically as you move away from the line (assuming that there is at least some quadratic dependence in the function).

the value at a nearby point, displaced by the vector (dx, dy, dz). What (approximately, to first order) is the change in f between the two points? Well, as you march a distance dx in the x direction, the (first-order) change in f is $(\partial f/\partial x)\, dx$, by the definition of the partial derivative (just as in the one-variable case). If you then march a distance dy in the y direction, the function changes by an additional amount of $(\partial f/\partial y)\, dy$. And likewise the z direction gives a change of $(\partial f/\partial z)\, dz$. Adding up these three changes in f, we see that the total first-order change in f is[2]

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz. \tag{15.53}$$

Using the dot product, this can be written concisely as

$$df = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \cdot (dx, dy, dz) \equiv \nabla f \cdot d\mathbf{r}. \tag{15.54}$$

We can now make use of eq. (15.40) to say that the change in f is $df = |\nabla f||d\mathbf{r}|\cos\theta$, where θ is the angle between ∇f and $d\mathbf{r}$. The meaning of this is the following. Consider a given point (x, y, z). The ∇f gradient vector at this point is a particular vector. Imagine marching along little $d\mathbf{r}$ vectors in various direction and seeing how much f changes (assume that all the $d\mathbf{r}$ vectors have the same length, to be consistent). What direction should you march in, in order to have f change the most? Or not change at all? In the $df = |\nabla f||d\mathbf{r}|\cos\theta$ expression, $|\nabla f|$ has a definite value at the point in question, and we're assuming that we pick $|d\mathbf{r}|$ to always be the same, so it comes down to the $\cos\theta$ factor. Therefore, if you march directly along the ∇f gradient vector at the point, then f increases the most. And if you march in any direction in the plane perpendicular to ∇f, then f doesn't change at all (to first order). And if you march in the direction anti-parallel to ∇f, then f decreases the most.

This is more easily visualized in the case of a function of only two variables, $f(x, y)$, because then we can picture the value of f as being the height in the z direction. The graph of f is just a surface of mountains and valleys above (or below) the x-y plane. The gradient $\nabla f = (\partial f/\partial x,\, \partial f/\partial y)$ then gives the direction of steepest ascent. That is, f changes at the greatest rate if you march in the direction of ∇f in the x-y plane; the slope of the surface in this direction in the x-y plane is larger than in any other direction. And if you march in either direction along the line in the x-y plane that is perpendicular to ∇f, then f has zero change. By continuing to march along the direction perpendicular to ∇f wherever you are, you will form a curve in the x-y plane for which all the points give the same value of f. In other words, if you slice the surface of f with a horizontal plane whose height equals this particular value of f, and if you look at the intersection of this plane with the surface, then the projection of this intersection onto the x-y plane is the above curve you formed in the x-y plane.

15.2.5 Divergence

Consider a vector whose components are functions of the coordinates. For example, let $\mathbf{F} = (F_x, F_y, F_z) = (3xz,\, 2y^2 + xyz,\, x^2 + z^3)$. Then the *divergence* of $\mathbf{F}$ is defined to be the dot product of the ∇ operator with $\mathbf{F}$, that is,

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\,. \tag{15.55}$$

[2] There is technically an ambiguity in where each of these partial derivative is evaluated, because the three little steps you took started at three different points. But these ambiguities involve first-order corrections to the partial derivatives, and since these partial derivatives are already being multiplied by the first-order dx, dy, and dz terms in eq. (15.53), any ambiguities will result in second-order effects and can therefore be ignored.

The divergence takes a vector and produces a number. The above **F** has a divergence of $(3z) + (4y + xz) + (3z^2)$.

What is the physical meaning of the divergence? Consider an infinitesimal box with sides of length dx, dy, and dz. Then the divergence measures the net flux of the vector field out of the box, divided by the volume of the box. (The flux through a surface is defined to be the integral of the area times the component of the vector perpendicular to the surface.) For example, if a certain vector field gives the velocity at each point in a fluid flow, and if the divergence is nonzero, then there must be a source (or sink) that creates (or destroys) fluid, because otherwise whatever fluid goes into the little box would have to come out of it somewhere, yielding zero net flux.

Let's see why the divergence equals the flux per volume. Consider the "left" $dy \times dz$ face of the little box. The amount of flux from the vector field into the box through this face equals the area $dy\, dz$ times the F_x component. (The F_y and F_z components are parallel to this face and therefore contribute nothing to the flux through it.) The amount of flux *out* of the "right" $dy \times dz$ face equals the area $dy\, dz$ times the value of the F_x component there. But this value equals (to first order) the original F_x plus $(\partial F_x/\partial x)\, dx$, by the definition of the partial derivative. The F_x part of this cancels the inward flux from the left face, so the net flux out of the little box through these two faces is $\big((\partial F_x/\partial x)\, dx\big)\, dy\, dz$. Similar calculations work for the other two pairs of parallel faces, so the total flux out of the box is

$$\text{Net flux} \;=\; \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) dx\, dy\, dz. \tag{15.56}$$

Therefore, as promised, the net flux per volume equals the divergence. The integrated form of this result is the *divergence theorem,* or *Gauss' theorem,* which takes the form,

$$\int_V \nabla \cdot \mathbf{F}\, dV = \int_S \mathbf{F} \cdot d\mathbf{A}. \tag{15.57}$$

The integral on the left runs over a given volume, and the integral on the right runs over the surface that encloses this volume. The vector $d\mathbf{A}$ has a magnitude equal to an infinitesimal piece of area of S and a direction defined to be perpendicular to the plane containing this piece (with the positive direction being outward from the volume). Dotting $d\mathbf{A}$ with **F** has the effect of picking out only the component of **F** that is perpendicular to the piece (which is what is relevant in calculating the flux).

We'll skip the fine details here, but the basic idea of the proof is to divide the volume up into many infinitesimal cubes and look at the total flux through all the cubes. From above, the integral of the divergence over one little cube (which is essentially the divergence times the volume, because the divergence is essentially constant over the tiny volume) equals the flux through that cube. The integral of the divergence over the whole volume therefore equals the sum of the fluxes through all the cubes. But all the faces of the cubes that lie in the interior of the volume are shared by two cubes, so the flux through these faces cancels when taking the sum (because the flux though a given face is counted positive for one cube and negative for the other). So we are left with only the flux through the faces on the boundary of the volume (because these faces appear only once in the total integral). We are therefore left with the flux through the surface S, which is what appears on the right-hand side of eq. (15.57).

Example (Flux through a sphere): Verify the divergence theorem in the case where the surface is a sphere of radius R centered at the origin, and $\mathbf{F} = (x, y, z)$.

Solution: On the left-hand side of eq. (15.57), the divergence of (x, y, z) is 1+1+1=3, so the integral of this over the volume of the sphere is simply $3(4\pi R^3/3) = 4\pi R^3$. On the right-hand side, the unit vector perpendicular to the surface is $(x, y, z)/R$, so $d\mathbf{A} = (dA)(x, y, z)/R$. The dot product of this with (x, y, z) is $(dA)(x^2 + y^2 + z^2)/R = (dA)R$. The integral of this over the surface of the sphere is just $(4\pi R^2)R = 4\pi R^3$. The two sides are therefore equal, as desired.

15.2.6 Curl

Consider a vector whose components are functions of the coordinates. For example, let $\mathbf{F} = (F_x, F_y, F_z) = (3xz,\ x^2yz,\ x + z)$. Then the *curl* of $\mathbf{F}$ is defined to be the cross product of the ∇ operator with $\mathbf{F}$, that is,

$$\begin{aligned}\nabla \times \mathbf{F} &\equiv \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix} \\ &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z},\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x},\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right). \end{aligned} \tag{15.58}$$

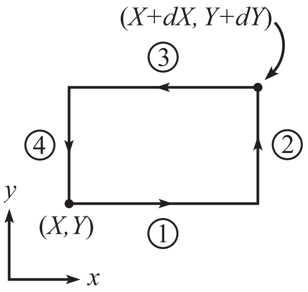

Figure 15.2

The curl takes a vector and produces another vector. The above $\mathbf{F}$ has a curl of $(-x^2y, 3x - 1, 2xyz)$.

