

Chapter 1

Problems

The Green-Eyed Dragons and Other Mathematical Monsters (Draft version, September 2018)
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TO THE READER: This book is available as both a paperback and an eBook. I have made the first chapter (the problems) available on the web, but it is possible (based on past experience) that a pirated version of the complete book will eventually appear on the web. In the event that you are reading such a version, I have a request:

If you don't find this book useful (in which case you probably would have returned it, if you had bought it), or if you do find it useful but aren't able to afford it, then no worries; carry on. However, if you do find it useful and are able to afford the Kindle eBook (priced below \$10), then please consider purchasing it (available on Amazon). If you don't already have the Kindle reading app for your computer, you can download it free from Amazon. I chose to self-publish this book so that I could keep the cost low. The resulting eBook price of under \$10 is less than a movie and a bag of popcorn, with the added bonus that the book lasts for more than two hours and has zero calories (if used properly!).

– David Morin

As mentioned in the preface (which you should be sure to read before tackling the problems), the most important advice for using this book is:

Don't look at the solutions too soon!

The problems in this book are in general extremely difficult and are designed to be brooded over for a significant amount of time. Most of them are classics that have been around for ages, and there are only so many such problems in the world. If you look at a solution too soon, the opportunity to solve the problem is gone, and it's never coming back. Don't waste it!

1.1 General

1. Green-eyed dragons

You visit a remote desert island inhabited by one hundred very friendly dragons, all of whom have green eyes. They haven't seen a human for many centuries and are very excited about your visit. They show you around their island and tell you all about their dragon way of life (dragons can talk, of course).

They seem to be quite normal, as far as dragons go, but then you find out something rather odd. They have a rule on the island that states that if a dragon ever finds out that he/she has green eyes, then at precisely midnight at the end of the day of this discovery, he/she must relinquish all dragon powers and transform into a long-tailed sparrow. However, there are no mirrors on the island, and the dragons never talk about eye color, so they have been living in blissful ignorance throughout the ages.

Upon your departure, all the dragons get together to see you off, and in a tearful farewell you thank them for being such hospitable dragons. You then decide to tell them something that they all already know (for each can see the colors of the eyes of all the other dragons): You tell them all that at least one of them has green eyes. Then you leave, not thinking of the consequences (if any). Assuming that the dragons are (of course) infallibly logical, what happens? If something interesting does happen, what exactly is the new information you gave the dragons?

2. Simpson's paradox

During the baseball season in a particular year, player A has a higher batting average than player B. In the following year, A again has a higher average than B. But to your great surprise when you calculate the batting averages over the combined span of the two years, you find that A's average is *lower* than B's! Explain, by giving a concrete example, how this is possible.

3. Verifying weights

- (a) You have a balance scale and wish to verify the weight of an item that can take on any integral value from 1 to 121. What is the minimum number of fixed weights (with known values) you need, in order to cover all 121 possibilities? What are the weights? ("Verify" here means making the scale be balanced, so that you can be certain of the given item's weight.)
- (b) Using n wisely-chosen fixed weights, what is the largest integer W for which you can verify all the integral weights less than or equal to W ? What fixed weights should you choose?

4. Counterfeit coin

- (a) You are given twelve coins, eleven of which have the same weight, and one of which has a weight different from the others (either heavier or lighter, you do not know). You have a balance scale. What is the minimum number of weighings required in order to guarantee that you can determine which coin has the different weight, and also whether it is heavier or lighter than the rest?

- (b) You are given N coins, $N - 1$ of which have the same weight, and one of which has a weight different from the others (either heavier or lighter, you do not know). You are allowed W weighings on a balance scale. What is the maximum value of N , as a function of W , for which you are guaranteed to be able to determine which coin has the different weight, and also whether it is heavy or light?

5. The game of Nim

Determine the best strategy for each player in the following two-player game. There are three piles, each of which contains some number of coins. Players alternate turns, each turn consisting of removing any (non-zero) number of coins from a single pile. The player who removes the last coin(s) wins.

6. Monochromatic triangle

- (a) Seventeen points, no three of which are collinear, are connected by all the possible lines between them ($\binom{17}{2} = 136$, in fact). Each line is colored either red, green, or blue (your choice for each line). Prove that within the resulting network of lines, there is at least one triangle all of whose sides are the same color.
- (b) Let $\lceil a \rceil$ denote the smallest integer greater than or equal to a . Let $\lceil n!e \rceil$ points, no three of which are collinear, be connected by all the possible lines between them. Each line is colored one of n colors (your choice for each). Prove that within the resulting network of lines, there is at least one triangle all of whose sides are the same color.

7. AM-GM inequality

Prove the Arithmetic-Mean Geometric-Mean inequality:

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}, \quad (1.1)$$

where the x_i are non-negative real numbers. As a hint, the inequality can be proved by induction in the following way. Let I_n represent the above inequality. Show that I_2 is true, and then show that I_n implies I_{2n} , and then finally show that I_n implies I_{n-1} .

8. Crawling ant

A rubber band with initial length L has one end attached to a wall. At $t = 0$, the other end is pulled away from the wall at constant speed V . (Assume that the rubber band stretches uniformly.) At the same time, an ant located at the end not attached to the wall begins to crawl toward the wall, with constant speed u relative to the band. Will the ant reach the wall? If so, how much time will it take?

1.2 Geometry

9. Apple core

You find an apple core of height h . What volume of apple was eaten? (In this problem, an apple is a perfect sphere, and the height of the core is the height of the cylindrical part of its boundary.)

10. Viewing the spokes

A wheel with radial spokes rolls (without slipping) on the ground. From off to the side, a stationary camera takes a picture of the wheel. If the exposure time is non-negligible, the spokes will in general appear blurred. At what locations in the picture do the spokes *not* appear blurred?

11. Painting a funnel

Consider the curve $y = 1/x$, from $x = 1$ to $x = \infty$. Rotate this curve around the x -axis to create a funnel-like surface of revolution, as shown in Fig. 1.1.

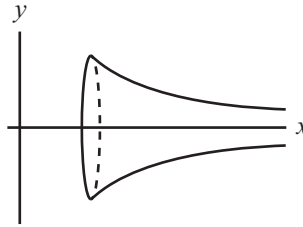


Figure 1.1

By slicing up the funnel into disks with radii $r = 1/x$ and thickness dx (and hence volume $(\pi r^2) dx$) stacked side by side, we see that the volume of the funnel is

$$V = \int_1^{\infty} \frac{\pi}{x^2} dx = -\frac{\pi}{x} \Big|_1^{\infty} = \pi, \quad (1.2)$$

which is finite. The surface area, however, involves the circumferential area of the disks, which is $(2\pi r) dx$ multiplied by a $\sqrt{1 + y'^2}$ factor accounting for the tilt of the area. The surface area of the funnel is therefore

$$A = \int_1^{\infty} \frac{2\pi\sqrt{1 + y'^2}}{x} dx > \int_1^{\infty} \frac{2\pi}{x} dx, \quad (1.3)$$

which is infinite because the integral of $1/x$, which is $\ln x$, diverges. (So the square-root factor turns out to be irrelevant for the present purposes.) Since the volume is finite but the area is infinite, it therefore appears that you can fill up the funnel with paint but you can't paint it. However, we then have a problem, because filling up the funnel with paint implies that you can certainly paint the *inside* surface. But the inside surface is the same as the outside surface, because the funnel wall has no thickness. So we should be able to paint the outside surface too. What's going on here? Can you paint the funnel or not?

12. **Tower of circles**

Consider N circles stacked on top of each other inside an isosceles triangle, as shown in Fig. 1.2 for the case of $N = 4$. Let A_C be the sum of the areas of the N circles, and let A_T be the area of the triangle. In terms of N , what should the vertex angle α be so that the ratio A_C/A_T is maximized? Assume that N is large, and ignore terms in your answer that are of subleading order in N . (Eq. (1.5) in Problem 53 might be helpful.)

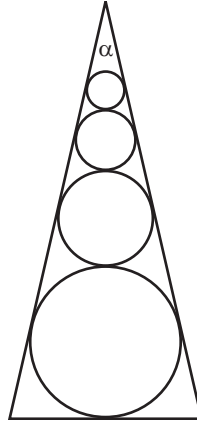


Figure 1.2

13. **Ladder envelope**

A ladder initially stands vertically against a wall. The bottom end is given a little sideways kick, causing the ladder to slide down. (The floor is slippery, so the ladder does in fact slide.) Assume that the bottom end is constrained to keep contact with the ground, and that the top end is constrained to keep contact with the wall. Describe the envelope of the ladder's positions.