What is the physical meaning of the curl? Consider the infinitesimal rectangle shown in Fig. 15.2. This rectangle lies in the x-y plane, so for the moment we will suppress the z component of all coordinates, for convenience. It turns out that the z component of the curl equals the counterclockwise integral $\int \mathbf{F} \cdot d\mathbf{r}$ around the closed loop, divided by the area of the loop (similar statements hold for the y and x components and the associated little rectangles in the x-z and y-z planes). Let's see why this is true.

The total counterclockwise integral of $\mathbf{F} \cdot d\mathbf{r}$ around the loop entails moving to the right on segment 1 and to the left on 3, and up on segment 2 and down on 4. On segments 1 and 3, both dy and dz are zero, so only the $F_x\, dx$ term survives in the dot product $\mathbf{F} \cdot d\mathbf{r}$. Likewise, $F_y\, dy$ is the only nonzero term on segments 2 and 4. If we pair up the two pairs of parallel sides, the total counterclockwise integral is

$$\begin{aligned}\int \mathbf{F} \cdot d\mathbf{r} &= \int_X^{X+dX} \Big(F_x(x, Y) - F_x(x, Y + dY) \Big)\, dx \\ &\quad + \int_Y^{Y+dY} \Big(F_y(X + dX, y) - F_y(X, y) \Big)\, dy. \end{aligned} \tag{15.59}$$

Let's approximate the differences in these parentheses. To first-order, we have

$$F_x(x, Y + dY) - F_x(x, Y) \approx dY \frac{\partial F_x(x, y)}{\partial y}\bigg|_{(x,Y)} \approx dY \frac{\partial F_x(x, y)}{\partial y}\bigg|_{(X,Y)}. \tag{15.60}$$

The first approximation here is valid due to the definition of the partial derivative. The second approximation (replacing x with X) is valid because our rectangle is small enough so that x is essentially equal to X. Any error in this approximation is second-order small, because we already have a factor of dY in our term. A similar treatment works for the F_y terms, so eq. (15.59) becomes

$$\int \mathbf{F} \cdot d\mathbf{r} = \int_Y^{Y+dY} dX \frac{\partial F_y(x, y)}{\partial x}\bigg|_{(X,Y)} dy - \int_X^{X+dX} dY \frac{\partial F_x(x, y)}{\partial y}\bigg|_{(X,Y)} dx. \tag{15.61}$$

The integrands are constants, so we can quickly perform the integrals to obtain

$$\int \mathbf{F} \cdot d\mathbf{r} = dX\, dY \left(\frac{\partial F_y(x,y)}{\partial x} - \frac{\partial F_x(x,y)}{\partial y} \right) \bigg|_{(X,Y)}. \tag{15.62}$$

As promised, z component of the curl equals the counterclockwise integral $\int \mathbf{F} \cdot d\mathbf{r}$ around the closed loop, divided by the area of the loop. The preceding analysis also works, of course, for little rectangles in the x-z and y-z planes. We therefore obtain the two other components of the curl.

The generalization of the above result to tilted and wavy surfaces is *Stokes' theorem*, which states that

$$\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{A} = \int_C \mathbf{F} \cdot d\mathbf{r}. \tag{15.63}$$

The integral on the left runs over a given surface, and the integral on the right runs over the curve that is the boundary of this surface. The vector $d\mathbf{A}$ has a magnitude equal to an infinitesimal piece of area of S and a direction defined to be perpendicular to the plane containing this piece (with its orientation defined via the orientation along C and the right-hand rule). We'll skip the details here, but the basic idea of the proof is similar to the idea behind the divergence theorem above, except with certain words replaced by other words ("volume" becomes "surface," and "surface" becomes "curve", etc.). We'll divide the surface up into many infinitesimal rectangles and look at the total integral around all the rectangles. For simplicity, let's just deal with a flat surface in the x-y plane.

From above, the integral of the z component of the curl over one little rectangle equals the counterclockwise integral of $\mathbf{F} \cdot d\mathbf{r}$ around the edges of the rectangle. The integral of the z component of the curl over the whole surface therefore equals the sum of the integrals around all the rectangles. But all the edges of the rectangles that lie in the interior of the surface are shared by two rectangles, so the integral along these edges cancels when taking the sum (because the integral along a given edge is counted positive for one rectangle and negative for the other). So we are left with only the integral along the edges on the boundary of the surface (because these edges appear only once in the total integral). We are therefore left with the integral along the curve C, which is what appears on the right-hand side of eq. (15.63).

Note that if the surface is closed, so that it has no boundary (in other words, there is no curve C), then the right-hand side of eq. (15.63) is zero, and hence the left-hand side is also. Such is the case with, for example, a sphere. Having no boundary means that if you're a little bug walking on the surface, you can't walk off it.

Example (Integral around a circle): Verify Stokes' theorem in the case where the curve is a circle of radius R in the x-y plane, centered at the origin, and $\mathbf{F} = (-y, x, 0)$.

Solution: On the left-hand side of eq. (15.63), the curl of $(-y, x, 0)$ is $(0, 0, 2)$. The $d\mathbf{A}$ vector also points in the z direction, so the dot product is just $2(dA)$. The integral of this over the interior of the circle is simply $2(\pi R^2)$. On the right-hand side, the dot product equals $-y\,dx + x\,dy$. Integrating this along the circumference of the circle is most easily done in polar coordinates. With $x = R\cos\theta$ and $y = R\sin\theta$, we have $dx = -R\sin\theta\, d\theta$ and $dy = R\cos\theta\, d\theta$. So $-y\,dx + x\,dy = R^2\, d\theta$. The integral of this as θ ranges from 0 to 2π is $R^2(2\pi)$. The two sides are therefore equal, as desired.

There are some handy facts that deal with combinations of the gradient, divergence, and curl. One is that the curl of a gradient is identically zero. That is, $\nabla \times \nabla f = 0$. You can verify this explicitly by using the definitions of the curl and the gradient, and also the fact that partial differentiation is commutative (that is, $\partial^2 f/\partial x\,\partial y = \partial^2 f/\partial y\,\partial x$). Alternatively, you can let $\mathbf{F} \equiv \nabla f$ in Stokes' theorem, which gives $\int_S (\nabla \times \nabla f) \cdot d\mathbf{A} = \int_C \nabla f \cdot d\mathbf{r}$. The right-hand side of this is simply the net change in the function f around the closed curve C, which is always zero. The integrand on the left-hand side must therefore identically be zero.

Also, the divergence of a curl is identically zero. That is, $\nabla \cdot (\nabla \times \mathbf{F}) = 0$. Again, you can verify this explicitly by using the definitions of the divergence and the curl, and also the fact that partial differentiation is commutative. Alternatively, you can combine Gauss' theorem and Stokes' theorem to write $\int_V \nabla \cdot (\nabla \times \mathbf{F})\, dV = \int_C \mathbf{F} \cdot d\mathbf{r}$. The right-hand side is always zero, because the boundary surface S of any given volume V is closed, so there is no curve C. The integrand on the left-hand side must therefore identically be zero.

15.3 Appendix C: $F = ma$ vs. $F = dp/dt$

In nonrelativistic mechanics,[3] the equations $F = ma$ and $F = dp/dt$ say exactly the same thing if m is constant. But if m is not constant, then $dp/dt = d(mv)/dt = ma + (dm/dt)v$, which doesn't equal ma. So if a system has a changing mass, should we use $F = ma$ or $F = dp/dt$? Which equation correctly describes the physics? The answer to this depends on what you label as the system to which you associate the quantities m, p, and a. You can generally do a problem using either $F = ma$ or $F = dp/dt$, but you must be very careful about how you label things and how you treat them. The subtleties are best understood through two examples.

Example 1 (Sand dropping into cart): Consider a cart into which sand is dropped vertically at a rate $dm/dt = \sigma$. With what force must you push on the cart to keep it moving horizontally at a constant speed v? (This was the setup in the first example in Section 5.8.)