14. **Equal segments**

You are given a line segment, an (infinite) line parallel to it, and a straightedge. Show how to divide the segment into N equal segments, for any integer N . (With a straightedge, you are allowed only to draw straight lines and create intersections. You are not allowed to mark off distances on the straightedge.)

15. **Collinear points**

You are given a finite number of points in space with the property that any line that contains two of the points contains three of them. Prove that all the points must lie on a common line.

16. **Attracting bugs**

N bugs are initially located at the vertices of a regular N -gon whose sides have length ℓ . At a given moment, they all begin walking with equal speeds in the clockwise direction, directly toward the adjacent bug. They continue to walk

directly toward the adjacent bug (whose position is continually changing, of course), until they finally all meet at the center of the original N -gon. What is the total distance each bug walks? How many times does each bug spiral around the center?

17. **Find the foci**

Using a straightedge and compass, construct (1) the foci of a given ellipse, (2) the focus of a given parabola, and (3) the foci of a given hyperbola.

18. **Construct the center**

Construct the center of a given circle, using only a compass. With a compass, you are allowed to mark points with the needle, and to draw arcs of circles (which may intersect at new points)

19. **Find the angles**

Quadrilateral $ABCD$ in Fig. 1.3 has angles $\angle BAC = 80^\circ$, $\angle CAD = 20^\circ$, $\angle BDA = 50^\circ$, and $\angle CDB = 50^\circ$. Find angles $\angle BCA$ and $\angle CBD$.

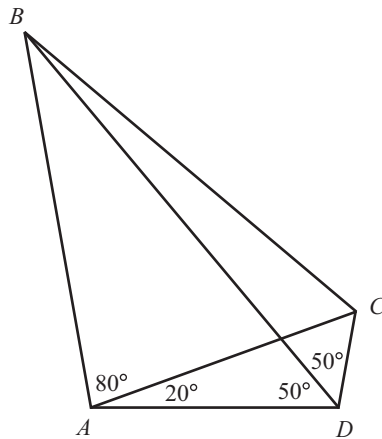


Figure 1.3

20. **Rectangle in a circle**

Given a cyclic quadrilateral $ABCD$, draw the diagonals AC and BD . Prove that the centers of the inscribed circles of triangles ABC , BCD , CDA , and DAB are the vertices of a rectangle, as shown in Fig. 1.4.

21. **Product of lengths**

Inscribe a regular N -gon in a circle of radius 1. Draw the $N - 1$ segments connecting a given vertex to the $N - 1$ other vertices. Show that the product of the lengths of these $N - 1$ segments equals N . Fig. 1.5 shows the case where $N = 10$; the product of the lengths of the nine segments is 10.

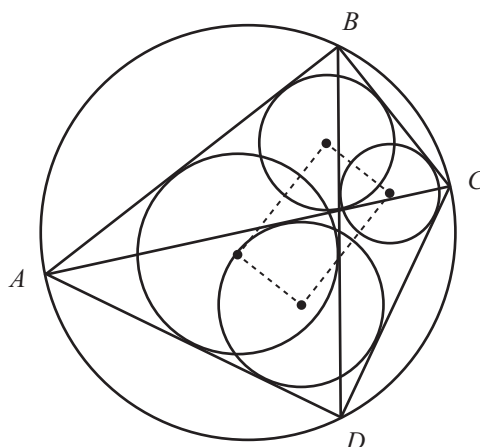


Figure 1.4

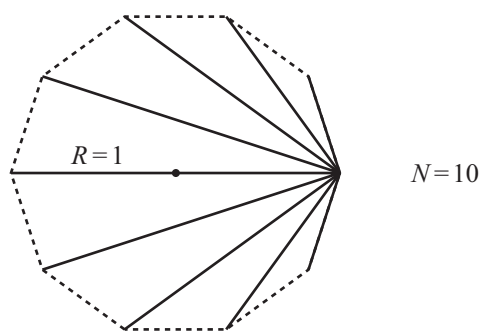


Figure 1.5

22. Mountain climber

A mountain climber wishes to climb up a frictionless conical mountain. He wants to do this by throwing a lasso (a rope with a loop) over the top and climbing up along the rope. Assume that the climber is of negligible height, so that the rope lies along the mountain, as shown in Fig. 1.6.

At the bottom of the mountain are two stores. One sells “cheap” lassos (made of a segment of rope tied to a loop of *fixed* length). The other sells “deluxe” lassos (made of one piece of rope with a loop of *variable* length; the loop’s length may change without any friction of the rope with itself). See Fig. 1.7.

When viewed from the side, the conical mountain has an angle α at its peak. For what angles α can the climber climb up along the mountain if he uses a cheap lasso? A deluxe lasso? (*Hint*: The answer in the cheap case isn’t $\alpha < 90^\circ$.)

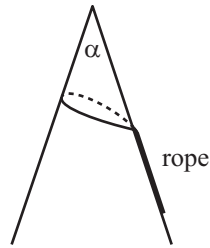


Figure 1.6

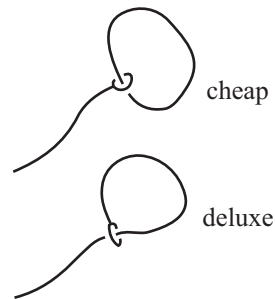


Figure 1.7

1.3 Probability

23. Passing the spaghetti

At a dinner party, n people are seated around a table. A plate of spaghetti starts at the head of the table. The person sitting there takes some spaghetti and then passes the (very large) plate at random to their right or left, with a 50-50 chance of either direction. Henceforth each person receiving the plate takes some spaghetti and then passes the plate at random to their right or left. (Diners who have already received the plate can simply pass it on, without taking any more.) When all the diners have finally received their spaghetti, the plate stops being passed, and the eating begins.

- What is the probability of being the last to be served, as a function of position (relative to the head) at the table of n people?
- If this procedure is repeated over the course of many dinners, what is the average number of times the plate is passed?

24. How many trains?

A train station consists of n parallel train tracks. The trains come at random times on each track, at equal average time intervals on each track. How many trains (including yours) will you see, on average, by the time the train on your track comes? The phrasing of this question is slightly ambiguous, so consider two possibilities:

- (a) You arrive at the station at a random time and then count the trains until your train arrives. You repeat this process on many different days and take an average.
- (b) You hang out at the station for a long time and count the trains (including yours) that arrive between each arrival of the train on your track. You then take an average.

25. Flipping a coin

- (a) Consider the following game. You flip a coin until you get a tails. The number of dollars you win equals the number of coins you end up flipping. (So if you immediately get a tails, you win one dollar; if you get one heads before a tails, you win two dollars, etc.) What is the expectation value of your winnings?
- (b) Play the same game, except now let the number of dollars you win be equal to 2^{n-1} , where n is the number of coins you end up flipping. What is the expectation value of your winnings now? Does your answer make sense?

26. Trading envelopes

- (a) I give you an envelope containing a certain amount of money, and you open it. I then put into a second envelope either twice this amount or half this amount, with a 50-50 chance of each. You are given the opportunity to trade envelopes. Should you?
- (b) I put two sealed envelopes on a table. One contains twice as much money as the other. You pick an envelope and open it. You are then given the opportunity to trade envelopes. Should you?
- (c) If your answers to (a) and (b) are the same, explain why. If they are different, explain why.

27. Waiting for an ace

How many cards do you need to deal from a standard deck, on average, to get your first ace? A standard deck contains 52 cards, four of which are aces.

28. Drunken walk

A drunk performs a random walk along a street. (His steps are all of the same length and equally likely to be in either direction along the street.) At one end of the street is a river, and at the other end is a police station. If he gets to either of these ends, he remains there. He starts n steps from the river, and there are N total steps between the river and the police station.

- (a) What is the probability that he ends up at the river? At the police station?
- (b) What is the expected total number of steps he takes?

29. **HTH and HTT**

A coin is flipped repeatedly, and the resulting string of Heads and Tails is listed out. For example, the string might look like THHTHTTTTHHT. . . .