First solution: Let $m(t)$ be the mass of the cart-plus-sand-inside system (which we'll just call the "cart"). If we use $F = ma$ (where a is the acceleration of the cart, which is zero), then we obtain $F = 0$, which is incorrect. The correct expression to use is $F = dp/dt$. This gives

$$F = \frac{dp}{dt} = ma + \frac{dm}{dt}v = 0 + \sigma v. \tag{15.64}$$

This makes sense, because your force is what increases the momentum of the cart, and this momentum increases simply because the mass of the cart increases.

Second solution: It is possible to solve this problem by using $F = ma$ if we let our system be a small piece of mass that is being added to the cart. Your force is what accelerates this mass from rest to speed v. Consider a mass Δm that falls into the cart during a time Δt. Imagine that it falls into the cart in one lump at the start of the Δt, and then accelerates up to speed v during the time Δt. (It accelerates due to friction. But if you want, you can eliminate the cart as the intermediate object and just push directly on the mass.) This process repeats during each successive Δt interval. We can use $F = ma$ here because the mass of the small piece is constant. So we have $F = ma = \Delta m(v/\Delta t)$. Writing this as $(\Delta m/\Delta t)v$ gives the σv result we found above.

Third solution: As in the second solution, let's imagine the process occurring in discrete steps, but now with the cart as the system. Assume that a mass Δm falls into the cart and instantaneously decreases its speed (by conservation of momentum) to $v' = mv/(m + \Delta m)$, which is $\Delta v = v - v' = v\Delta m/(m + \Delta m)$ less than the original v. Assume that you then push on the cart for a time Δt (during which the mass remains constant at $m + \Delta m$, so $F = ma$ is the relevant expression) and bring it back up to speed v. The acceleration is $a = \Delta v/\Delta t = v(\Delta m/\Delta t)/(m + \Delta m) = \sigma v/(m + \Delta m)$, so your force is

$$F = (m + \Delta m)a = (m + \Delta m)\left(\frac{\sigma v}{m + \Delta m}\right) = \sigma v. \tag{15.65}$$

Example 2 (Sand leaking from cart): Consider a cart that leaks sand out of the bottom at a rate $dm/dt = \sigma$. If you apply a force F to the cart, what is its acceleration?

Solution: Let $m(t)$ be the mass of the cart-plus-sand-inside system (the "cart"). In this example, the correct expression to use is $F = ma$, so the acceleration is

$$a = \frac{F}{m}\,. \tag{15.66}$$

[3] We won't bother with relativity in this appendix, because nonrelativistic mechanics contains all the critical aspects we want to address.

Note that since m decreases with time, a increases with time. We used $F = ma$ here because at any instant, the mass m is what is being accelerated by the force F. As above, you can imagine the process occurring in discrete steps: You push on the mass for a short period of time, then a little piece instantaneously leaks out; then you push again on the new (smaller) mass, then another little piece leaks out; and so on. In this discretized scenario, it is clear that $F = ma$ is the appropriate formula, because it holds for each step in the process. The only ambiguity is whether to use m or $m + dm$ at a certain time, but this yields a negligible error.

REMARKS: It is still true that $F = dp/dt$ in this second example, provided that you let F be *total* force, and let p be the *total* momentum. In this example, F is the only force. However, the total momentum consists of both the sand in the cart and the sand that has leaked out and is falling through the air.[4] A common mistake is to use $F = dp/dt$, with p being only the cart's momentum. The leaked sand still has momentum.

There is an simple example that demonstrates why $F = dp/dt$ doesn't work when p refers only to the cart. Choose $F = 0$, so that the cart moves with constant speed v. Cut the cart in half, and label the back part as the "leaked sand" and the front part as the "cart." If you want the cart's p to have $dp/dt = F = 0$, then the cart's speed must double if its mass gets cut in half. But this is nonsense. Both halves simply continue to move at the same rate. ♣

To sum up, $F = dp/dt$ is always valid, provided that you use the *total* force and *total* momentum of a given system of particles. This approach, however, can get messy in certain situations. So in some cases it is easier to use an $F = ma$ argument, but you must be careful to correctly identify the system that is being accelerated by the force. The asymmetry in the above two examples is that in the first example, the force does indeed accelerate the incoming sand. But in the second example, the force does *not* accelerate (or decelerate) the outgoing sand. F has nothing to do with the leaked sand.

[4]If there were air resistance, we would have to worry about its effect on the falling sand if we wanted to use $F = dp/dt$ to solve the problem, where p is the total momentum. This is clearly not the best way to do the problem. If complicated things happen with the sand in the air, it would be foolish to consider this part of the sand when we don't have to.

15.4 Appendix D: Existence of principal axes

In this Appendix, we will prove Theorem 9.4. That is, we will show that an orthonormal set of principal axes does indeed exist for any object, and for any choice of origin. It isn't crucial that you study this proof. If you want to just accept the fact that principal axes exist, that's perfectly fine. But the method we will use in this proof is one you will see again and again in your physics studies, in particular when you study quantum mechanics (see the remark following the proof).

Theorem 15.1 *Given a real symmetric* 3×3 *matrix,* $\mathbf{I}$*, there exist three orthonormal real vectors,* $\hat{\boldsymbol{\omega}}_k$*, and three real numbers,* I_k*, with the property that*

$$\mathbf{I}\hat{\boldsymbol{\omega}}_k = I_k \hat{\boldsymbol{\omega}}_k. \tag{15.67}$$

Proof: This theorem holds more generally with 3 replaced by N (all the steps below easily generalize), but we'll work with $N = 3$, to be concrete. Consider a general 3×3 matrix, $\mathbf{I}$ (we don't need to assume yet that it's real or symmetric). Assume that $\mathbf{I}\mathbf{u} = I\mathbf{u}$ for some vector $\mathbf{u}$ and some number I.[5] This may be rewritten as

$$\begin{pmatrix} (I_{xx} - I) & I_{xy} & I_{xz} \\ I_{yx} & (I_{yy} - I) & I_{yz} \\ I_{zx} & I_{zy} & (I_{zz} - I) \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{15.68}$$

In order for there to be a nontrivial solution for the vector $\mathbf{u}$ (that is, one where $\mathbf{u} \neq (0,0,0)$), the determinant of this matrix must be zero.[6] Taking the determinant, we see that we get an equation for I of the form

$$aI^3 + bI^2 + cI + d = 0. \tag{15.69}$$

The constants a, b, c, and d are functions of the matrix entries I_{ij}, but we won't need their precise form to prove this existence theorem. The only thing we need this equation for is to say that there do exist three (generally complex) solutions for I, because the equation is a cubic.

We will now show that the solutions for I are real. This will imply that there exist three real vectors $\mathbf{u}$ satisfying $\mathbf{I}\mathbf{u} = I\mathbf{u}$, because we can plug the real I's back into eq. (15.68) and solve for the real components u_x, u_y, and u_z, up to an overall constant. We will then show that these vectors are orthogonal.

- *Proof that the I's are real:* This follows from the real and symmetric conditions on $\mathbf{I}$. Start with the equation $\mathbf{I}\mathbf{u} = I\mathbf{u}$, and take the dot product with $\mathbf{u}^*$ to obtain

$$\begin{aligned} \mathbf{u}^* \cdot \mathbf{I}\mathbf{u} &= \mathbf{u}^* \cdot I\mathbf{u} \\ &= I\mathbf{u}^* \cdot \mathbf{u}. \end{aligned} \tag{15.70}$$

  The vector $\mathbf{u}^*$ is the vector obtained by complex conjugating each component of $\mathbf{u}$ (we don't know yet that $\mathbf{u}$ can be chosen to be real). On the right side, I is a scalar, so we can take it out from between the $\mathbf{u}^*$ and $\mathbf{u}$. The fact that $\mathbf{I}$ is real implies that if we complex conjugate the equation $\mathbf{I}\mathbf{u} = I\mathbf{u}$, we obtain $\mathbf{I}\mathbf{u}^* = I^*\mathbf{u}^*$ (we know that $\mathbf{I}$ is real, but we don't know yet that I is real). If we then take the dot product of this equation with $\mathbf{u}$, we obtain

$$\mathbf{u} \cdot \mathbf{I}\mathbf{u}^* = I^*\mathbf{u} \cdot \mathbf{u}^*. \tag{15.71}$$

[5] Such a vector $\mathbf{u}$ is called an *eigenvector* of $\mathbf{I}$, and I is the associated *eigenvalue*. But don't let these names scare you. They're just definitions.