- Consider the two sequences of letters, HTH and HTT. Which sequence is more likely to occur first? Or are they equally likely to occur first?
- Let E_{HTH} be the expectation value for the number of flips needed to complete the first occurrence of HTH. For example, in the above string, the first HTH sequence is completed on the 5th flip. Likewise for E_{HTT} ; in the above string, the first HTT sequence is completed on the 7th flip. Which of E_{HTH} and E_{HTT} is larger? Or are they equal?
- What are the values of E_{HTH} and E_{HTT} ?
- In a large number of flips, how many times on average will each of HTH and HTT appear? (In the case of HTH sequences, a given H is allowed to count twice. For example, the string HTHTH contains two HTH sequences.)

30. **Staying ahead**

In a two-way election, candidate A receives a votes and candidate B receives b votes, with $a \geq b$. If the ballots are removed one at a time from the ballot box and a running total of the score is kept, what is the probability that at all times A 's sub-total is greater than or equal to B 's sub-total?

31. **Random walk**

Consider a random walk in one dimension. Each step has unit length, with equal probabilities of being rightward or leftward.

- What is the probability, p_{2n} , of returning to the origin (not necessarily for the first time) on the $(2n)$ th step? (If you are at the origin, the number of steps must be even, of course.)
- What is the probability, f_{2n} , of returning to the origin for the first time on the $(2n)$ th step? The technique used in Problem 30 is helpful here.
- Show that $f_{2n} = p_{2n-2} - p_{2n}$. A quick corollary is that $f_2 + f_4 + f_6 + \dots = 1$, which means that you are guaranteed to eventually return to the origin in a 1-D random walk.
- Show that the probability, a_{2n} , of not returning to the origin at any time (even at the end) during a walk of length $2n$ equals p_{2n} . One method is to use the $f_{2n} = p_{2n-2} - p_{2n}$ result, but try to also think of a method that uses the technique from Problem 30.

32. **Standing in a line**

N people are standing in random order in a line, facing forward down the line. How many of them, on average, are able to say, "I am taller than everyone in front of me."?¹

¹There is a semantics issue with the first person in the line (who has no one in front of them). But let's declare that they are able to make the statement. Equivalently, we can work instead with the statement, "There is no one in front of me who is taller than I am." (Assume that no two people have exactly the same height.)

33. Rolling the die

Two players alternately roll an N -sided die. The player who fails to improve upon the previous roll loses. What is the probability that the first player wins?

34. Strands of spaghetti

A bowl contains N spaghetti noodles. You reach into the bowl and grab two free ends at random and attach them. You do this N times until there are no free ends left. On average, how many loops are formed by this process?

35. How much change?

You are out shopping one day with $\$N$, and you find an item whose price takes on a random value between $\$0$ and $\$N$. (You may assume that $\$N$ is large compared with a penny, so that the distribution of prices is essentially continuous.) You buy as many of these items as you can with your $\$N$. What is the expectation value of the money you have left over?

36. Relatively prime numbers

What is the probability that two randomly chosen positive integers are relatively prime (that is, they have no common factor, aside from 1)?

37. The hotel problem

You are driving down a one-way road and pass a strip of a large number, N , of hotels. These all have different rates, arranged randomly. You want to maximize your chance of choosing the cheapest hotel, but you can't return to one you've passed up. Assume that your only goal is to obtain the cheapest hotel (the second cheapest is of no more value to you than the most expensive). If your strategy is to proceed past a certain fraction, x , of the hotels and then pick the next one that is cheaper than all the ones you've seen so far, what should x be? What, then, is your probability of success? Assume that N is very large, and ignore terms in your answer that are of subleading order in N .

38. Decreasing numbers

Pick a random number (evenly distributed) between 0 and 1. Continue picking random numbers as long as they keep decreasing; stop picking when you obtain a number that is greater than the previous one you picked. What is the expected number of numbers you pick?

39. Sum over 1

- (a) You are given a random number (evenly distributed) between 0 and 1. To this, you add a second such random number. Keep adding numbers until the sum exceeds 1, and then stop. How many numbers, on average, will you need?
- (b) When the sum finally exceeds 1 and the game stops, what is the expectation value for the sum?

40. **Convenient migraines**

A probability course at a particular college has two exam days and 18 lecture days. At the end of the semester, the teacher notes that a certain student claimed to have a migraine headache on both of the exam days, thereby asking for the exams to be postponed. The teacher also notes that this student never had a headache during any of the 18 lectures.

The teacher is understandably suspicious of this coincidence, so she tries to calculate the probability P of having a headache on, and only on, the two exam days. But she quickly realizes, of course, that it is impossible to calculate P without knowing the probability p of a migraine occurring on a given day (which we'll assume is completely random and not based on real-life effects such as stress, etc.). Nevertheless, she realizes that it is possible to calculate the *maximum possible* value of P . And since any argument of plausibility on the student's part can use at most this best-case result, the teacher ends up coming to a fairly certain conclusion, as you will too.

- (a) Let P be the probability of having a migraine on, and only on, the days of exams. If there are a exams and b lectures, what probability p of having a migraine on a given day leads to the maximum possible P ?
- (b) What is this maximum probability P_{\max} in terms of a and b ? What is P_{\max} if $a = 2$ and $b = 18$?
- (c) In the approximation where a is much smaller than b , and where b is assumed to be given, how does P_{\max} depend on a ?

41. **Letters in envelopes**

You are given N addressed letters and N addressed envelopes. If you randomly put one letter in each envelope, what is the probability that no letter ends up in the correct envelope?

42. **Leftover dental floss**

Two rolls of dental floss initially have equal lengths, L . Each day, a person chooses one of the rolls at random and cuts off a fixed small length, d . This continues until one of the rolls runs out of floss. How much floss, on average, is left on the other roll at this time? Assume that $N \equiv L/d$ is large, and ignore terms in your answer that are of subleading order in N . You will need to use a result from Problem 56.

43. **Comparing the numbers**

The integers 1 through N are put in a hat. (Technically, any set of N distinct numbers would work just as well.) You and $N - 1$ other people each pick a number. You then compare your number with the others, one at a time, until you find one that is smaller than yours. This procedure is repeated many times. How many numbers, on average, will you need to check in order to find one that is smaller than yours? Consider two cases:

- (a) You ask the other people randomly. That is, at all times you have equal probabilities of asking each person. This could be arranged, for example,

by demanding that you have a very bad memory, so that you might ask a given person more than once.

- (b) You have a good memory. In other words, you don't ask a given person more than once.

Ignore the scenarios where you have the number 1, because otherwise in part (a) the average would be infinite, and in part (b) you would always end up checking $N - 1$ numbers and never finding a smaller one.

44. Shifted intervals

Let $\epsilon \equiv 1/N$, where N is large. Choose a number at random between 0 and 1. Choose a second number between ϵ and $1 + \epsilon$. Choose a third number between 2ϵ and $1 + 2\epsilon$. Continue this process, until you choose an N th number between $1 - \epsilon$ and $2 - \epsilon$. What is the probability that the first number you choose is the smallest of all the numbers? Assume that N is very large, and make suitable approximations.

45. Intervals between independent events

Consider a repeating event that happens completely randomly in time. Such a process can be characterized by the probability per unit time (call it p) of an event happening. The definition of p is that the probability of an event happening in an infinitesimal² time dt equals $p dt$.

From Problem 54 (you should solve that problem before this one), we know that starting at any particular time (not necessarily the time of an event), the probability that the next event happens between t and $t + dt$ later equals $pe^{-pt} dt$. (You can quickly show that the integral of this probability equals 1, as it must.) We're using p here in place of the λ from Problem 54.

- Using the $pe^{-pt} dt$ probability, show that starting at any particular time (not necessarily the time of an event), the average waiting time to the next event equals $1/p$. Explain why this is also the average time between events.
- Pick a random point in time, and look at the length of the time interval (between successive events) that it belongs to. Explain, using the above results, why the average length of this interval is $2/p$, and not $1/p$.
- We have found that the average time between events is $1/p$, and also that the average length of the interval surrounding a randomly chosen point in time is $2/p$. Someone might think that these two results should be the same. Explain intuitively why they are not.
- By correctly incorporating the probability distribution $pe^{-pt} dt$ mentioned above, show mathematically why $2/p$ is the correct result for the average length of the interval surrounding a randomly chosen point in time.