[6] If the determinant were not zero, then we could explicitly construct the inverse of the matrix, which involves cofactors divided by the determinant. Multiplying both sides by this inverse would yield $\mathbf{u} = \mathbf{0}$.

We now claim that if $\mathbf{I}$ is symmetric, then $\mathbf{a} \cdot \mathbf{I}\mathbf{b} = \mathbf{b} \cdot \mathbf{I}\mathbf{a}$, for any vectors $\mathbf{a}$ and $\mathbf{b}$. (We'll leave this for you to show by simply multiplying each side out.) In particular, $\mathbf{u}^* \cdot \mathbf{I}\mathbf{u} = \mathbf{u} \cdot \mathbf{I}\mathbf{u}^*$, so eqs. (15.70) and (15.71) give

$$(I - I^*)\mathbf{u} \cdot \mathbf{u}^* = 0. \tag{15.72}$$

And since $\mathbf{u} \cdot \mathbf{u}^* = |u_1|^2 + |u_2|^2 + |u_3|^2 \neq 0$, we must have $I = I^*$. Therefore, I is real.

- *Proof that the* $\mathbf{u}$ *are orthogonal:* This follows from the symmetric condition on $\mathbf{I}$. Let $\mathbf{I}\mathbf{u}_1 = I_1\mathbf{u}_1$, and $\mathbf{I}\mathbf{u}_2 = I_2\mathbf{u}_2$. Take the dot product of the former equation with $\mathbf{u}_2$ to obtain

$$\mathbf{u}_2 \cdot \mathbf{I}\mathbf{u}_1 = I_1\mathbf{u}_2 \cdot \mathbf{u}_1, \tag{15.73}$$

and take the dot product of the latter equation with $\mathbf{u}_1$ to obtain

$$\mathbf{u}_1 \cdot \mathbf{I}\mathbf{u}_2 = I_2\mathbf{u}_1 \cdot \mathbf{u}_2. \tag{15.74}$$

As above, the symmetric condition on $\mathbf{I}$ implies that the left-hand sides of eqs. (15.73) and (15.74) are equal. Therefore,

$$(I_1 - I_2)\mathbf{u}_1 \cdot \mathbf{u}_2 = 0. \tag{15.75}$$

There are two possibilities here: (1) If $I_1 \neq I_2$, then we are done, because $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$, which says that $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal. (2) If $I_1 = I_2 \equiv I$, then we have $\mathbf{I}(a\mathbf{u}_1 + b\mathbf{u}_2) = I(a\mathbf{u}_1 + b\mathbf{u}_2)$, for any a and b. So any linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ has the same property that $\mathbf{u}_1$ and $\mathbf{u}_2$ have (namely, that applying $\mathbf{I}$ is the same as just multiplying by I). We therefore have a whole plane of such vectors, so we can pick any two orthogonal vectors in this plane to be called $\mathbf{u}_1$ and $\mathbf{u}_2$. ■

This theorem proves the existence of principal axes, because the inertia tensor in eq. (9.8) is indeed a real and symmetric matrix.

Remark: (Warning: This remark has nothing to do with classical mechanics. It's simply an ill-disguised excuse to get a limerick on quantum mechanics into the book.) In quantum mechanics, it turns out that any observable quantity, such as position, energy, momentum, angular momentum, etc., can be represented by a *Hermitian* matrix, with the observed value being an eigenvalue of the matrix. A Hermitian matrix is a (generally complex) matrix with the property that the transpose of the matrix equals the complex conjugate of itself. For example, a 2×2 Hermitian matrix must be of the form,

$$\begin{pmatrix} a & b + ic \\ b - ic & d \end{pmatrix}, \tag{15.76}$$

for real numbers a, b, c, and d. Now, if observed values are to be given by the eigenvalues of such a matrix, then the eigenvalues had *better* be real, because no one (in this world, at least) is about to go for a jog of $4 + 3i$ miles, or pay an electric bill for $17 - 43i$ kilowatt-hours. And indeed, you can show, via a slightly modified version of the above "Proof that the I's are real" procedure, that the eigenvalues of any Hermitian matrix are in fact real. (And likewise, the eigenvectors are orthogonal.) This is, to say the least, very fortunate.

God's first tries were hardly ideal,
For complex worlds have no appeal.
So in the present edition,
He made things Hermitian,
And *this* world, it seems, is quite real. ♣

15.5 Appendix E: Diagonalizing matrices

This appendix is relevant to Section 9.3, which covers principal axes. The process of diagonalizing matrices (that is, finding the *eigenvectors* and *eigenvalues*, defined below) has countless applications in a wide variety of subjects. We'll describe the process here as it applies to principal axes and moments of inertia.

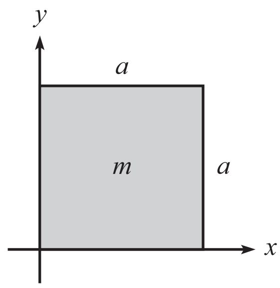

Figure 15.3

Let's find the three principal axes and moments of inertia for a square with side length a, mass m, and one corner at the origin. The square lies in the x-y plane, with sides along the x and y axes (see Fig. 15.3). We'll choose the given x, y, and z axes as our initial basis axes. Using eq. (9.8), you can show that the matrix $\mathbf{I}$ (with respect to this initial basis) is

$$\mathbf{I} = \rho \begin{pmatrix} \int y^2 & -\int xy & 0 \\ -\int xy & \int x^2 & 0 \\ 0 & 0 & \int (x^2+y^2) \end{pmatrix} = ma^2 \begin{pmatrix} 1/3 & -1/4 & 0 \\ -1/4 & 1/3 & 0 \\ 0 & 0 & 2/3 \end{pmatrix}, \qquad (15.77)$$

where ρ is the mass per unit area, so that $a^2\rho = m$. We have used the fact that $z = 0$, and we have not bothered to write the $dx\,dy$ in the integrals.

Our goal is to find the basis in which $\mathbf{I}$ is diagonal. That is, we want to find three solutions[7] for $\mathbf{u}$ (and I) for the equation $\mathbf{I}\mathbf{u} = I\mathbf{u}$. Letting $I \equiv \lambda ma^2$ to make things look a little cleaner, and using the above explicit form of $\mathbf{I}$, the equation $(\mathbf{I} - I)\mathbf{u} = 0$ becomes

$$ma^2 \begin{pmatrix} 1/3-\lambda & -1/4 & 0 \\ -1/4 & 1/3-\lambda & 0 \\ 0 & 0 & 2/3-\lambda \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \qquad (15.78)$$

In order for there to be a nonzero solution for the components u_x, u_y, u_z, the determinant of this matrix must be zero (see Footnote 6). The resulting cubic equation for λ is easy to solve, because the determinant is $[(1/3-\lambda)^2 - (1/4)^2](2/3-\lambda) = 0$. The solutions are $\lambda = 1/3 \pm 1/4$, and $\lambda = 2/3$. So our three principal moments, $I \equiv \lambda ma^2$, are

$$I_1 = \frac{7}{12}ma^2, \qquad I_2 = \frac{1}{12}ma^2, \qquad I_3 = \frac{2}{3}ma^2. \qquad (15.79)$$

These are the *eigenvalues* of $\mathbf{I}$.

What are the vectors, $\mathbf{u}_1$, $\mathbf{u_2}$, and $\mathbf{u}_3$, associated with each of these I's? Plugging $\lambda = 7/12$ into eq. (15.78) gives the three equations (one for each component), $-u_x - u_y = 0$, $-u_x - u_y = 0$, and $u_z = 0$. These are redundant equations (that was the point of setting the determinant equal to zero). So $u_x = -u_y$, and $u_z = 0$. The vector may therefore be written as $\mathbf{u}_1 = (c, -c, 0)$, where c is any constant.[8] If we want a normalized vector, then $c = 1/\sqrt{2}$. In a similar manner, plugging $\lambda = 1/12$ into eq. (15.78) gives $\mathbf{u_2} = (c, c, 0)$. And finally, plugging $\lambda = 1/3$ into eq. (15.78) gives $\mathbf{u}_3 = (0, 0, c)$, as claimed in the above footnote. Our three orthonormal principal axes corresponding to the moments in eq. (15.79) are therefore

$$\hat{\boldsymbol{\omega}}_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right), \qquad \hat{\boldsymbol{\omega}}_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \qquad \hat{\boldsymbol{\omega}}_3 = (0, 0, 1). \qquad (15.80)$$

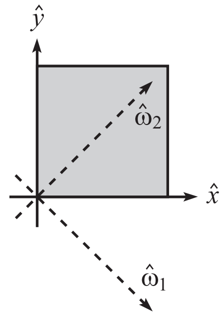

Figure 15.4

These are the *eigenvectors* of $\mathbf{I}$. These axes are shown in Fig. 15.4. In the new basis of the

[7]One obvious solution is $\mathbf{u} = \hat{\mathbf{z}}$, because $\mathbf{I}\hat{\mathbf{z}} = (2/3)ma^2\hat{\mathbf{z}}$. From the orthogonality result of Theorem 9.4, we know that the other two vectors must lie in the x-y plane. So we could quickly reduce this problem to a two-dimensional one, but let's forge ahead with the general method.

[8]We can solve for $\mathbf{u}$ only up to an overall constant, because if $\mathbf{I}\mathbf{u} = I\mathbf{u}$ is true for a certain $\mathbf{u}$, then it is also true that $\mathbf{I}(c\mathbf{u}) = I(c\mathbf{u})$, where c is any constant.

principal axes, the matrix $\mathbf{I}$ takes the form,

$$\mathbf{I} = ma^2 \begin{pmatrix} 7/12 & 0 & 0 \\ 0 & 1/12 & 0 \\ 0 & 0 & 2/3 \end{pmatrix}. \tag{15.81}$$

In other words, we have "diagonalized" the matrix. The basic idea is that from now on we should use the principal axes as our basis vectors. We can forget that we ever had anything to do with the original x, y, and z axes.

REMARKS:

1. $I_1 + I_2 = I_3$, as the perpendicular-axis theorem demands.
2. I_2 is the moment around one diagonal through the center of the square, which of course equals the moment around the other diagonal through the center. But the latter is related to I_1 by the parallel-axis theorem. And indeed, $I_1 = I_2 + m(a/\sqrt{2}\,)^2$.
3. Any axis through the center of the square, in the plane of the square, has the same moment (by Theorem 9.5 or 9.6). So I_2 equals the moment around an axis through the center, parallel to a side. But this is the same as the moment of a stick of length a around its center (the extent of the square in the direction of the axis is irrelevant). Hence the factor of $1/12$ in I_2. ♣

15.6 Appendix F: Qualitative relativity questions

1. Is there such a thing as a perfectly rigid body?

   Answer: No. Since information can move no faster than the speed of light, it takes time for the atoms in the body to communicate with each other. If you push on one end of a rod, then the other end won't move right away.

2. How do you synchronize two clocks that are at rest with respect to each other?

   Answer: One way is to put a light source midway between the two clocks and send out signals, and then set the clocks to a certain value when the signals hit them. Another way is to put a watch right next to one of the clocks and synchronize it with this clock, and then move the watch very slowly over to the other clock and synchronize that clock with it. Any time-dilation effects can be made arbitrarily small by moving the watch sufficiently slowly, because the time-dilation effect is second order in v.

3. Moving clocks run slow. Does this result have anything to do with the time it takes light to travel from the clock to your eye?

   Answer: No. When we talk about how fast a clock is running in a given frame, we are referring to what the clock actually reads in that frame. It will certainly take time for the light from the clock to reach an observer's eye, but it is understood that the observer subtracts off this transit time in order to calculate the time at which the clock actually shows a particular reading. Likewise, other relativistic effects, such as length contraction and the loss of simultaneity, have nothing to do with the time it takes light to reach your eye. They deal only with what really *is*, in your frame. One way to avoid the complication of the travel time of light is to use the lattice of clocks and meter sticks described at the end of Section 11.3.3.

4. Does time dilation depend on whether a clock is moving across your vision or directly away from you?

   Answer: No. A moving clock runs slow, no matter which way it is moving. This is clearer if you think in terms of the lattice of clocks and meter sticks from Section 11.3.3. If you imagine a million people standing at the points of the lattice, then they all observe the clock running slow. Time dilation is an effect that depends on the *frame* and the speed of a clock with respect to it. It doesn't matter where you are in the frame (as long as you're at rest in it).

5. Does special-relativistic time dilation depend on the acceleration of the moving clock?

   Answer: No. The time-dilation factor is $\gamma = 1/\sqrt{1 - v^2/c^2}$, which doesn't depend on a. The only relevant quantity is the v at a given instant. It doesn't matter if v is changing. But if *you* are accelerating, then you can't naively apply the results of special relativity. (To do things correctly, it is perhaps easiest to think in terms of general relativity. But GR is actually not required; see Chapter 14 for a discussion of this.) But as long as you represent an inertial frame, then the clock you are viewing can undergo whatever motion it wants, and you will observe it running slow by the simple factor of γ.

6. Someone says, "A stick that is length-contracted isn't *really* shorter, it just *looks* shorter." How do you respond?

Answer: The stick really *is* shorter in your frame. Length contraction has nothing to do with how things look. It has to do with where the ends of the stick are at simultaneous times in your frame. (This is, after all, how you measure the length of something.) At a given instant in time in your frame, the distance between the ends of the stick is indeed less than the proper length of the stick.

7. Consider a stick that moves in the direction in which it points. Does its length contraction depend on whether this direction is across your vision or directly away from you?

   Answer: No. The stick is length-contracted in both cases. Of course, if you look at the stick in the latter case, then all you see is the end, which is just a dot. But the stick is indeed shorter in your reference frame. As in Question 4 above, length contraction depends on the frame, not where you are in it.

8. A mirror moves toward you at speed v. You shine a light toward it and the light beam bounces back at you. What is the speed of the reflected beam?

   Answer: The speed is c, as always. You will observe the light having a higher frequency, due to the Doppler effect. But the speed is still c.

9. In relativity, the order of two events in one frame may be reversed in another frame. Does this imply that there exists a frame in which I get off a bus before I get on it?

   Answer: No. The order of two events can be reversed in another frame only if the events are spacelike separated. That is, if $\Delta x > c\Delta t$ (in other words, the events are too far apart for even light to get from one to the other). The two relevant events here (getting on the bus, and getting off the bus) are not spacelike separated, because the bus travels at a speed less than c, of course. They are timelike separated. Therefore, in all frames it is the case that I get off the bus after I get on it.

   There would be causality problems if there existed a frame in which I got off the bus before I got on it. If I break my ankle getting off a bus, then I wouldn't be able to make the mad dash that I made to catch the bus in the first place, in which case I wouldn't have the opportunity to break my ankle getting off the bus, in which case I could have made the mad dash to catch the bus and get on, and, well, you get the idea.

10. You are in a spaceship sailing along in outer space. Is there any way you can measure your speed without looking outside?

    Answer: There are two points to be made here. First, the question is meaningless, because absolute speed doesn't exist. The spaceship doesn't have a speed; it only has a speed relative to something else. Second, even if the question asked for the speed with respect to, say, a piece of stellar dust, the answer would be "no." Uniform speed is not measurable from within the spaceship. Acceleration, on the other hand, is measurable (assuming there is no gravity around to confuse it with).

11. If you move at the speed of light, what shape does the universe take in your frame?

    Answer: The question is meaningless, because it's impossible for you to move at the speed of light. A meaningful question to ask is: What shape does the universe take if you move at a speed very close to c? The answer is that in your frame everything would be squashed along the direction of your motion, due to length contraction. Any given region of the universe would be squashed down to a pancake.