²The probability of an event happening in a noninfinitesimal time t is *not* equal to pt . If t is large enough, then pt is larger than 1, so it certainly can't represent a probability. pt is the *average number* of events that occur in time t . But this doesn't equal the probability that an event occurs, because there can be double, triple, etc. events in the time t . We don't have to worry about multiple events if dt is infinitesimal.

46. The prosecutor's fallacy

Consider the following scenario. Detectives in a city, say, Boston (whose population we will assume to be one million), are working on a crime and have put together a description of the perpetrator, based on things such as height, a tattoo, a limp, an earring, etc. Let's assume that only one person in 10,000 fits the description. On a routine patrol the next day, police officers see a person fitting the description. This person is arrested and brought to trial based solely on the fact that he fits the description.

During the trial, the prosecutor tells the jury that since only one person in 10,000 fits the description (a true statement), it is highly unlikely (far beyond a reasonable doubt) that an innocent person fits the description (again a true statement), and therefore it is highly unlikely that the defendant is innocent. If you were a member of the jury, would you cast a "guilty" vote? If yes, what is your level of confidence? If no, what is wrong with the prosecutor's reasoning?

47. The game-show problem

A game-show host offers you the choice of three doors. Behind one of these doors is the grand prize, and behind the other two are goats. The host (who knows what is behind each of the doors) announces that after you select a door (without opening it), he will open one of the other two doors and purposefully reveal a goat. You select a door. The host then opens one of the other doors and reveals the promised goat. He then offers you the chance to switch your choice to the remaining door. To maximize the probability of winning the grand prize, should you switch or not? Or does it not matter?

48. A random game-show host

Consider the following variation of Problem 47. A game-show host offers you the choice of three doors. Behind one of these doors is the grand prize, and behind the other two are goats. The host announces that after you select a door (without opening it), he will *randomly* open one of the other two doors. You select a door. The host then randomly opens one of the other doors, and the result happens to be a goat. He then offers you the chance to switch your choice to the remaining door. Should you switch or not? Or does it not matter?

49. The birthday problem

- (a) How many people need to be in a room in order for there to be a greater than $1/2$ probability that at least two of them have the same birthday? By "same birthday" we mean the same day of the year; the year may differ. Ignore leap years.
- (b) Assume that there is a large number N of days in a year. How many people are now necessary for the odds to favor a common birthday? Equivalently, assuming a normal 365-day year, how many people are required for there to be a greater than $1/2$ probability that at least two of them were born in the same hour on the same date? Or in the same minute of the same hour on the same date? Neglect terms in your answer that are of subleading order in N .

50. The boy/girl problem

The classic “boy/girl” problem can be stated in many different ways, with answers that may or may not be the same. Three different formulations are presented below, and a fourth is given in Problem 51. Assume in all of them that any process involved in the scenario is completely random. That is, assume that any child is equally likely to be a boy or a girl (even though this isn’t quite true in real life), and assume that there is nothing special about the person you’re talking with, and assume that there are no correlations between children (as there are with identical twins), and so on.

- (a) You bump into a random person on the street who says, “I have two children. At least one of them is a boy.” What is the probability that the other child is also a boy?
- (b) You bump into a random person on the street who says, “I have two children. The older one is a boy.” What is the probability that the other child is also a boy?
- (c) You bump into a random person on the street who says, “I have two children, one of whom is this boy standing next to me.” What is the probability that the other child is also a boy?

51. Boy/girl problem with general information

This problem is an extension of the preceding problem. You should study that one thoroughly before tackling this one. As in the original versions of the problem, assume that all processes are completely random. The new variation is the following:

You bump into a random person on the street who says, “I have two children. At least one of them is a boy whose birthday is in the summer.” What is the probability that the other child is also a boy?

What if the clause is changed to “a boy whose birthday is on August 11th”? Or, “a boy who was born during a particular minute on August 11th”? Or more generally, “a boy who has a particular characteristic that occurs with probability p ”?

1.4 Foundational**52. Stirling’s formula**

- (a) Using $N! = \int_0^\infty x^N e^{-x} dx$ (which you can prove by induction), derive Stirling’s formula,

$$N! \approx N^N e^{-N} \sqrt{2\pi N}. \quad (1.4)$$

Hint: Write $x^N e^{-x}$ as $e^{N \ln x - x} \equiv e^{f(x)}$, and then expand $f(x)$ in a Taylor series about its maximum, which you can show occurs at $x = N$. (See the appendix for a review of Taylor series.)

- (b) Find also the first-order (in $1/N$) correction to Stirling’s formula. (This calculation is a bit tedious.)

53. A handy formula

Expressions of the form $(1 + a)^n$ come up often in mathematics, especially in probability. Show that for small a ,

$$(1 + a)^n \approx e^{na}. \quad (1.5)$$

Under what condition is this expression valid? Show that a more accurate approximation to $(1 + a)^n$ is

$$(1 + a)^n \approx e^{na} e^{-na^2/2}. \quad (1.6)$$

Under what condition is this expression valid? How should the righthand side be modified to make it even more accurate? There are various ways to answer these questions, but the cleanest way is to integrate both sides of the formula for the sum of an infinite geometric series,

$$1 - a + a^2 - a^3 + a^4 - \dots = \frac{1}{1 + a}. \quad (1.7)$$

54. Exponential distribution

Consider a repeating event that happens completely randomly in time. By “completely randomly” we mean that there is a uniform probability that an event happens at any given instant (or more precisely, in any small time interval of a given length), independent of what has already happened. That is, the process has no “memory.” Let the average time between events be τ . Equivalently, let $\lambda = 1/\tau$ be the average rate at which the events occur (the number per second, or whatever unit of time is being used).

The task of this problem is to derive the probability distribution $\rho(t)$ for the waiting time until the next event occurs is, that is, to determine the function $\rho(t)$ for which $\rho(t) dt$ is the probability that the next event occurs at a time between t and $t + dt$ (with $t = 0$ being when you start waiting). Show that $\rho(t)$ is given by

$$\rho(t) = \lambda e^{-\lambda t}. \quad (1.8)$$

This is (naturally) called the exponential distribution. You will need to use a result from Problem 53.

55. Poisson distribution

As with the exponential distribution in Problem 54, consider a repeating event that happens completely randomly in time. Show that the probability distribution for the number k of events that occur during a given time interval takes the form of the Poisson distribution,

$$P(k) = \frac{a^k e^{-a}}{k!}, \quad (1.9)$$

where a is the expected (average) number of events in the given interval. You will need to use a result from Problem 53. Note that whereas the exponential distribution deals with the *waiting time* until the next event, the Poisson distribution deals with the *number of events* in a given time (or space, etc.) interval.

56. Gaussian approximation to the binomial distribution

If n coins are flipped, the probability of obtaining k Heads is given by the binomial distribution, $P(k) = \binom{n}{k}/2^n$. This follows from the fact that there are 2^n equally likely outcomes for the string of n flips, and $\binom{n}{k}$ of these outcomes have k Heads.

To make the math more tractable in this problem, let's replace n with $2n$, and k with $n + x$. So we're flipping $2n$ coins, and we want to get x Heads relative to the average (which is n). The probability is then

$$P_B(x) = \frac{1}{2^{2n}} \binom{2n}{n+x} \quad (\text{for } 2n \text{ coin flips}) \quad (1.10)$$

where the subscript B is for binomial. Show that if n is large, $P_B(x)$ takes approximately the Gaussian form,

$$P_B(x) \approx \frac{e^{-x^2/n}}{\sqrt{\pi n}} \equiv P_G(x) \quad (\text{for } 2n \text{ coin flips}) \quad (1.11)$$

where the subscript G is for Gaussian. You will need to use the results from Problems 52 and 53.

57. Gaussian approximation to the Poisson distribution

Show that in the limit of large a , the Poisson distribution from Problem 55,

$$P_P(k) = \frac{a^k e^{-a}}{k!}, \quad (1.12)$$

takes approximately the Gaussian form,

$$P_P(x) \approx \frac{e^{-x^2/2a}}{\sqrt{2\pi a}}, \quad (1.13)$$

where $x \equiv k - a$ is the deviation from the average, a . You will need to use the results from Problems 52 and 53.