12. Two objects fly toward you, one from the east with speed u, and the other from the west with speed v. Is it correct that their relative speed, as measured by you, is $u + v$? Or should you use the velocity-addition formula, $V = (u + v)/(1 + uv/c^2)$? Is it possible for their relative speed, as measured by you, to exceed c?

    Answer: Yes, no, yes, to the three questions. It is legal to simply add the two speeds to obtain $u+v$. There is no need to use the velocity-addition formula, because both speeds here are measured with respect to the *same thing*, namely you. It's perfectly legal for the result to be greater than c, but it must be less than (or equal to, for photons) $2c$.

    You need to use the velocity-addition formula when, for example, you are given the speed of a ball with respect to a train, and also the speed of the train with respect to the ground, and your goal is to find the speed of the ball with respect to the ground. The point is that now the two given speeds are measured with respect to *different* things, namely the train and the ground.

13. Two clocks at the ends of a train are synchronized with respect to the train. If the train moves past you, which clock shows the higher time?

    Answer: The rear clock shows the higher time. It shows Lv/c^2 more than the front clock, where L is the proper length of the train.

14. A train moves at speed $4c/5$. A clock is thrown from the back of the train to the front. As measured in the ground frame, the time of flight is 1 second. Is the following reasoning correct? "The γ factor between the train and the ground is $\gamma = 1/\sqrt{1-(4/5)^2} = 5/3$. And since moving clocks run slow, the time elapsed on the clock during the flight is $3/5$ of a second."

    Answer: No. It is incorrect, because the time-dilation result holds only for two events that happen at the *same place* in the relevant reference frame (the train, here). The clock moves with respect to the train, so the above reasoning is invalid.

    Another way of seeing why it must be incorrect is the following. A certainly valid way to calculate the clock's elapsed time is to find the speed of the clock with respect to the ground (more information would have to be given to determine this), and to then apply time dilation with the associated γ factor to arrive at the answer of $1/\gamma$. Since the clock's v is definitely not $4c/5$, the correct answer is definitely not $3/5$ s.

15. Person A chases person B. As measured in the ground frame, they have speeds $4c/5$ and $3c/5$, respectively. If they start a distance L apart (as measured in the ground frame), how much time will it take (as measured in the ground frame) for A to catch B?

    Answer: As measured in the ground frame, the relative speed is $4c/5 - 3c/5 = c/5$. Person A must close the initial gap of L, so the time it takes is $L/(c/5) = 5L/c$. There is no need to use any fancy velocity-addition or length-contraction formulas, because all quantities in this problem are measured with respect to the *same* frame. So it quickly reduces to a simple "(rate)(time) = (distance)" problem.

16. Is the "the speed of light is the same in all inertial frames" postulate really necessary? That is, is it not already implied by the "the laws of physics are the same in all inertial frames" postulate?

    Answer: Yes, it is necessary. The speed-of-light postulate is definitely not implied by the laws-of-physics postulate. The latter doesn't imply that baseballs have the same speed in all inertial frames, so it likewise doesn't imply it for light.

It turns out that nearly all the results in special relativity can be deduced by using only the laws-of-physics postulate. What you can find (with some work) is that there is some limiting speed, which may or may not be infinite (see Section 11.10). But you still have to say whether this speed is finite or infinite. The speed-of-light postulate does the trick.

17. Imagine closing a very large pair of scissors. It is quite possible for the point of intersection of the blades to move faster than the speed of light. Does this violate anything in relativity?

    Answer: No. If the angle between the blades is small enough, then the tips of the blades (and all the other atoms in the scissors) can move at a speed well below c, while the intersection point moves faster than c. But this doesn't violate anything in relativity. The intersection point is not an actual object, so there is nothing wrong with it moving faster than c.

    You might be worried that this result allows you to send a signal down the scissors at a speed faster than c. However, since there is no such thing as a rigid body, it is impossible to get the far end of the scissors to move right away, when you apply a force at the handle. The scissors would have to already be moving, in which case the motion is independent of any decision you make at the handle to change the motion of the blades.

18. Two twins travel away from each other at relativistic speed. The time-dilation result says that each twin sees the other twin's clock running slow, so each says the other has aged less. How would you reply to someone who asks, "But which twin really *is* younger?"

    Answer: It makes no sense to ask which twin really is younger, because the two twins aren't in the same reference frame; they are using different coordinates to measure time. It's as silly as having two people run away from each other into the distance (so that each person sees the other become small), and then asking: Who is really smaller?

19. A particular event has coordinates (x, t) in one frame. How do you use a Lorentz transformation to find the coordinates of this event in another frame?

    Answer: You don't. Lorentz transformations have nothing to do with single events. They deal only with *pairs* of events and the *separation* between them. As far as a single event goes, its coordinates in another frame can be anything you want, simply by defining your origin to be wherever and whenever you please. But for pairs of events, their separation is a well-defined thing, independent of any definition of origin. It is therefore a meaningful question to ask how the separations in two different frames are related, and the Lorentz transformations answer this question.

20. When using the Lorentz transformations, how do you tell which frame is the moving "primed" frame?

    Answer: You don't. There is no preferred frame, so it doesn't make sense to ask which frame is moving. We used the "primed" notation in the derivation in Section 11.4.1 for ease of notation, but don't take it to imply that there is a preferred frame S, and a less fundamental frame S'. In general, a better notation is to use subscripts that describe the two frames, such as "g" for ground and "t" for train. For example, if you know the values of Δt_t and Δx_t on a train (which we'll assume is moving in the positive x direction with respect to the ground), and if you want to find the values of Δt_g and Δx_g on the ground, then you can write down:

$$\begin{aligned} \Delta x_g &= \gamma(\Delta x_t + v\,\Delta t_t), \\ \Delta t_g &= \gamma(\Delta t_t + v\,\Delta x_t/c^2). \end{aligned} \tag{15.82}$$

The sign is a "+" because the frame associated with the left side of the equation (the ground) sees the frame associated with the right side (the train) moving to the right. If instead you know the intervals on the ground and you want to find them on the train, then you just need to switch the subscripts "g" and "t" and change the sign to "$-$", by the reasoning in the previous sentence.

21. The momentum of an object with mass m and speed v is $p = \gamma mv$. "A photon has zero mass, so it should have zero momentum." Correct or incorrect?

    Answer: Incorrect. True, m is zero, but the γ factor is infinite because $v = c$. Infinity times zero is undefined. A photon does indeed have momentum, and it equals E/c (which happens to equal $h\nu/c$, where ν is the frequency of the light).

22. It is not necessary to postulate the impossibility of accelerating an object to speed c. It follows as a consequence of the relativistic form of energy. Explain.

    Answer: $E = \gamma mc^2$, so if $v = c$ then $\gamma = \infty$, and the object must have an infinite amount of energy (unless $m = 0$, as for a photon). All the energy in the universe, let alone all the king's horses and all the king's men, can't accelerate something to speed c.

15.7 Appendix G: Derivations of the Lv/c^2 result

In Section 11.3.1, we showed that if a train with proper length L moves at speed v with respect to the ground, then in the ground frame the rear clock reads Lv/c^2 more than the front clock (assuming that the clocks are synchronized in the train frame). There are various other ways to derive this result, so for the fun of it I've listed here all the derivations I can think of. The explanations are terse, but I refer you to the specific problem or section in the text where things are discussed in more detail. Many of these derivations are slight variations on each other, so perhaps they shouldn't all count as separate ones, but here's my list:

1. **Light source on train**: Put a light source on a train, at distances $d_f = L(c-v)/2c$ from the front and $d_b = L(c+v)/2c$ from the back. You can show that the photons hit the ends of the train simultaneously in the ground frame. But they hit the ends at different times in the train frame; the difference in the readings on clocks at the ends when the photons run into them is $(d_b - d_f)/c = Lv/c^2$. Therefore, at a given instant in the ground frame (for example, the moment when clocks at the ends are simultaneously illuminated by the photons), a person on the ground sees the rear clock read Lv/c^2 more than the front clock. (See the example in Section 11.3.1.)

2. **Lorentz transformation**: The second of eqs. (11.17) is $\Delta t_g = \gamma(\Delta t_t + v\,\Delta x_t/c^2)$, where the subscripts refer to the ground and train frames. If two events (for example, two clocks flashing their times) located at the ends of the train are simultaneous in the ground frame, then we have $\Delta t_g = 0$. And $\Delta x_t = L$, of course. The above Lorentz transformation therefore gives $\Delta t_t = -Lv/c^2$. The minus sign here means that the event with the larger x_t value has the smaller t_t value. In other words, the front clock reads Lv/c^2 less time than the rear clock, at a given instant in the ground frame.

3. **Invariant interval**: This is actually just a partial derivation, because it determines only the magnitude of the Lv/c^2 result, and not the sign. The invariant interval says that $c^2\Delta t_g^2 - \Delta x_g^2 = c^2\Delta t_t^2 - \Delta x_t^2$, where the subscripts refer to the ground and train frames. If two events (for example, two clocks flashing their times) located at the ends of the train are simultaneous in the ground frame, then we have $\Delta t_g = 0$. And $\Delta x_t = L$, of course. And we also know from length contraction that $\Delta x_g = L/\gamma$. The invariant interval then gives $c^2(0)^2 - (L/\gamma)^2 = c^2\Delta t_t^2 - L^2$, which yields $c^2\Delta t_t^2 = L^2(1 - 1/\gamma^2) \Longrightarrow c^2\Delta t_t^2 = L^2v^2/c^2 \Longrightarrow \Delta t_t = \pm Lv/c^2$. As mentioned above, the sign isn't determined by this method.

4. **Minkowski diagram**: The task of Exercise 11.63 is to use a Minkowski diagram to derive the Lv/c^2 result. The basic goal is to determine how many ct' units fit in the segment BC in Figure 11.27, and also how many ct units fit in the segment BE in Figure 11.28.

5. **Walking slowly on a train**: In Exercise 11.58, a person walks very slowly at speed u from the back of a train of proper length L to the front. In the frame of the train, the time-dilation effect is second order in u/c and therefore negligible (because the total time is only first order in $1/u$). But in the frame of the ground, the time-dilation effect is (as you can show) *first* order in u/c and therefore has a nonzero effect; an observer on the ground sees the person's clock advance by less than a clock that is fixed on the train. Now, the person's clock agrees with clocks at the rear and front at the start and finish, because of the negligible time dilation in the train frame. Therefore, since less time elapses on the person's clock than on the front clock, the person's clock must have started out reading more time than the front clock. This then implies that the

rear clock must show more time than the front clock. A quantitative analysis shows that this excess time is in fact Lv/c^2.

6. **Consistency arguments**: There are many setups (a few examples are Problems 11.2, 11.3, 11.8, and Exercise 11.35) where the Lv/c^2 result is an ingredient in explaining a result. Without it, you would encounter a contradiction, such as two different frames giving two different answers to a frame-independent question. So if you wanted to, you could work backwards (under the assumption that everything is consistent in relativity) and let the rear-clock-ahead effect be some unknown time T (which might be zero, for all you know), and then solve for the T that makes everything consistent. You would arrive at $T = Lv/c^2$.

7. **Gravitational time dilation**: The task of Exercise 14.13 is to derive the Lv/c^2 result by making use of the fact that Lv/c^2 looks a lot like the gh/c^2 term in the GR time-dilation result. If you stand on the ground near the front of a train of length L and then accelerate toward the back with acceleration g, you will see a clock at the back running faster by a factor $(1 + gL/c^2)$, which will cause it to read $(gL/c^2)t = Lv/c^2$ more than a clock at the front. (Assume that you accelerate for a short period of time, so that the distance in the gL/c^2 term remains essentially L.)

8. **Accelerating rocket**: The task of Problem 14.6 is to show that if a rocket with proper length L accelerates at g and reaches a speed v, then in the ground frame the readings on the front and rear clocks are related by $t_f = t_b(1 + gL/c^2) - Lv/c^2$. In other words, the front clock reads $t_b(1 + gL/c^2) - Lv/c^2$ simultaneously with the rear clock reading t_b, in the ground frame. But in the rocket frame, gravitational time dilation tells us that the front clock reads $t_b(1 + gL/c^2)$ simultaneously with the rear clock reading t_b. The difference in clock readings (front minus rear) is therefore smaller in the ground frame than in the rocket frame, by an amount equal to Lv/c^2. This is the desired result.

15.8 Appendix H: Resolutions to the twin paradox

The twin paradox appeared in Chapters 11 and 14, both in the text and in various problems. To summarize, the twin paradox deals with twin A who stays on the earth,[9] and twin B who travels quickly to a distant star and back. When they meet up again, they discover that B is younger. This is true because A can use the standard special-relativistic time-dilation result to say that B's clock runs slow by a factor γ.

The "paradox" arises from the fact that the situation seems symmetrical. That is, it seems as though each twin should be able to consider herself to be at rest, so that she sees the other twin's clock running slow. So why does B turn out to be younger? The resolution to the paradox is that the setup is in fact *not* symmetrical, because B must turn around and thus undergo acceleration. She is therefore not always in an inertial frame, so she cannot always apply the simple special-relativistic time-dilation result.

While the above reasoning is sufficient to get rid of the paradox, it isn't quite complete, because (a) it doesn't explain how the result from B's point of view quantitatively agrees with the result from A's point of view, and (b) the paradox can actually be formulated without any mention of acceleration, in which case slightly different reasoning applies.

Below is a list of all the complete resolutions I can think of. The descriptions are terse, but I refer you to the specific problem or section in the text where things are discussed in more detail. As with the Lv/c^2 derivations in Appendix G, many of these resolutions are slight variations on each other, so perhaps they shouldn't all count as separate ones, but here's my list:

1. **Rear-clock-ahead effect**: Let the distant star be labeled as C. Then on the outward part of the journey, B sees C's clock ahead of A's by Lv/c^2, because C is the rear clock in the universe as the universe flies by. But after B turns around, A becomes the rear clock and is therefore now ahead of C. This means that A's clock must jump forward very quickly, from B's point of view. (See Section 11.3.1 and Problem 11.2.)

2. **Looking out the portholes**: Imagine many clocks lined up between the earth and the star, all synchronized in the earth-star frame. And imagine looking out the portholes of the spaceship and making a movie of the clocks as you fly past them. Although you see each individual clock running slow, you see the "effective" clock in the movie (which is really many successive clocks) running fast. This effect is just a series of small applications (see Problem 11.2) of the rear-clock-ahead effect mentioned above.

3. **Minkowski diagram**: Draw a Minkowski diagram with the axes in A's frame perpendicular. Then the lines of simultaneity (that is, the successive x axes) in B's frame are titled in different directions for the outward and inward parts of the journey. The change in the tilt at the turnaround causes a large amount of time to advance on A's clock, as measured in B's frame. (See Section 11.7 and Figure 11.68.)

4. **General Relativistic turnaround effect**: The acceleration that B feels when she turns around may equivalently be thought of as a gravitational field. Twin A on the earth is high up in the gravitational field, so B sees A's clock run very fast during the turnaround. This causes A's clock to show more time in the end. (See Problem 14.9.)

5. **Doppler effect**: By equating the total number of signals one twin sends out with the total number of signals the other twin receives, we can relate the total times on their clocks. (See Exercise 11.67.)

[9] We should actually have A floating in space, to avoid any GR time-dilation effects from the earth's gravity. But if B travels quickly enough, the SR effects will dominate the gravitational ones.

15.9 Appendix I: Lorentz transformations

In this Appendix, we will give an alternate derivation of the Lorentz transformations in eq. (11.17). The goal here is to derive them from scratch, using only the two postulates of relativity. We will *not* use any of the results derived in Section 11.3. Our strategy will be to use the relativity postulate ("all inertial frames are equivalent") to figure out as much as we can, and to then invoke the speed-of-light postulate at the end. The main reason for doing things in this order is that it will allow us to derive a very interesting result in Section 11.10.

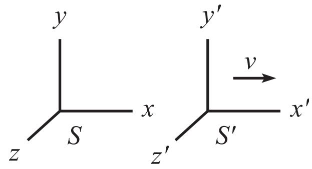

Figure 15.5

As in Section 11.4, consider a coordinate system S' moving relative to another system S (see Fig. 15.5). Let the constant relative speed between the frames be v. Let the corresponding axes of S and S' point in the same direction, and let the origin of S' move along the x axis of S, in the positive direction. As in Section 11.4, we want to find the constants, A, B, C, and D, in the relations,

$$\begin{aligned}\Delta x &= A\,\Delta x' + B\,\Delta t',\\ \Delta t &= C\,\Delta t' + D\,\Delta x'.\end{aligned} \qquad (15.83)$$

The four constants will end up depending on v (which is constant, given the two inertial frames). Since we have four unknowns, we need four facts. The facts we have at our disposal (using only the two postulates of relativity) are the following.

1. The physical setup: S' travels with velocity v with respect to S.

2. The principle of relativity: S should see things in S' in exactly the same way as S' sees things in S (except perhaps for a minus sign in some relative positions, but this just depends on our arbitrary choice of directional signs for the axes).

3. The speed-of-light postulate: A light pulse with speed c in S' also has speed c in S.

The second statement here contains two independent bits of information. (It contains at least two, because we will indeed be able to solve for our four unknowns. And it contains no more than two, because then our four unknowns would be over-constrained.) The two bits that are used depend on personal preference. Three that are commonly used are: (a) the relative speed looks the same from either frame, (b) time dilation (if any) looks the same from either frame, and (c) length contraction (if any) looks the same from either frame. It is also common to recast the second statement in the form: The Lorentz transformations are the same as their inverse transformations (up to a possible minus sign). We'll choose to work with (a) and (b). Our four independent facts are then:

1. S' travels with velocity v with respect to S.

2. S travels with velocity $-v$ with respect to S'. The minus sign here is due to the convention that we picked the positive x axes of the two frames to point in the same direction.

3. Time dilation (if any) looks the same from either frame.

4. A light pulse with speed c in S' also has speed c in S.

Let's see what these imply, in the above order.[10]

[10]In what follows, we could obtain the final result a little quicker if we invoked the speed-of-light fact prior the time-dilation one. But we'll do things in the above order so that we can easily carry over the results of this appendix to the discussion in Section 11.10.

- (1) says that a given point in S' moves with velocity v with respect to S. Letting $x' = 0$ (which is understood to be $\Delta x' = 0$, but we'll drop the Δ's from here on) in eqs. (15.83) and dividing them gives $x/t = B/C$. This must equal v. Therefore, $B = vC$, and the transformations become

$$\begin{aligned} x &= Ax' + vCt', \\ t &= Ct' + Dx'. \end{aligned} \tag{15.84}$$

- (2) says that a given point in S moves moves velocity $-v$ with respect to S'. Letting $x = 0$ in the first of eqs. (15.84) gives $x'/t' = -vC/A$. This must equal $-v$. Therefore, $C = A$, and the transformations become

$$\begin{aligned} x &= Ax' + vAt', \\ t &= At' + Dx'. \end{aligned} \tag{15.85}$$

Note that these are consistent with the Galilean transformations, which have $A = 1$ and $D = 0$.

- (3) can be used in the following way. How fast does a person in S see a clock in S' tick? (The clock is assumed to be at rest with respect to S'.) Let our two events be two successive ticks of the clock. Then $x' = 0$, and the second of eqs. (15.85) gives

$$t = At'. \tag{15.86}$$

In other words, one second on S''s clock takes a time of A seconds in S's frame.

Consider the analogous situation from S''s point of view. How fast does a person in S' see a clock in S tick? (The clock is now assumed to be at rest with respect to S, in order to create the analogous setup. This is important.) If we invert eqs. (15.85) to solve for x' and t' in terms of x and t, we find

$$\begin{aligned} x' &= \frac{x - vt}{A - Dv}, \\ t' &= \frac{At - Dx}{A(A - Dv)}. \end{aligned} \tag{15.87}$$

Two successive ticks of the clock in S satisfy $x = 0$, so the second of eqs. (15.87) gives

$$t' = \frac{t}{A - Dv}. \tag{15.88}$$

In other words, one second on S's clock takes a time of $1/(A - Dv)$ seconds in S''s frame.

Both eqs. (15.86) and (15.88) apply to the same situation (someone looking at a clock flying by). Therefore, the factors on the right-hand sides must be equal, that is,

$$A = \frac{1}{A - Dv} \qquad \Longrightarrow \qquad D = \frac{1}{v}\left(A - \frac{1}{A}\right). \tag{15.89}$$

Our transformations in eq. (15.85) therefore take the form

$$\begin{aligned} x &= A(x' + vt'), \\ t &= A\left(t' + \frac{1}{v}\Big(1 - \frac{1}{A^2}\Big)x'\right). \end{aligned} \tag{15.90}$$

These are consistent with the Galilean transformations, which have $A = 1$.

- (4) may now be used to say that if $x' = ct'$, then $x = ct$. In other words, if $x' = ct'$, then

$$c = \frac{x}{t} = \frac{A\big((ct') + vt'\big)}{A\left(t' + \frac{1}{v}\left(1 - \frac{1}{A^2}\right)(ct')\right)} = \frac{c+v}{1 + \frac{c}{v}\left(1 - \frac{1}{A^2}\right)} \,. \tag{15.91}$$

Solving for A gives

$$A = \frac{1}{\sqrt{1 - v^2/c^2}} \,. \tag{15.92}$$

We have chosen the positive square root so that the positive x and x' axes point in the same direction. The transformations are now no longer consistent with the Galilean transformations, because c is not infinite, which means that A is not 1.

The constant A is commonly denoted by γ, so we may finally write our Lorentz transformations, eqs. (15.90), in the form,

$$\begin{aligned} x &= \gamma(x' + vt'), \\ t &= \gamma(t' + vx'/c^2), \end{aligned} \tag{15.93}$$

where

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}} \,, \tag{15.94}$$

in agreement with eq. (11.17).

15.10 Appendix J: Physical constants and data

Earth

Mass	$M_E = 5.98 \cdot 10^{24}$ kg
Mean radius	$R_E = 6.37 \cdot 10^6$ m
Mean density	5.52 g/cm^3
Surface acceleration	$g = 9.81$ m/s^2
Mean distance from sun	$1.5 \cdot 10^{11}$ m
Orbital speed	29.8 km/s
Period of rotation	23 h 56 min 4 s $= 8.6164 \cdot 10^4$ s
Period of orbit	365 days 6 h $= 3.16 \cdot 10^7$ s

Moon

Mass	$M_L = 7.35 \cdot 10^{22}$ kg
Radius	$R_L = 1.74 \cdot 10^6$ m
Mean density	3.34 g/cm^3
Surface acceleration	1.62 m/s$^2 \approx g/6$
Mean distance from earth	$3.84 \cdot 10^8$ m
Orbital speed	1.0 km/s
Period of rotation	27.3 days $= 2.36 \cdot 10^6$ s
Period of orbit	27.3 days $= 2.36 \cdot 10^6$ s

Sun

Mass	$M_S = 1.99 \cdot 10^{30}$ kg
Radius	$R_S = 6.96 \cdot 10^8$ m
Surface acceleration	274 m/s$^2 \approx 28g$

Fundamental constants

Speed of light	$c = 2.998 \cdot 10^8$ m/s
Gravitational constant	$G = 6.673 \cdot 10^{-11}$ N m^2/kg^2
Planck's constant	$h = 6.63 \cdot 10^{-34}$ J s
	$\hbar \equiv h/2\pi = 1.05 \cdot 10^{-34}$ J s
Electron charge	$-e = -1.602 \cdot 10^{-19}$ C
Electron mass	$m_e = 9.11 \cdot 10^{-31}$ kg $= 0.511$ MeV$/c^2$
Proton mass	$m_p = 1.673 \cdot 10^{-27}$ kg $= 938.3$ MeV$/c^2$
Neutron mass	$m_n = 1.675 \cdot 10^{-27}$ kg $= 939.6$ MeV$/c^2$